SOME THEOREMS CONNECTED WITH ABEL'S THEOREM ON THE CONTINUITY OF POWER SERIES

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1. It will probably make the object of this paper more easily intelligible if, at the risk of repeating a certain number of well known facts, I preface it with a brief historical *résumé*.

In his famous memoir on the Binomial Series Abel proved that, *if a series Σa_n is convergent, the series $\Sigma a_n x^n$ is convergent for all positive values of x less than unity, and represents a function $f(x)$ which is continuous for all such values of x, unity included.**

An alternative proof of Abel's theorem was given later by Dirichlet.†

Stated in the language of the modern theory of functions, Abel's theorem runs : "If a power series in x converges to the sum s at a point P on its circle of convergence, and $f(x)$ is the function represented by the series within the circle, then $f(x)$ tends to the limit s when x tends to P along a radius vector from the origin."

This theorem has proved the starting point for a considerable number of later researches. Stolz was the first to prove that the result still holds if x tends to P along any path which lies entirely within the circle of convergence.‡ At a later date Pringsheim returned to the subject in a very instructive memoir,§ in which he shows that Abel's proof suffices to prove not only the continuity of $f(x)$, but also the *uniform convergence* of the series $\Sigma a_n x^n$ throughout the interval (0, 1). Of this the continuity of $f(x)$ for $x = 1$ is a corollary ; but Abel had really proved more than mere continuity, and Pringsheim justly remarks that Dirichlet's proof is inferior to Abel's in that it obscures this fundamental point.

This is not the only direction in which Abel's theorem has been generalised. The property of the special function x^n, upon which Abel's

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* *Crelle*, Bd. I. ; *Œuvres*, T. I., p. 223.

† *Liouville*, Sér. 2, T. VII. ; *Werke*, Bd. II., p. 305.

‡ *Zeitschr. f. Math.*, Bd. XX., p. 370, and Bd. XXIX., p. 127. This statement is somewhat loose ; see § 4.

§ *Münchener Sitzungsberichte*, 1897, p. 343.

proof was based, was simply that expressed by the inequality

$$x^n \geqslant x^{n+1} \quad (0 \leqslant x \leqslant 1),$$

and it was at once suggested that similar theorems must hold for more general classes of series of the type $\Sigma a_n f_n(x)$. And, in fact, Dirichlet and Dedekind* arrived at the following results, which for the sake of brevity I state on the hypothesis that the functions $f_n(x)$ are real functions of x defined for the interval $0 \leqslant x \leqslant 1$.

(*a*) If $$f_n(x) \geqslant f_{n+1}(x) \geqslant 0 \quad (0 \leqslant x \leqslant 1),$$

and Σa_n is convergent, then $\Sigma a_n f_n(x)$ is convergent and, if every f_n is continuous, the sum of the series is a continuous function of x.

(*b*) If Σa_n oscillates between finite limits of indetermination,

$$f_n(x) \geqslant f_{n+1}(x), \quad \text{and} \quad \lim f_n = 0,$$

then $\Sigma a_n f_n(x)$ is convergent; and, if every f_n is continuous, the sum of the series is a continuous function of x.

Dirichlet and Dedekind were concerned mainly with applications of these theorems to Dirichlet's series, and pass somewhat lightly over the general properties of series which are involved in them. Their exposition is also obscured to some extent by the fact that they do not utilize the notion of *uniform convergence*. I have therefore discussed the question further in § 2, and have stated a few theorems which summarize the conclusions which can be drawn from the discussion. I cannot claim any particular originality for these theorems, but, so far as I know, they have not, in the form in which I state them, been included in any published work. They would naturally suggest themselves to any one who undertook a careful analysis of the various theorems stated in this section, and Prof. Bromwich informs me that he has himself included Theorem I. *a* in a tract on the theory of series which will ultimately form one of the *Cambridge Tracts in Mathematics and Mathematical Physics*.

I have also included in §§ 3, 4 some applications of these theorems which do not appear to have been noticed hitherto, and in § 5 I have discussed a passage in Kronecker's *Vorlesungen über Integrale* which is concerned with the subject, but appears to contain serious errors.

There is yet another form of generalisation of Abel's theorem which has occupied the attention of mathematicians. It may happen that the series $\Sigma a_n x^n$ is divergent at a point on the circle of convergence, but is capable of "summation" by one or other of the methods furnished by

* *Vorlesungen über Zahlentheorie*, §§ 100 and 143–4.

the theory of divergent series, Cesàro's method of mean values, or Borel's method of exponential summation, or one of the various generalisations of either method. And it results from the combined researches of a number of writers that, *if* Σa_n *has the sum s when summed according to any of these methods, then* $f(x)$ *tends to the limit s when x tends to the point in question on the circle of convergence by any path subject to certain restrictions.* In the latter part of the paper I have occupied myself with series summable by Cesàro's method. The theorem for such series which corresponds to Abel's original theorem was first proved by Frobenius,* and states that, if

$$s_n = a_0+a_1+\ldots+a_n,$$

and

$$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = s,$$

then

$$\lim_{x=1} f(x) = s.$$

I have attempted to prove a general theorem which shall stand to this theorem in the same relation as Theorem I. to Abel's theorem. This theorem (Theorem II.) is the principal result of the paper: it will be found in § 6.

Finally, I have illustrated some of the most obvious applications of this general theorem, and I have indicated some further questions which are naturally suggested, but which I cannot profess to have completely solved.

I may remark that I was led to this investigation by considering various problems concerning the limits approached by the q-series of elliptic functions, when q tends to a point on the unit circle, and a number of my illustrations are furnished by q-series. But I have not in this paper attempted to treat any such particular class of problems systematically.

2. THEOREM I. *a.—If* $f_0(x)$, $f_1(x)$, $f_2(x)$, ... *is a series of real finite positive functions*† *such that*

(1) $$f_n(x) \geqslant f_{n+1}(x) \quad (0 \leqslant x \leqslant 1),$$

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* *Crelle*, Bd. LXXXIX., p. 262.

