

So far I have not been able to get any other definite results than those I have given above, which correspond to those given by Hicks (*Phil. Trans.*, 1885).

It is true that all have the same peculiarity, which causes the difficulty of accounting for the masses of the elements as compared with the ether on the vortex hypothesis. In no one of the cases I have obtained does our criterion of stability allow the density of the core to be greater than that of the fluid outside.

Yet, in the face of the period equation, it would be rash to say that this must hold in all cases. Though my work has not brought to light any arrangement of density and circulation which would satisfy the conditions for stability and yet allow for the excessive masses of the elements, the question is reduced to the consideration of this equation. At any rate, before we reject the vortex atom theory because of the difficulty in certain types of ring of accounting for the difference in mass between gross matter and the ether, we must have more proof that its roots are unreal, when we have that distribution of density for which this difference seems to call.

*Note on the Symmetric Group.** By W. BURNSIDE. Received
November 9th, 1896. Read November 12th, 1896.

The symmetric groups of three and four symbols are capable of abstract definition by the relations

$$A^3 = 1, \quad B^3 = 1, \quad (AB)^2 = 1,$$

and
$$A^3 = 1, \quad B^3 = 1, \quad (AB)^4 = 1,$$

respectively; while the alternating group of five symbols can be defined by

$$A^5 = 1, \quad B^5 = 1, \quad (AB)^5 = 1.$$

* [On November 23rd, 1896, I received from Prof. F. H. Moore, of the University of Chicago, for communication to the Society, a paper dealing with the abstract definition of the symmetric and the alternating groups. As the results contained in Prof. Moore's paper are much more complete than mine, I asked permission from the Council to withdraw my communication. Its appearance in the *Proceedings* is due to the Council having expressed an opinion that it was desirable that both papers should be printed.—W. B.]

So far as I know, no attempt has hitherto been made to define in abstract form the symmetric group of more than four symbols, or the alternating group of more than five. For the investigations of the various types of group which can be constructed from given factor-groups, a definition of the factor-groups in abstract form is, except in the simplest cases, essential.

In the present note I have made one step towards this definition for the case of the symmetric group of n symbols, whatever n may be, by showing that, in addition to

$$S_1^2 = 1, \quad S_n^n = 1, \quad (S_n S_1)^{n-1} = 1,$$

$\frac{1}{2}(3n-10)$ or $\frac{1}{2}(3n-11)$ additional equations, according as n is even or odd, are certainly sufficient to insure that the group generated by S_1 and S_n shall be the symmetric group of n symbols. In the last paragraph I show that the alternating group of six symbols (the simple group of lowest order for which an abstract definition has not hitherto been given) is defined by

$$A^3 = 1, \quad S^3 = 1, \quad (AS)^3 = 1, \quad (AS^{-1})^4 = 1, \quad (AS^3 AS^{-3})^2 = 1.$$

1. Let s_1, s_2, \dots, s_n represent the substitutions

$$(a_1 a_2), (a_1 a_2 a_3), \dots, (a_1 a_2 a_3 \dots a_{n-1} a_n).$$

Whatever n may be, it is easy to verify that

$$s_{n-1} = s_n^2 s_2 s_n^{-1};$$

and therefore that s_3, s_4, \dots, s_{n-1} can be expressed in terms of s_1 and s_n . The expressions thus obtained are rather complicated. It is simple, however, to verify directly that

$$s_n^{r+1} (s_n^{-1} s_2)^r$$

is a circular substitution of $n-r$ letters for all values of r from 0 to $n-2$; and that therefore $s_n^{r+1} (s_n^{-1} s_2)^r$ and s_{n-r} are conjugate substitutions in the symmetric group. The relations

$$s_3^3 = 1, \quad s_4^4 = 1, \quad \dots, \quad s_{n-1}^{n-1} = 0,$$

are therefore equivalent to

$$[s_n^{r+1} (s_n^{-1} s_2)^r]^{n-r} = 1. \quad (r = 1, 2, \dots, n-3)$$

The form of these relations is capable of considerable modification.

