Review
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Many reproductions of antique forms are given, including a photographic one of an inscription by the Buddhist King Asoka, and a table from Mr. Hill's paper in *Archæologia*, 1910. We give two quotations.

(i) On the discussion of the Boethius question. "It is true we have no records of the interchange of learning, in any large way, between Eastern Asia and Central Europe in the century preceding the time of Boethius. But it is one of the mistakes of scholars to believe that they are the sole transmitters of knowledge. As a matter of fact there is abundant reason for believing that the Hindu numerals would naturally have been known to the Arabs, and even along every trade route to the remote west long before the zero entered to make the place value possible, and that the characters, the methods of calculating, the improvements that took place from time to time, the zero when it appeared, and the customs in solving business problems, would all have been made known from generation to generation along the same trade routes from the Orient to the Occident. It must always be kept in mind that it was to the tradesman and the wandering scholar that the spread of such learning was due rather than to the schoolman."

(ii) On the spread of the numerals in Europe. "From his (Mr. Hill's) investigations it appears that the earliest occurrence of a date in these numerals on a coin is found in the reign of Roger of Sicily, in 1138 A.D. Until recently it was thought that the earliest such date was 1217 A.D. for an Arabic piece, and 1388 for a Turkish coin. Most of the seals and medals containing dates that were at one time thought to be very early have been shown by Mr. Hill to be of comparatively late workmanship. There are, however, in European manuscripts numerous instances of the use of these numerals before the twelfth century. Besides the example in the Codex Vigilanus, another of the tenth century has been found in the St. Gall MS. now in the University Library at Zürich, the forms differing materially from those in the Spanish Codex. . . . It is of interest to add that among the earliest dates of European coins or medals in these numerals are the following: Austria, 1484; Germany, 1489 (Cologne); Switzerland, 1424 (St. Gall); Netherlands, 1474; France, 1485; Italy, 1390." E. M. LANGLEY.

Die complexen Veränderlichen und ihre Funktionen. By G. KOWALEWSKI. Pp. ii + 455. 12 m. 1911. (Teubner.)

This is a charming book, written not only with extreme clearness and precision, but also with a freshness and originality seldom to be found in books which purport to be elementary text-books. When I began reading it I was not acquainted with any of Prof. Kowalewski's writings, but I had not spent an hour over it before I went out and ordered his *Grundzüge der Differential- und Integralrechnung*, of which it is a continuation. I recommend it with confidence to all who are interested in the foundations of the theory of functions of a complex variable.

The book opens with a long chapter dealing with the algebra and geometry of complex numbers, and containing a good deal of elementary group-theory. There are sections, for example, on finite groups of linear transformations, on the equivalence of positive quadratic forms, and on the modular group.

The second chapter is short, but particularly interesting. Some classical results in *Mengenlehre* are proved, first for a sphere, and then for a plane whose points have a (1, 1) correspondence with the points of the sphere. We are then introduced to the notions of a *convex* set of points, and of the least convex set which includes a given set; and these conceptions are used for the purpose of extending the first and second mean-value theorems to integrals involving complex functions of a real variable. Finally Taylor's theorem is proved for such functions. All this work is presented in a way very novel in a text-book, and shows Prof. Kowalewski quite at his best.

The third chapter deals with the definition of functions of a complex variable, elementary properties of complex power-series, the logarithmic and exponential functions, and so on, and is on more familiar lines. In the fourth the notions of a *path*, a *rectifiable path*, and an *integral along a path* are discussed with extreme care, and applied to functions defined by power-series in z or $1/z$. The fifth is the most valuable chapter in the book, containing as it does about the best and the most complete discussion of Cauchy's Theorem that I have seen.

The theorem is first established for a rectangle, by a modification of Goursat's method due in substance to Prof. E. H. Moore. We then pass to a series of theorems concerning real functions of two real variables. A function $f(x, y)$ is said to have an *eigentliches Differential* if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist, and

$$\frac{f(x+h, y+k)-f(x, y)-h\frac{\partial f}{\partial x}-k\frac{\partial f}{\partial y}}{|h|+|k|}\to 0$$

when $|h|+|k|\to 0$. This definition, it should be observed, is practically the same as that given by Dr. W. H. Young in his papers in the *Proceedings of the London Mathematical Society*. It is then shown that if u and v have differentials, and $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$, then $u\,dx+v\,dy$ is a differential, and Cauchy's Theorem is deduced by what in the long run amounts to an accurate statement of the classical "double-integral" proof. After this the theorem is extended from a rectangle to any "normal region": finally a more direct proof is given which applies directly to such a region. The contents of the rest of the chapter are of a more ordinary type. But we realise the thoroughly modern character of the book when we find integral equations introduced in connection with the "Randwertaufgabe" for a circle.

I have left myself no space to say much of the last two chapters (theory of series and products, simply and doubly periodic functions, Weierstrass's and Mittag-Leffler's theorem). But this part of the book, though consistently sound and good, does not exhibit the peculiar merits of the author in as striking a way as does the earlier part. It only remains that I should wish the book every success in England as well as in Germany. G. H. HARDY.

Estudio elemental de la prolongación analitica. Por PATRICIO PEÑALVER Y BACHILLER. (Tesis, Madrid, 1911.)

The author gives a careful account of certain well known regions of function theory. He makes no pretence to originality; but this thesis is interesting as showing an interest in modern mathematics in Spain. G. H. HARDY.

Leçons sur les principes de l'Analyse. Par R. D'ADHEMAR. Tome I. (Paris: Gauthier-Villars, 1911.)

This volume seems to me hardly worthy of the firm by whom it is published. It is slap-dash and inaccurate, and although it has some novel features, its originality is not of so striking an order as to justify an addition to the long list of *Cours* and *Traités* already on the market. Moreover, it is written in what is to me at any rate a very irritating style. French is admittedly the language best adapted for scientific purposes, and the French of the best French mathematicians is unrivalled for lucidity and charm. But it is a language which has the defects of its qualities: its lucidity can wear thin, and its nervous terseness become jerky. M. d'Adhemar's mathematical style is not like that of Picard or Goursat.

I should add a few criticisms of details to justify these remarks. M. d'Adhemar remarks in his preface: "peut-être l'ordre suivi n'est-il pas assez logique. Dans la théorie des intégrales doubles et des potentiels, je me sers de quelques propositions qui sont démontrées plus loin, dans le Chapitre X. Mais l'exposition paraît ainsi moins lourde..." And at the beginning of Chapter VII. (Les Potentiels) we are told that "dans ce Chapitre, nous admettons les règles de dérivation des séries et des intégrales, théorie qui sera faite, en détail, dans le Chapitre X." What one wants in particular is certain theorems concerning the differentiation of multiple integrals. Now these theorems are not contained in Chapter X. (or elsewhere in the book). Only simple integrals are considered there. Moreover, the discussion in Chapter X. is erroneous. M. d'Adhemar gives as sufficient conditions for the truth of the equation,

$$\phi'(\alpha)=\frac{d}{d\alpha}\int_a^b f(x, \alpha)\,dx^2\int_a^b \frac{\partial f}{\partial \alpha}\,dx,$$