† A finite function (*fonction bornée*) is a function whose absolute value is, throughout the interval of variation of the independent variable, less than a constant K. It would obviously be enough to assert that $|f_0| < K$.

and Σa_n *is any convergent series, then the series* $\Sigma a_n f_n(x)$ *is uniformly convergent throughout the interval* (0, 1).

For

$$(1)\quad \sum_{\nu=m}^{n} a_\nu f_\nu = \sum_{\nu=m}^{n-1} (a_m+a_{m+1}+\ldots+a_\nu)(f_\nu-f_{\nu+1})+(a_m+a_{m+1}+\ldots+a_n) f_n.$$

Choose m_0 so that, for $\nu \geqslant m \geqslant m_0$,

$$| a_m+a_{m+1}+\ldots+a_\nu | < \epsilon.$$

Then
$$\left| \sum_{\nu=m}^{n} a_\nu f_\nu \right| < \epsilon f_m < \epsilon M,$$

where M is the maximum of $f_0(x)$ in the range (0, 1). The theorem is therefore proved.

COROLLARY.—*If the functions* $f_n(x)$ *are continuous, the series* $\Sigma a_n f_n(x)$ *represents a function of* x *continuous throughout the interval* $0 \leqslant x \leqslant 1$.

THEOREM I. *a* 1.—*If the restriction that* f_n *is real and positive is removed, and the condition* (1) *is replaced by the condition that*

$$(1a)\qquad \sum_{m}^{n} | f_\nu(x)-f_{\nu+1}(x) | < K,$$

where K *is a constant, then the series* $\Sigma a_n f_n$ *is still uniformly convergent.**

We first observe that the existence of such a constant K involves that of a constant L, such that $|f_n(x)| < L$, for all values of x and n. For

$$|f_n(x)| \leqslant |f_0(x)| + \sum_{0}^{n-1} |f_\nu(x)-f_{\nu+1}(x)| < M+K.$$

Hence
$$\left| \sum_{m}^{n} a_\nu f_\nu \right| < \epsilon \left\{ \sum_{m}^{n-1} |f_\nu-f_{\nu+1}| + |f_n| \right\} < \epsilon(M+2K),$$

and the result follows as before.

COROLLARY.—*If the functions* f_n *are continuous, the sum of the series is continuous.*

An obvious generalisation is—

THEOREM I. *a* 2. — *The conclusions of the preceding theorems and corollaries still hold if the terms of the series* Σa_n *are functions of* x, *provided the series is uniformly convergent, and* (*in the corollaries*) *the functions* a_n *are continuous.*

* We may suppose either that f_n is a complex function of a real variable, or a function of a complex variable; in the latter case the interval (0, 1) must be replaced by a region.

These theorems all arise from the Theorem (*a*) of Dirichlet-Dedekind. It is with this rather than with Theorem (*b*) that I am concerned in this paper ; but the latter also raises interesting questions.

THEOREM I. *b*.—*If the functions $f_n(x)$ satisfy, in addition to the conditions of I., the condition $\lim_{n=\infty} f_n(x) = 0$, and if Σa_n oscillates between finite limits of indetermination,* then the series $\Sigma a_n f_n$ is uniformly convergent.*

In the first place there is a number K such that

$$|a_m + a_{m+1} + \ldots + a_\nu| < K$$

for all values of m and ν. In the second place $f_n(x)$ is a function of x which never increases as n increases, and whose limit zero is a continuous function of x. The convergence of $f_n(x)$ to its limit is therefore *uniform*,† and we can choose m_0 so that, for $m \geqslant m_0$, and for all values of x,

$$|f_m(x)| < \epsilon.$$

The theorem now follows immediately from (1).

COROLLARY.—*If the functions f_n are continuous, the sum of the series $\Sigma a_n f_n(x)$ is a continuous function of x.*

THEOREM I. *b* 1.—*If the restriction that the functions $f_n(x)$ are real and positive is removed, and the conditions to which they are subject are replaced by the condition that the series $\Sigma |f_n(x) - f_{n+1}(x)|$ is convergent, the series $\Sigma a_n f_n$ is convergent.*

THEOREM I. *b* 2.—*If in addition the functions f_n are continuous and either of the equivalent conditions* (i.) *that the series $\Sigma|f_n - f_{n+1}|$ is uniformly convergent, or* (ii.) *that its sum represents a continuous function of x, is satisfied, the series $\Sigma a_n f_n$ will be uniformly convergent and continuous.*

THEOREM I. *b* 3.—*The preceding conclusions are not affected if the a_n's are functions of x, provided a constant K exists such that*

$$|a_0 + a_1 + \ldots + a_n| < K$$

for all values of n and x, and (*if the continuity of the series is asserted*) *the functions a_n are continuous.*

These theorems follow at once by trifling modifications of the preceding arguments. It will be seen that the series of theorems I. *b*, *b* 1, *b* 2, *b* 3 runs almost, though not exactly, parallel to the series I. *a*, *a* 1, *a* 2.

* *I.e.*, $|a_0 + a_1 + \ldots + a_n| < K$.

† Dini, *Grundlagen*, pp. 148, 149. The corollary is substantially Dedekind's theorem: his proof is less simple, owing to the fact that he does not employ the notion of uniform convergence.

3. Of the preceding theorems those of which the applications are most interesting are I. a and its extension I. a 1.

Since $f_n \geqslant f_{n+1}$, f_n tends to a limit for $n = \infty$ for all values of x; but in general it will not tend *uniformly* to this limit, and the limit will not be a continuous function of x. In the most important applications such a non-uniformity or discontinuity occurs at one or other end of the interval (0, 1), and the interest of the theorem lies in its application to establish the continuity of the series $\Sigma a_n f_n$ at this end. Thus

(i.) If
$$f_n(x) = x^n, \quad f_n \geqslant f_{n+1},$$
$$\lim f_n = 0 \ (0 \leqslant x < 1), \quad \lim f_n = 1 \ (x = 1),$$
and we obtain Pringsheim's form of Abel's theorem.