Thus $s_n^{-1}s_2$ is the inverse of s_2s_n , a substitution of order $n-1$, so that

$$(s_n^{-1}s_2)^r = (s_2s_n)^{n-r-1}.$$

For $n-r=3$, the relation is

$$[s_n^{-2}(s_2s_n)^2]^3 = 1,$$

or $(s_n^{-2}s_2s_ns_2s_n)^3 = 1,$

or $(s_n^{-1}s_2s_ns_2)^3 = 1.$

For $n-r=4$, it is $[s_n^{-3}(s_2s_n)^3]^4 = 1,$

or $(s_n^{-3}s_2s_ns_2s_ns_2)^4 = 1 ;$

and so on.

2. Let S_2 and S_{n-1} be two operations of orders 2 and $n-1$; and suppose that

$$f_1(S_2, S_{n-1}) = 1, \quad f_2(S_2, S_{n-1}) = 1, \quad \dots, \quad f_i(S_2, S_{n-1}) = 1,$$

form a set of relations connecting S_2 and S_{n-1} , sufficient and necessary to insure that the group generated by S_2 and S_{n-1} shall be simply isomorphic with the group generated by the two substitutions s_2 and s_{n-1} (i.e., with the symmetric group of $n-1$ symbols); S_2 and s_2 , S_{n-1} and s_{n-1} , being corresponding operations.

Also let S_n be an operation of order n , connected with S_2 and S_{n-1} by the two relations

$$S_{n-1} = S_n^2 S_2 S_n^{-1}, \quad (1)$$

and $(S_n^{-2} S_2 S_n^2 S_2)^2 = 1. \quad (2)$

We may then show that S_2 and S_n generate a group simply isomorphic with the symmetric group of n symbols.

To this end, we first prove that the cyclical group generated by S_n and the group generated by S_2 and S_{n-1} are permutable with each other; i.e., that, if the operations of the latter group are represented by the symbols T_i , then, for every value of r and i , $S_n^r T_i$ can be expressed in the form $T_i S_n^r$. Since every T_i can be expressed in terms of S_2 and S_{n-1} , it is clearly sufficient to show that, for all values of r , $S_n^r S_2$ and $S_n^r S_{n-1}$ can each be expressed in the form $T_i S_n^r$.

Now, from (1), it follows that

$$S_n S_2 = S_{n-1}^{-1} S_n^2,$$

$$S_n^2 S_2 = S_{n-1} S_n,$$

$$S_n^{-1} S_{n-1} = S_{n-1}^{-1} S_n,$$

and

$$S_n^{-2} S_{n-1} = S_2 S_n^{-1}.$$

Also

$$S_n^r S_{n-1} = S_n^{r+2} S_2 S_n^{-r-2} S_n^{r+1}.$$

Hence, if $S_n^{r+2} S_2 S_n^{-r-2}$ belongs to the group generated by S_2 and S_{n-1} for all values of r from 1 to $n-3$, the two groups in question are permutable with each other. We shall prove that these conditions are satisfied by showing that

$$S_n^{r+2} S_2 S_n^{-r-2} = S_{n-1}^{r+1} S_2 S_{n-1}^{-r-1}, \quad (r = 1, 2, \dots, n-3)$$

or that $S_n^{-r-2} S_{n-1}^{r+1}$ is permutable with S_2 .

Expressed at length this condition is

$$S_n^{-r-2} (S_n^2 S_2 S_n^{-1})^{r+1} S_2 (S_n^2 S_2 S_n^{-1})^{-r-1} S_n^{r+2} S_2 = 1,$$

$$\text{or} \quad S_n^{-r-1} (S_n S_2)^{r+1} S_{n-1}^{-1} S_2 S_n (S_2 S_n^{-1})^{r+1} S_n^{r+1} S_2 = 1. \quad (3)$$

$$\text{Now} \quad S_{n-1}^{-1} S_2 S_{n-1} S_2 = S_n S_2 S_n^{-2} S_2 S_n^2 S_n^{-1} S_2,$$

$$\text{and, from (2), } S_2 S_n^{-2} S_2 S_n^2 S_2 = S_n^{-2} S_2 S_n^2,$$

$$\begin{aligned} \text{so that} \quad S_{n-1}^{-1} S_2 S_{n-1} S_2 &= S_n S_n^{-2} S_2 S_n^2 S_n^{-1} S_2 \\ &= S_n^{-1} S_2 S_n S_2. \end{aligned}$$