(ii.) If
$$f_n(x) = n^{-x}, \quad f_n \geqslant f_{n+1},$$
$$\lim f_n = 0 \ (0 < x \leqslant 1), \quad \lim f_n = 1 \ (x = 0),$$
and we deduce that the Dirichlet's series
$$\frac{a_1}{1^x} + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \dots$$
is uniformly convergent throughout (0, 1), and so continuous for $x = 0$, which is one of the Dirichlet-Dedekind theorems.

(iii.) If (denoting the independent variable now by q) we take
$$f_n(q) = \frac{q^n}{1+q^n},$$
so that
$$f_n - f_{n+1} = \frac{q^n(1-q)}{(1+q^n)(1+q^{n+1})} \geqslant 0,$$
and
$$\lim_{n=\infty} f_n = 0 \ (q < 1), \quad = \frac{1}{2} \ (q = 1),$$
and we deduce that, if Σa_n is convergent,
$$\lim_{q=1} \Sigma \frac{a_n q^n}{1+q^n} = \tfrac{1}{2}\Sigma a_n;$$
numerous applications of this result [and the similar results for $\Sigma a_n q^n/(1+q^{2n})$, ...] may be made in the theory of elliptic functions. For instance, from
$$\log k = \log 4\sqrt{q} + 4\Sigma \frac{(-q)^n}{n(1+q^n)} *$$
we deduce
$$\lim_{q=1} \log k = 2\log 2 - 4\left(\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \dots\right) = 0,$$
as may be verified independently.

(iv.) Let us next consider the series
$$\Sigma \frac{n a_n q^n (1-q)}{1-q^n} = \Sigma \frac{n a_n q^n}{1+q+q^2+\dots+q^{n-1}}.$$
Here
$$f_n(q) = \frac{nq^n}{1+q+q^2+\dots+q^{n-1}},$$
$$f_n(q) - f_{n+1}(q) = \frac{(1-q)^2 q^n}{(1-q^n)(1-q^{n+1})}(n-1-q-\dots-q^{n-1}) \geqslant 0.$$

* Jacobi, *Fundamenta Nova*, p. 103.

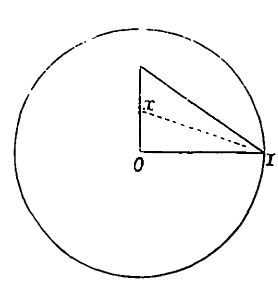
We deduce that
$$\lim_{q=1}(1-q)\Sigma\frac{na_n q^n}{1-q^n} = \Sigma a_n,$$
provided only the latter series is convergent. This result has been proved (by a special method depending upon integrals) by Franel.* Similar results may, of course, be proved for such series as
$$\Sigma\frac{2na_n q^n}{1-q^{2n}},\quad \Sigma\frac{(2n+1)\,a_n q^{2n+1}}{1-q^{4n+2}},\quad \ldots.$$

For instance, from
$$-\log k' = 8\sum_0^\infty \frac{q^{2n+1}}{(2n+1)(1-q^{4n+2})}\dagger$$
we deduce
$$-\log_{q=1} k' \sim \frac{\pi^2}{2(1-q)},$$
and from
$$\frac{2\omega}{\pi}\sqrt{(\wp u-e_3)} = \operatorname{cosec}\frac{u\pi}{2\omega}+4\Sigma\frac{q^{2n+1}}{1-q^{2n+1}}\sin\left\{(2n+1)\frac{u\pi}{2\omega}\right\}\ddagger$$
we deduce
$$\frac{2\omega}{\pi}\sqrt{(\wp u-e_3)} \sim \frac{4}{1-q}\Sigma\frac{\sin\left\{(2n+1)\frac{u\pi}{2\omega}\right\}}{2n+1} = \pm\frac{\pi}{1-q},$$
according to the value of u. In the last equation we must suppose that ω is constant and that ω' varies in such a way that q tends to 1 along the real axis.

In an interesting note recently published in the *Messenger of Mathematics*,§ Prof. Bromwich establishes the asymptotic equality
$$f(\theta) = \sum_1^\infty \frac{(-)^{n-1}}{\sinh n\theta} \sim \frac{\log 2}{\theta}$$
for $\theta = 0$. This result follows immediately from what precedes if we write q for e^θ. I shall refer later on to Prof. Bromwich's further results.

4. I shall now consider some examples of the use of Theorem I. a 1.

(i.) Suppose that $f_n(x) = x^n$, and that the region of variation of x is a triangle formed by joining 0 and 1 to any point inside the unit circle.



It is easily verified that a constant K (depending only on the triangle) can be found such that for all points within or on the boundary of the triangle
$$\frac{|1-x|}{1-|x|} < K.$$

Hence, if $|x| = r$,
$$\sum_m^n |f_\nu(x)-f_{\nu+1}(x)| = \sum_m^n r^\nu|1-x| < K\sum_m^n r^\nu(1-r) < K,$$

* *Math. Annalen*, Bd. LII.

† *Fundamenta Nova*, *l.c.*

‡ Halphen, *Fonctions Elliptiques*, t. I., p. 431.

§ "Some Contributions to the Theory of Two Electrified Spheres," *Messenger*, Vol. XXXV., p. 1.

and the conditions of the theorem are satisfied. We thus obtain Pringsheim's generalisation of Abel's theorem.*

(ii.) The theorem may be applied to q-series such as those previously considered when q moves (let us say) along a radius vector to a *rational point* on the unit circle, *i.e.*, a point $e^{\pi ib/a}$, where a and b are integers. Take, *e.g.*, the series for $\log k$ considered above,† and suppose that $q = re^{\pi ib/a}$, where b is even and a odd, and that r tends to unity along the radius vector $(0, 1)$. Then none of the terms of the series become infinite in the limit; also

$$\sum_1^\infty \frac{(-q)^n}{n(1+q^n)} = \sum_{s=1}^{a} \sum_{m=0}^{\infty} \frac{(-q)^{ma+s}}{(ma+s)(1+q^{ma+s})} = \sum_{s=1}^{a} (-)^s r^s e^{s\pi ib/a} F_s(r^a),$$

where
$$F_s(\rho) = \sum_0^\infty \frac{(-\rho)^m}{(ma+s)(1+\rho^{m+s/a}e^{s\pi ib/a})}.$$

This last series satisfies the criteria of I. a 1 for uniform convergence throughout the interval $(0, 1)$ of values of ρ. For, if $a_m = (-)^m/(ma+s)$, Σa_m is convergent. Also, if

$$f_m(\rho) = \frac{\rho^m}{1+\rho^{m+s/a}e^{s\pi ib/a}},$$

$$f_m(\rho) - f_{m+1}(\rho) = \frac{\rho^m(1-\rho)}{(1+A\rho^{m+s/a})(1+A\rho^{m+1+s/a})},$$

where $A = e^{s\pi ib/a}$. Now

$$|1 + A\rho^{m+s/a}| = \sqrt{\{1 + \rho^{2(m+s/a)} + 2\rho^{m+s/a}\cos(s\pi b/a)\}}.$$

If $\cos(s\pi b/a) > 0$, this is greater than unity; if $\cos(s\pi b/a) < 0$, it has a minimum when $\rho^{m+s/a} = -\cos(s\pi b/a)$, this minimum being $|\sin(s\pi b/a)|$. And in any case

$$|f_m(\rho) - f_{m+1}(\rho)| < K\rho^m(1-\rho),$$

from which it follows at once that the conditions of I. a 1 are satisfied.

Hence the original series for $\log k$ converges uniformly when $q = re^{\pi ib/a}$, $0 \leqslant r \leqslant 1$. For $r = 1$ it assumes the form

$$2\log 2 + \frac{\pi ib}{2a} + 2\sum_1^\infty \frac{(-)^n}{n}\left(1 + i\tan\frac{n\pi b}{a}\right) = \frac{\pi ib}{a} + 2i\sum_1^\infty \frac{1}{n}\tan\frac{n\pi b}{a},$$

and this is therefore the value to which $\log k$ tends as r approaches unity. The series on the right may be summed in finite terms.‡

5. In a passage in his *Vorlesungen über Integrale*, which has doubtless puzzled many readers besides myself, Kronecker apparently essays to prove a theorem designed to be a generalisation of Abel's theorem somewhat on the lines of Theorem I. a, except that there is no mention of uniform convergence. The whole passage is obscure; but the suggested

* *Münchener Sitzungsberichte*, l.c.

† § 3, iii.

‡ See H. J. S. Smith, "On some Discontinuous Series considered by Riemann" (*Messenger*, Vol. XI., pp. 1–11; *Collected Math. Papers*, Vol. II., p. 312); Dedekind's Note in Riemann's *Werke*, pp. 427–447; G. H. Hardy, "Note on the Limiting Values of the Elliptic Modular Functions," *Quarterly Journal*, Vol. XXXIV., pp. 76–86.

theorem seems to be as follows:—* "If

(i.) Σa_n is a convergent series,

(ii.) the functions $f_n(x)$ are positive and continuous throughout (a, A),

(iii.) $f_n(x) \geqslant f_{n+1}(x)$,

(iv.) $\lim_{x=A} f_m(x) = \lim_{x=A} f_n(x)$, for all values of m and n,

then $\Sigma a_n f_n(x)$ will be convergent and continuous for $x = A$."

My criticisms on the passage are in brief (i.) that the conditions are redundant, the fourth of them being quite unnecessary and having nothing to do with the essence of the matter; and (ii.) that the proof is altogether unsound. The unsoundness of the proof appears to have arisen from a mistaken idea of the importance of condition (iv.). Kronecker argues as follows. Starting from Abel's partial summation lemma, the origin of all these theorems, viz.,

$$c_0 f_0 + \sum_1^n (c_\nu - c_{\nu-1}) f_{\nu-1} = \sum_1^n c_\nu (f_{\nu-1} - f_\nu) + c_n f_n,$$

and putting $$c_\nu = -(a_\nu + a_{\nu+1} + \ldots),$$

he deduces

$$-f_0 \sum_0^\infty a_\nu + \sum_1^n a_{\nu-1} f_{\nu-1} = -\sum_1^n (f_{\nu-1} - f_\nu) \sum_\nu^\infty a_\kappa - f_n \sum_n^\infty a_\kappa$$

$$= -(f_0 - f_n) M_n - f_n \sum_n^\infty a_\kappa,$$

where M_n lies between the least and greatest of the values of

$$\sum_\nu^\infty a_\kappa \quad (\nu = 1, 2, \ldots, n).$$

Making n tend to infinity, and observing that $\sum_1^\infty a_{\nu-1} f_{\nu-1}$ is convergent, we obtain

$$-f_0 \sum_0^\infty a_\nu + \sum_1^\infty a_{\nu-1} f_{\nu-1} = -(f_0 - \lim_{n=\infty} f_n) M,$$

where M lies between the least and greatest of all the values of $\sum_\nu^\infty a_\kappa$.

He then makes x tend to A, and (unless his meaning has been entirely obscured by misprints), argués that, because

$$\lim_{x=A} f_0 = \lim_{x=A} f_n$$

* I have altered Kronecker's notation so as to agree with my own (Kronecker, *l.c.*, pp. 88, 89).

for all values of n, therefore

$$\lim_{x=A} (f_0 - \lim_{n=\infty} f_n) = 0;$$

and therefore $$\lim_{x=A} \sum_1^\infty a_{\nu-1} f_{\nu-1} = \lim_{x=A} f_0 \times \sum_0^\infty a_\nu.$$

But it is obvious that all that he is justified in asserting is that

$$\lim_{x=A} f_0 = \lim_{n=\infty} (\lim_{x=A} f_n),$$

and *not* $$\lim_{x=A} f_0 = \lim_{x=A} (\lim_{n=\infty} f_n),$$

the two repeated limits only being equal in exceptional circumstances. And, in fact, in the very simplest case, when $f_n(x) = x^n$ and $A = 1$,

$$\lim_{n=\infty} \lim_{x=1} x^n = 1, \qquad \lim_{x=1} \lim_{n=\infty} x^n = 0;$$

so that his argument does not even suffice to prove Abel's theorem itself. And a careful examination of the passage will, I think, lead any reader to the conclusion that the flaw in it is fundamental and not to be repaired by any alterations merely of detail.