Also, since the group generated by S_2 and S_{n-1} is simply isomorphic with the substitution group of § 1 when $n-1$ is written for n ,

$$(S_{n-1}^{-1} S_2 S_{n-1} S_2)^3 = 1.$$

Hence also

$$(S_n^{-1} S_2 S_n S_2)^3 = 1.$$

Writing now the equation (3) in the form

$$S_n^{-r-1} (S_n S_2)^r S_n S_2 S_n^{-1} S_2 S_n S_2 S_n^{-1} S_2 S_n^{-1} (S_2 S_n^{-1})^{r-1} S_n^{r+1} S_2 = 1,$$

and using the result just obtained, we find that

$$S_n^{-r-1} (S_n S_2)^r S_2 S_n S_2 S_n^{-2} (S_2 S_n^{-1})^{r-1} S_n^{r+1} S_2 = 1,$$

$$\text{or} \quad S_n^{-r-1} (S_n S_2)^{r-1} S_n^2 S_2 S_n^{-2} (S_2 S_n^{-1})^{r-1} S_n^{r+1} S_2 = 1. \quad (4)$$

This condition then must be satisfied for all values of r from 1 to $n-3$.

3. Since S_2 and S_{n-1} generate a group simply isomorphic with the group generated by the substitutions s_2 and s_{n-1} , it may be verified by forming the corresponding substitutions that

$$(S_{n-1}^r S_2 S_{n-1}^{-r} S_2)^2 = 1. \quad (r = 2, 3, \dots, n-3)$$

Hence, writing for S_{n-1} its value in terms of S_n ,

$$[S_n (S_n S_2)^r S_n^{-1} S_2 S_n (S_2 S_n^{-1})^r S_n^{-1} S_2]^2 = 1,$$

and, using in this the relation

$$(S_n S_2 S_n^{-1} S_2)^2 = S_2 S_n S_2 S_n^{-1},$$

$$\text{it becomes} \quad [S_n (S_n S_2)^{r-2} S_n^2 S_2 S_n^{-2} (S_2 S_n^{-1})^{r-2} S_n^{-1} S_2]^2 = 1. \quad (5)$$

For $r = 2$, this gives

$$(S_n^3 S_2 S_n^{-3} S_2)^2 = 1.$$

Again, using the relation

$$S_2 S_n^2 S_2 S_n^{-2} = S_n^2 S_2 S_n^{-2} S_2,$$

derived from (2), in (5), it becomes

$$[S_n (S_n S_2)^{r-3} S_n^3 S_2 S_n^{-3} (S_2 S_n^{-1})^{r-3} S_n^{-1} S_2]^2 = 1.$$

For $r = 3$, this gives

$$(S_n^4 S_2 S_n^{-4} S_2)^2 = 1.$$

This process may clearly be continued step by step so that, from the equation at the beginning of this paragraph with equation (2), we derive

$$(S_n^r S_2 S_n^{-r} S_2)^2 = 1. \quad (r = 3, 4, \dots, n-2)$$

These equations are not all distinct, the values $r = t$ and $r = n-t$ clearly giving the same equation.

4. Returning to equation (4), we may now verify step by step that it is satisfied for all values of r concerned. Thus, if we put $r = 1$, it is satisfied identically.

Making use of the relation

$$S_2 S_n^2 S_2 S_n^{-2} = S_n^2 S_2 S_n^{-2} S_2,$$

(4) may be written

$$S_n^{-r-1} (S_n S_2)^{r-2} S_n^3 S_2 S_n^{-3} (S_2 S_n^{-1})^{r-2} S_n^{r+1} S_2 = 1,$$

and in this form it is satisfied identically for $r = 2$.