6. I shall now consider the case in which the series Σa_n is divergent but summable by Cesàro's method of mean values. I use the following notation and terminology. We shall say that Σa_n is *summable* if

$$\frac{s_0+s_1+\ldots+s_n}{n+1},$$

where $$s_n = a_0+a_1+\ldots+a_n,$$

tends to a finite limit for $n=\infty$; and, if the terms a_n are functions of a variable x, and the convergence of this mean value to its limit is uniform throughout a certain interval or region, we shall say that Σa_n is *uniformly summable*. It is evident that the sum of a uniformly summable series of continuous terms is a continuous function of x.

THEOREM 2.—*If the functions f_n are finite, real, and positive, and $f_n - f_{n+1}$ and $f_n - 2f_{n+1} + f_{n+2}$, their first and second differences, are positive for $0 \leqslant x \leqslant 1$ and for all values of n, and if the series Σa_n is summable, then the series $\Sigma a_n f_n$ is uniformly summable throughout* (0, 1).

COROLLARY.—*If the functions f_n are continuous, the sum of the series $\Sigma a_n f_n$ is a continuous function of x.*

The proof of this theorem presents somewhat greater difficulties than those of the simpler theorems of § 2. We shall find it a necessary preliminary to establish a series of lemmas.

LEMMA 1.—*If* s_n *tends uniformly to a limit* s, *the series* Σa_n *is uniformly summable and has the sum* s.

If we omit "uniformly," this is a well known theorem* asserting the consistency of the new definition with the old. The insertion of "uniformly" in no way affects the proof.

LEMMA 2.—*If*
$$\lim \frac{s_0+s_1+\ldots+s_n}{n+1} = 0,$$
we can determine a series of positive quantities $\epsilon_1, \epsilon_2, \ldots$, *whose limit is zero, such that*
$$\left|\frac{s_p+s_{p+1}+\ldots+s_{p+r}}{p+r+1}\right| < \epsilon_p$$
for all values of r.

For we may write $s_0+s_1+\ldots+s_n = (n+1)\,\eta_n$,
where $\lim \eta_n = 0$. And then
$$s_p+s_{p+1}+\ldots+s_{p+r} = (p+r+1)\,\eta_{p+r} - p\eta_{p-1},$$
from which the lemma follows; for we can choose p so that, for $\nu \geqslant p-1$, $|\eta_\nu| < \epsilon$, however small be ϵ, and then
$$\left|\frac{s_p+s_{p+1}+\ldots+s_{p+r}}{p+r+1}\right| < 2\epsilon$$
for all values of r. In particular, as is well known,
$$\lim s_p/(p+1) = 0.$$

LEMMA 3.—*If* f_n *is finite, real, and positive and* $f_n \geqslant f_{n+1}$ *for all values of* n *and* x, *and*
$$\lim \frac{s_0+s_1+\ldots+s_n}{n+1} = 0;$$
then
$$\lim \frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1} = 0$$
uniformly for all values of x.

For
$$s_0 f_0+\ldots+s_n f_n = \sum_{\nu=0}^{n-1}(s_0+\ldots+s_\nu)(f_\nu-f_{\nu+1})+(s_0+\ldots+s_n)f_n$$
$$= \left(\sum_{\nu=0}^{r-1}+\sum_{r}^{n-1}\right)(s_0+\ldots+s_\nu)(f_\nu-f_{\nu+1})+(s_0+\ldots+s_n)f_n$$
$$= (f_0-f_r)\,M_{0,\,r-1}+f_r M_{r,\,n},$$

* See, *e.g.*, Bromwich and Hardy, *Proceedings*, Vol. II., p. 172.

where $M_{0,r-1}$ lies between the least and greatest of

$$s_0,\ s_0+s_1,\ \ldots,\ s_0+s_1+\ldots+s_{r-1},$$

and $M_{r,n}$ between the least and greatest of

$$s_0+s_1+\ldots+s_r,\ \ldots,\ s_0+s_1+\ldots+s_n.$$

Let ϵ be an assigned positive small quantity. We can choose r so that for $\nu \geqslant r$

$$\left|\frac{s_0+s_1+\ldots+s_\nu}{\nu+1}\right| < \epsilon,$$

and, *a fortiori*,

$$\left|\frac{s_0+s_1+\ldots+s_\nu}{n+1}\right| < \epsilon$$

for $n \geqslant \nu \geqslant r$; and therefore we can choose r so that

$$\left|\frac{M_{r,n}}{n+1}\right| < \epsilon$$

for all values of $n \geqslant r$. But when r is fixed we can obviously choose n so that

$$\left|\frac{M_{0,r-1}}{n+1}\right| < \epsilon.$$

When r and n are thus chosen

$$\left|\frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1}\right| < 2M\epsilon,$$

where M is the maximum of $f_0(x)$. The lemma is therefore proved.

LEMMA 4.—*If the conditions of* 3 *are satisfied except that*

$$\lim_{n=\infty}\frac{s_0+s_1+\ldots+s_n}{n+1} = s\ (\neq 0),$$

then

$$\lim_{n=\infty}\frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1} = s \lim_{n=\infty} f_n\,;$$

but the convergence to this limit will in general not be uniform.

For let $s_0 = s+t_0$, $s_1 = s+t_1$, ... Then

$$\lim_{n=\infty}\frac{t_0+t_1+\ldots+t_n}{n+1} = 0\,;$$

and therefore

$$\frac{t_0 f_0+t_1 f_1+\ldots+t_n f_n}{n+1}$$

converges *uniformly* to zero. Also

$$\lim_{n=\infty} \frac{s(f_0+f_1+\ldots+f_n)}{n+1} = s \lim_{n=\infty} f_n;$$

but the convergence to this limit will not in general be uniform unless f_n converges to its limit uniformly, which will not generally be the case.

LEMMA 5.—*If*
$$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = 0,$$

and the f_n's satisfy the further condition

$$f_n - f_{n+1} \geqslant f_{n+1} - f_{n+2}$$

for all values of n and x in question, then the series

$$\sum_0^\infty s_n (f_n - f_{n+1})$$

is uniformly convergent.

In the first place

$$f_0 - f_n = (f_0 - f_1) + \ldots + (f_{n-1} - f_n) \geqslant n(f_{n-1} - f_n).$$

Hence a constant K can be assigned so that for all values of x and n

$$f_{n-1} - f_n < K/n.$$

Now
$$\begin{aligned} & s_p(f_p - f_{p+1}) + s_{p+1}(f_{p+1} - f_{p+2}) + \ldots + s_{q-1}(f_{q-1} - f_q) \\ &= s_p(f_p - 2f_{p+1} + f_{p+2}) + (s_p + s_{p+1})(f_{p+1} - 2f_{p+2} + f_{p+3}) \\ &\qquad + \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\qquad + (s_p + s_{p+1} + \ldots + s_{q-2})(f_{q-2} - 2f_{q-1} + f_q) \\ &\qquad + (s_p + s_{p+1} + \ldots + s_{q-1})(f_{q-1} - f_q), \end{aligned}$$

the modulus of which is less than

$$\begin{aligned} & \epsilon_p \{(p+1)(f_p - 2f_{p+1} + f_{p+2}) + (p+2)(f_{p+1} - 2f_{p+2} + f_{p+3}) + \ldots \\ &\qquad \ldots + (q-1)(f_{q-2} - 2f_{q-1} + f_q) + q(f_{q-1} - f_q)\} \\ &= \epsilon_p \{p(f_p - f_{p+1}) + f_p - f_q\} < \epsilon_p \{K + 2M\}, \end{aligned}$$

where M is the maximum of $f_0(x)$. The lemma is therefore proved.

LEMMA 6.—*If the f_n's satisfy the conditions of* 5, *but*

$$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = s\,(\neq 0),$$

the series

$$\sum_0^\infty s_n(f_n - f_{n+1})$$

is convergent (*but, in general, not uniformly convergent*).

Let

$$s_n = s + t_n\,;$$

then, by 3, the series $\Sigma t_n(f_n - f_{n+1})$ is uniformly convergent. On the other hand, the series $\Sigma s(f_n - f_{n+1})$ is convergent, but not uniformly convergent, unless f_n tends to its limit uniformly.

7. *Proof of Theorem* 2.—Let s be the sum of the divergent series Σa_n, and let

$$a_0' = a_0 - s,\ a_1' = a_1,\ a_2' = a_2,\ \ldots,\ s_n' = a_0' + a_1' + \ldots + a_n' = s_n - s\,;$$

then $\Sigma a_n'$ is summable, and its sum is zero; *i.e.*,

$$\lim \frac{s_0'+s_1'+\ldots+s_n'}{n+1} = 0.$$

By Lemma 3,

$$\frac{s_0'f_0 + s_1'f_1 + \ldots + s_n'f_n}{n+1}$$

tends uniformly to 0 for $n=\infty$; and, by Lemma 5, the series

$$\Sigma s_n'(f_n - f_{n+1})$$

is uniformly convergent. Hence, if

$$S_n' = \sum_0^n s_\nu (f_\nu - f_{\nu+1}),$$

S_n' tends uniformly to a limit for $n=\infty$, and so, by Lemma 1,

$$\frac{S_0'+S_1'+\ldots+S_n'}{n+1}$$

does the same.

Now $\quad a_\nu' f_\nu = (s_\nu - s_{\nu-1}')f_\nu = s_\nu' f_\nu - s_{\nu-1}' f_{\nu-1} + s_{\nu-1}'(f_{\nu-1} - f_\nu).$

Hence, if $\sigma_n = a_0 f_0 + a_1 f_1 + \ldots + a_n f_n$, $\sigma_n' = a_0' f_0 + a_1' f_1 + \ldots + a_n' f_n$,

$$\sigma_n' = s_n' f_n + \sum_1^n s_{\nu-1}'(f_{\nu-1} - f_\nu) = s_n' f_n + S_{n-1}',$$

and
$$\frac{\sigma_0'+\sigma_1'+\ldots+\sigma_n'}{n+1} = \frac{s_0' f_0 + \ldots + s_n f_n}{n+1} + \left(\frac{n}{n+1}\right)\frac{S_0'+S_1'+\ldots+S_{n-1}}{n},$$

and therefore tends uniformly to a limit for $n=\infty$. But

$$\frac{\sigma_0+\sigma_1+\ldots+\sigma_n}{n+1} = sf_0+\frac{\sigma_0'+\sigma_1'+\ldots+\sigma_n'}{n+1},$$

and therefore also tends uniformly to a limit for $n=\infty$. Hence *the series $\Sigma a_n f_n$ is uniformly summable, and, if the functions f_n are continuous, its sum is a continuous function of n.* The theorem is therefore proved.

8. In order to show more precisely the relations of the preceding lemmas and theorem I take a very simple example.

Let $$a_0=1,\quad a_1=-2,\quad a_2=2,\quad a_3=-2,\quad \ldots,$$

so that $$s_{2n}=1,\quad s_{2n+1}=-1,$$

and $$\lim\frac{s_0+s_1+\ldots+s_n}{n+1}=0;$$

and suppose $f_n(x)=x^n$. Then

(i.) $$s_n f_n = (-)^n x^n,$$

$$\frac{s_0f_0+s_1f_1+\ldots+s_nf_n}{n+1}=\frac{1+(-)^n x^{n+1}}{(n+1)(1+x)},$$

which converges *uniformly* to 0 for $n=\infty$ (*Lemma* 3).

(ii.) Again $\sum_1^n s_{\nu-1}(f_{\nu-1}-f_\nu)=\sum_1^n(-)^{\nu-1}x^{\nu-1}(1-x)=(1-x)\left\{1+(-)^{n-1}x^n\right\}/(1+x)$,

which tends *uniformly* to $(1-x)/(1+x)$ for $n=\infty$ (*Lemma* 5). For, although x^n does *not* tend uniformly to its limit,

$$x^n-x^{n+1}-(x^{n+1}-x^{n+2})=x^n(1-x)^2 \geqslant 0,$$

and $$1-x^{n+1}=(1-x)+(x-x^2)+\ldots+(x^n-x^{n+1}) \geqslant (n+1)(x^n-x^{n+1}),$$

so that $$x^n(1-x)<\frac{1}{n+1},$$

and therefore does tend uniformly to zero.