Using now the relation

$$S_2 S_n^3 S_2 S_n^{-3} = S_n^3 S_2 S_n^{-3} S_2,$$

obtained in the last paragraph, (4) becomes

$$S_n^{-r-1} (S_n S_2)^{r-3} S_n^4 S_2 S_n^{-4} (S_2 S_n^{-1})^{r-3} S_n^{r+1} S_2 = 1,$$

and it is satisfied identically for $r = 3$. Proceeding thus, we show by the aid of the results of the last paragraph that (4) is satisfied for all values of r up to $n-3$.

Hence the cyclical group generated by S_n is permutable with the group generated by S_2 and S_{n-1} , and therefore S_2 and S_n generate a group whose order is equal to or is a factor of $n!$. Now the substitutions s_2 and s_n generate a group of order $n!$ (the symmetric group of n symbols), and they satisfy all the relations satisfied by S_2 and S_n . Hence the order of the group generated by S_2 and S_n cannot be less than $n!$. It is therefore equal to $n!$; and at the same time it is proved that the group generated by S_2 and S_n is simply isomorphic with the substitution group generated by s_2 and s_n , i.e., with the symmetric group of n symbols.

5. The i independent relations,

$$f_r (S_2, S_{n-1}) = 1, \quad (r = 1, 2, \dots, i)$$

will not be increased in number, when in each of them S_{n-1} is replaced by $S_n^2 S_2 S_n^{-1}$. In addition to these i relations, S_2 and S_n satisfy the two further equations

$$S_n^n = 1,$$

and

$$(S_n^{-2} S_2 S_n^2 S_2)^3 = 1.$$

Hence, if S_2 and S_{n-1} , the generators of the symmetric group of $n-1$ symbols, are connected by i independent relations, then S_2 and S_n , the generators of the symmetric group of n symbols, are connected by at most $i+2$ independent relations.

6. Now for the symmetric group of five symbols we may show directly that

$$S_3^2 = 1, \quad (S_3^{-1} S_3 S_3 S_3)^3 = 1, \quad (S_3 S_3)^4 = 1, \quad S_3^5 = 1, \quad (6)$$

form a necessary and sufficient set of relations between S_3 and S_3 . For this purpose, consider the operations A and B defined by

$$A = S_3^{-2}, \quad B = (S_3 S_3)^3.$$

Since
$$S_3^{-1} S_3 S_3 S_3 = S_3^{-2} (S_3 S_3)^2,$$

A and B satisfy the relations

$$A^5 = 1, \quad B^3 = 1, \quad (AB)^3 = 1;$$

and therefore* A and B generate a group of order 60. Now this group contains the operation $A^3 B$ or $S_3 S_3 S_3$; and therefore it contains every operation of the group generated by S_3 and S_3 in which S_3 occurs an even number of times. The set of operations $S_3 T$, when for T is put in turn every operation of the group generated by A and B , must therefore give every operation of the group generated by S_3 and S_3 in which S_3 occurs an odd number of times. The order of the group generated by S_3 and S_3 is therefore 120. Now the substitutions s_3 and s_3 satisfy the relations which S_3 and S_3 satisfy and generate a group of order 120. Hence S_3 and S_3 , when connected by the relations (6), generate a group simply isomorphic with the symmetric group of five symbols.

It remains to show that the relations (6) are independent. If they are not, either the second or third of the equations must be redundant. The second cannot be, for the relations

$$S_3^2 = 1, \quad S_3^5 = 1, \quad (S_3 S_3)^4 = 1,$$

do not define a group of finite order.†

Again, the relations

$$S_3^2 = 1, \quad S_3^5 = 1, \quad (S_3^{-1} S_3 S_3 S_3)^3 = 1,$$

* Hamilton, *Phil. Mag.*, 1856, p. 446.

† Dyck, "Gruppentheoretische Studien," *Math. Ann.*, **xx** (1882), § 15.

certainly cannot define a group simply isomorphic with the symmetric group of five symbols, for the substitutions

$$\sigma_2 = (a_1 a_3)(a_3 a_5), \quad \sigma_5 = (a_1 a_2 a_3 a_4 a_5)$$

satisfy these relations, and they generate the alternating group of five symbols.

Hence, finally, the relations (6) are necessary as well as sufficient to define the symmetric group of five symbols.

7. If we now define S_r in terms of S_{r+1} , for all values of r , by the relation

$$S_r = S_{r+1}^2 S_2 S_{r+1}^{-1},$$

the symmetric group of five symbols is given by

$$S_2^2 = 1, \quad S_3^2 = 1, \quad S_4^2 = 1, \quad S_5^2 = 1.$$

Hence S_2 and S_5 will generate a group simply isomorphic with the symmetric group of six symbols, if

$$S_2^3 = 1, \quad S_3^3 = 1, \quad S_4^3 = 1, \quad S_5^3 = 1, \quad S_6^3 = 1,$$

and

$$(S_6^{-2} S_2 S_6^2 S_5)^2 = 1.$$

Proceeding thus, step by step, we find that the relations

$$S_r^r = 1, \quad (r = 2, 3, \dots, n),$$

and

$$(S_r^{-2} S_2 S_r^2 S_5)^3 = 1, \quad (r = 6, 7, \dots, n),$$

are sufficient to insure that S_2 and S_n shall generate a group simply isomorphic with the symmetric group of n symbols.

These relations may be simplified, and some of them may be shown to be redundant, as follows:—

The equation
$$S_3^3 = 1$$

is equivalent, when $n = 5$, to

$$(S_3^{-1} S_2 S_5 S_2)^3 = 1.$$

Let us suppose that, when $n = r$, it is equivalent to

$$(S_r^{-1} S_2 S_r S_2)^3 = 1.$$

Writing in this $S_{r+1}^2 S_3 S_{r+1}^{-1}$ for S_r , it becomes

$$(S_{r+1} S_3 S_{r+1}^{-2} S_3 S_{r+1}^2 S_3 S_{r+1}^{-1} S_3)^3 = 1;$$

and since, when $n = r+1$, one of the defining relations is

$$(S_{r+1}^{-2} S_3 S_{r+1}^2 S_3)^3 = 1,$$

the previous equation becomes

$$(S_{r+1}^{-1} S_3 S_{r+1} S_3)^3 = 1.$$

Hence, for all values of n ,

$$S_3^3 = 1$$

may be replaced by $(S_n^{-1} S_3 S_n S_3)^3 = 1.$ (7)

In the next place, it follows from § 3 that the relation

$$(S_{n-1}^r S_3 S_{n-1}^{-r} S_3)^3 = 1,$$

which holds for all values of r other than ± 1 , is equivalent to

$$(S_n^{r+1} S_3 S_n^{-r-1} S_3)^3 = 1.$$

Hence

$$(S_{n-r}^{-3} S_3 S_{n-r}^2 S_3)^3 = 1$$

is equivalent to

$$(S_n^{-r-2} S_3 S_n^{r+2} S_3)^3 = 1.$$

The relations

$$(S_r^{-2} S_3 S_r^2 S_3)^3 = 1, \quad (r = 6, 7, \dots, n)$$

may therefore be replaced by

$$(S_n^{-t} S_3 S_n^t S_3)^3 = 1. \quad (t = 2, 3, \dots, n-4) \quad (8)$$

Now

$$(S_n^{-t} S_3 S_n^t S_3)^3 = 1$$

and

$$(S_n^{-n+t} S_3 S_n^{n-t} S_3)^3 = 1$$

are equivalent, and therefore of the relations (8) only those for which t is not greater than $\frac{1}{2}n$ can be independent.