(iii.) Finally, $\sigma_n=1-2x+2x^2-\ldots+(-)^n 2x^n=\dfrac{1-x}{1+x}+2(-)^n\dfrac{x^{n+1}}{1+x}$,

and $$\frac{\sigma_0+\sigma_1+\ldots+\sigma_n}{n+1}=\frac{1-x}{1+x}+\frac{2\{x+(-)^n x^{n+2}\}}{(n+1)(1+x)^2},$$

which tends *uniformly* to $(1-x)/(1+x)$ for $n=\infty$ (*Theorem* 2).

If the conditions were altered by changing a_0 into $1+\alpha$ ($\alpha\neq 0$), we should have

$$s_nf_n=\left\{\alpha+(-1)^n\right\}x^n,$$

and $$\frac{s_0f_0+s_1f_1+\ldots+s_nf_n}{n+1}=\phi+\frac{1+(-)^n x^{n+1}}{(n+1)(1+x)},$$

where $$\phi=\frac{\alpha}{n+1}\cdot\frac{1-x^{n+1}}{1-x}\quad (x<1),$$

$$\phi=\alpha \qquad (x=1),$$

and the convergence of ϕ to its limit is not uniform (*Lemma* 4). Similarly $\Sigma s_{\nu-1}(f_{\nu-1}-f_\nu)$ is increased by the addition of the non-uniformly convergent series $\Sigma\alpha(x^{n-1}-x^n)$ (*Lemma* 6); but it is easily verified that the uniformity of convergence which is prescribed by Theorem 2 is not affected, the two non-uniformities (so to say) cancelling one another.

9. *Applications of Theorem* 2.—(i.) If $f_n(x) = x^n$,

$$f_n - 2f_{n+1} + f_{n+2} = x^n(1-x)^2 \geqslant 0$$

for $0 \leqslant x \leqslant 1$ and all values of n. Hence, if Σa_n is summable, $\Sigma a_n x^n$ is uniformly summable for $0 \leqslant x \leqslant 1$; and its sum is a continuous function of x for $x = 1$, which is Frobenius's theorem cited in § 1.

(ii.) If $f_n(x) = n^{-x}$ $(n \geqslant 1,\ x \geqslant 0)$, it is easy to see that the first and second differences of f_n are positive (or zero). Hence we obtain the theorem that, if $\sum\limits_1^\infty a_n$ is summable, $\sum\limits_1^\infty a_n n^{-x}$ is uniformly summable for all positive values of x, including zero, and its sum is a continuous function of x for $x = 0$. That is to say

$$\lim_{x=0}\left(\frac{a_1}{1^x} + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \dots\right) = \lim_{n=\infty} \frac{s_0 + s_1 + \dots + s_n}{n+1},$$

if the latter limit exists. For example,

$$\lim_{x=0}\left(\frac{1}{1^x} - \frac{1}{2^x} + \frac{1}{3^x} - \dots\right) = \tfrac{1}{2}.$$

(iii.) If
$$f_n(q) = \frac{q^n}{1+q^n} \quad (0 \leqslant q \leqslant 1),$$

$$f_n - 2f_{n+1} + f_{n+2} = \frac{q^n(1-q)^2(1-q^{n+1})}{(1+q^n)(1+q^{n+1})(1+q^{n+2})} \geqslant 0.$$

Hence, if Σa_n is summable, $\Sigma a_n q^n/(1+q^n)$ is uniformly summable for $0 \leqslant q \leqslant 1$, and represents a continuous function of q, in particular for $q = 1$.

For instance, from the formula

$$\frac{2k'K}{\pi} = 1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^3} - \frac{4q^5}{1+q^5} + \dots *$$

we deduce that
$$\lim_{q=1} \frac{2k'K}{\pi} = 1 - 4\left(\tfrac{1}{2} - \tfrac{1}{2} + \dots\right) = 1 - 4\,.\,\tfrac{1}{4} = 0.\dagger$$

(iv.) Consider the series
$$\frac{q}{1-q^2} - \frac{2q^2}{1-q^4} + \frac{3q^3}{1-q^6} - \dots,$$

whose sum is easily found‡ to be
$$\frac{K}{2\pi^2}(E - k'^2K).$$

We may write this in the form
$$\frac{q}{1-q^2}\,\Sigma a_n f_n(q),$$

where
$$a_n = (-1)^n \quad \text{and} \quad f_n(q) = \frac{(n+1)q^n}{1+q^2+\dots+q^{2n}},$$

and it is easy to verify that the first and second differences of f_n are positive. Hence $\Sigma a_n f_n$ is uniformly summable. For $q = 1$ it takes the form

$$1 - 1 + 1 - \dots = \tfrac{1}{2}.$$

* *Fundamenta Nova*, § 40, (6).

† Strictly speaking, the divergent series should be written

$$\tfrac{1}{2} + 0 - \tfrac{1}{2} - 0 + \tfrac{1}{2} + 0 - \tfrac{1}{2} - \dots.$$

‡ *E.g.*, by making $x = \frac{1}{2}\pi$ in formula (1) of § 41 of the *Fundamenta Nova*.

We deduce that
$$K(E-k'^2K) \sim \frac{\pi^2}{2(1-q)}$$
for $q = 1$.

10. It would be easy to multiply instances of interesting applications of Theorem 2. Those which I have given are fair examples of some of the simplest types which naturally occur, and the length of this paper forbids that I should attempt to treat them in a more systematic manner. I shall conclude by indicating briefly certain actual or possible further generalisations.

In the first place we may at once enunciate

THEOREM 2 *a* 1.—*The conclusions of Theorem* 2 (*and the lemmas preliminary to it*) *are still valid if the functions* $f_n(x)$ *are not restricted to be real and positive, and the condition that the first and second differences of the functions are not negative is replaced by the conditions*

$$\sum_m^n |f_\nu - f_{\nu+1}| < K, \qquad \sum_m^n (\nu+1)\,|f_\nu - 2f_{\nu+1} + f_{\nu+2}| < K,$$

for all values of m, n, and x.

The course of the proof is unaffected save for slight modifications in the case of Lemmas 3 and 5.

Consider, for example, the series
$$\vartheta_4(v, q) = 1 + 2\sum_1^\infty (-)^n q^{n^2} \cos 2n\pi v.$$
Taking $a_n = 2(-)^n \cos 2n\pi v$ $(n > 0)$ and $f_n = q^{n^2}$, we may verify without difficulty that the conditions of the theorem are satisfied. Since the series
$$1 - 2\cos 2\pi v + 2\cos 4\pi v - \dots$$
has the sum zero when summed by Cesàro's method, we deduce that
$$\lim_{q=1} \vartheta_4(v, q) = 0.*$$

THEOREM 2 *a* 2.—*The preceding conclusions are not affected if the terms of the series* Σa_n *are functions of x, provided the series be uniformly summable.*

A much more interesting and more difficult question is that of the extension of Theorem II. to cases in which the summation of Σa_n requires

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* See Borel, *Leçons sur les Séries divergentes*, p. 7; L. Fejér, *Math. Annalen*, Bd. LVIII., p. 66; Hardy, "Note on Divergent Fourier Series," *Messenger*, Vol. XXXIII., p. 144. I refer later to Herr Fejér's investigations.

one of the extended forms of the mean value process, *e.g.*, when, if

$$s_n^{(1)} = \frac{s_0+s_1+\dots+s_n}{n+1},$$

$s_n^{(1)}$ oscillates for $n=\infty$, but

$$s_n^{(2)} = \frac{s_0^{(1)}+s_1^{(1)}+\dots+s_n^{(1)}}{n+1}$$

has a limit.

The following more general theorem is naturally suggested, and I have no doubt that it is true. We define "summable" to mean "summable by k repetitions of the mean value process." Then,

If the first, second, ..., $(k+1)$-th differences of the functions $f_n(x)$ are positive (or zero) for all values of x and n in question, and the series Σa_n is summable, then the series $\Sigma a_n f_n(x)$ is uniformly summable, and therefore its sum is a continuous function of x

—with corollaries and generalisations in every way analogous to those of Theorems I. *a* and II. Such a theorem would be related to Hölder's extensions of Frobenius's theorem as is II. to Frobenius's and I. *a* to Abel's theorem. But I have not up to the present succeeded in overcoming the algebraical difficulties attendant upon a complete and rigorous proof.

In the most interesting cases Theorem II. is generally sufficient. But the latter theorem does not cover such cases as those in which Σa_n is a series like $1-2+3-4+\dots$ or $1^2-2^2+3^2-4^2+\dots$.

An example in which a result more general than that of II. is needed may be found in the theory of two electrified spheres. In the paper already referred to, Prof. Bromwich, seeking a rigorous proof of Lord Kelvin's theorem that the force acting between two spheres in contact and at potential V is $\frac{1}{6}V^2(\log 2-\frac{1}{4})$, requires to show that, for small values of θ,

$$f(\theta) = \Sigma\frac{(-)^{n-1}}{\sinh n\theta} = \frac{\log 2}{\theta} - \tfrac{1}{24}\theta + \dots.$$

The first approximation was established in § 3 (iv.). To obtain the second we must prove that

$$\lim_{\theta=0}\frac{1}{\theta}\Sigma(-)^{n-1}\left(\frac{1}{n\theta}-\frac{1}{\sinh n\theta}\right) = \tfrac{1}{24}.$$

The limiting form of the series is $\frac{1}{6}(1-2+3-4+\dots)$,

which is summable by *two* repetitions of the mean value process, and has the sum $\frac{1}{24}$. Here we could take $a_n = (-)^{n-1}n$ and $f_n(\theta) = \frac{1}{n\theta}\left(\frac{1}{n\theta}-\frac{1}{\sinh n\theta}\right)$, and so obtain the result desired.

Although I have not succeeded in proving the suggested general theorem, I have, starting from a theorem of Herr Fejér's, succeeded in proving a number of theorems of a more special character which do enable us to deal effectively with cases such as these: *e.g.*, to assign the limit of

$$\frac{q}{1+q}-\frac{2q^2}{1+q^2}+\frac{3q^3}{1+q^3}-\dots$$

for $q = 1$. I confine myself at present to stating one of these theorems. Herr Fejér's theorem (modified so as to correspond to Theorem 2*) runs as follows:—*If*

(i.) Σa_n *is summable* (*to the sum s*),

(ii.) *the functions* $f_n(x)$ *and their first and second differences are positive* (*or zero*),

(iii.) $\Sigma n f_n(x)$ *is convergent for* $x > 0$,

(iv.) $\lim_{x=0} f_n(x) = 1$ *for all values of n,*

then $\Sigma a_n f_n(x)$ *is absolutely convergent for* $x > 0$, *and its limit for* $x = 0$ *is s.*

The more general theorem is that the same conclusion holds when k repetitions of the mean value process are necessary in order to sum the series Σa_n, and

(ii.)′ the first, second, ..., $(k+1)$-th differences of the functions $f_n(x)$ are positive (or zero),

(iii.)′ $\Sigma n^k f_n(x)$ is absolutely convergent.

The proof is not difficult. The other theorems relate to cases in which condition (ii.) or (ii.)′ is not satisfied. I have included proofs of these theorems in a paper which will be published in the *Mathematische Annalen.*

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* The conditions actually stated by Herr Fejér differ from the above in the restriction of $f_n(x)$ to be of the form $\phi(nx)$, and the substitution for (ii.) and (iii.) of the conditions

$$|\phi(t)| < \frac{K}{t^{2+\rho}}, \quad |\phi''(t)| < \frac{K}{t^{2+\rho}},$$

where $\rho > 0$. The proof of the theorem as I state it may be made a good deal simpler than Herr Fejér's proof.