Finally, the equations $S_r^r = 1$

may be expressed in a simple form in terms of S_n and S_3 . Thus

$$S_{n-1}^{n-1} = 1$$

is the same as $(S_n^2 S_2 S_n^{-1})^{n-1} = 1,$

and it is therefore equivalent to

$$(S_n S_2)^{n-1} = 1.$$

Again,

$$S_{n-2}^{n-2} = 1$$

is equivalent to

$$(S_{n-1} S_2)^{n-2} = 1,$$

or to

$$(S_n^2 S_2 S_n^{-1} S_2)^{n-2} = 1;$$

and

$$S_{n-3}^{n-3} = 1$$

is equivalent to $(S_{n-1}^2 S_2 S_{n-1}^{-1} S_2)^{n-3} = 1,$

which, expressed in terms of S_n and S_2 , is

$$(S_n^2 S_2 S_n S_2 S_n^{-1} S_2 S_n S_2 S_n^{-2} S_2)^{n-3} = 1.$$

Using (7) in this, it becomes

$$(S_n^2 S_n S_2 S_n^{-1} S_2 S_n^{-1} S_2)^{n-3} = 1,$$

or

$$(S_n^3 S_2 S_n^{-1} S_2 S_n^{-1} S_2)^{n-3} = 1.$$

Proceeding thus, step by step, we show that, for each value of r ,

$$S_r^r = 1$$

is equivalent to

$$[S_n^{-r+1} (S_n^{-1} S_2)^{-r+1}]^r = 1,$$

or

$$[S_n^{-r+1} (S_2 S_n)^{r-1}]^r = 1, \quad (9)$$

or

$$[S_n^{-r+1} (S_n S_2)^{r-1}]^r = 1.$$

Collecting the results thus obtained, we find that the relations

$$[S_n^{-r+1} (S_n S_2)^{r-1}]^r = 1, \quad (r = 2, 3, \dots, n)$$

and

$$(S_n^{-r} S_2 S_n^r S_2)^2 = 1, \quad \left(r = 2, 3, \dots, \frac{n-1}{2} \text{ or } \frac{n}{2}\right)$$

are sufficient to insure that S_n and S_2 shall generate a group simply isomorphic with the symmetric group of n symbols. Besides the equations

$$S_2^2 = 1, \quad S_n^n = 1, \quad (S_n S_2)^{n-1} = 1,$$

giving the orders of the generating operations and of their product, there are thus *at most* $\frac{3n-10}{2}$ or $\frac{3n-11}{2}$ additional relations according as n is even or odd.

8. When $n = 6$, the relations are

$$\begin{aligned} S_1^2 = 1, \quad (S_0 S_1 S_0^{-1} S_1)^5 = 1, \quad (S_0^2 S_1 S_0^{-1} S_1)^4 = 1, \\ (S_0 S_1)^5 = 1, \quad (S_0^2 S_1 S_0^{-2} S_1)^2 = 1, \quad S_0^6 = 1. \end{aligned}$$

It is easy to verify from these relations that $S_0 S_1 S_0^{-1}$ is permutable with $S_0^2 S_1 S_0^{-1} S_1$, so that

$$(S_0^2 S_1 S_0^{-1} S_1)^4 = 1$$

is equivalent to $[(S_0 S_1)^5 S_0^{-1} S_1]^4 = 1$.

The alternating group of six symbols will consist of those operations of the symmetric group which, when expressed in terms of S_0 and S_1 , contain an even number of factors. It may therefore be generated by S_0^2 and $S_0 S_1$.

If now we write $A = S_0^{-2}$, $O = S_0 S_1$,

all the defining relations of the symmetric group (the third being replaced by the equivalent form just found), except the first, may be written in terms of A and O . Moreover, since S_1 does not occur in the alternating group, the first of the above equations cannot be one of its defining relations. The defining relations of the alternating group of six symbols may therefore be written

$$A^3 = 1, \quad O^3 = 1, \quad (OAC)^3 = 1, \quad (O^2AC)^4 = 1,$$

and $(A^{-1}O^{-1}AC)^2 = 1$.

By taking O^2 instead of O for a generating operation, these relations become

$$A^3 = 1, \quad O^3 = 1, \quad (AO)^3 = 1, \quad (AO^{-1})^4 = 1, \quad (AO^2A^{-1}O^{-2})^2 = 1.$$