Writing, for shortness,

$$m \sin C - n \sin B = \lambda,$$
$$n \sin A - l \sin C = \mu,$$
$$l \sin B - m \sin A = \nu;$$

then, identically, $\lambda \sin A + \mu \sin B + \nu \sin C = 0.$

If, further, $a, b, c, f, g, h$ be written for $\frac{d^2u}{dx^2} \dots \frac{d^2u}{dx\,dy}$, then the equation to a diameter is

$$a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu = 0,$$

and the envelope of this line, where the parameters $\lambda, \mu, \nu$ vary, subject to the above relation, is at once seen to be

$$(bc - f^2) \sin^2 A + (ca - g^2) \sin^2 B + (ab - h^2) \sin^2 C + 2(gh - af) \sin B \sin C$$
$$+ 2(hf - bg) \sin C \sin A + 2(fg - ch) \sin A \sin B = 0;$$

which, in relation to these "diameters," may be called the "central conic."

The preceding method may obviously be generalised. The equation to any "diametral curve" is, whatever the order of $u$,

$$\lambda^r \frac{d^r u}{dx^r} + \dots + r\lambda^r \mu \frac{d^r u}{dx^{r-1}\,dy} + \dots = 0,$$

and the envelope of this, subject to

$$\lambda \sin A + \mu \sin B + \nu \sin C = 0,$$

may similarly be sought.

The "diametral conics" of a cubic, in particular, pass through the four fixed points common to the polar conics of all points on the line at infinity.

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## *On the Twenty-one Coordinates of a Conic in Space.*

*By* WILLIAM SPOTTISWOODE, P.R.S.

[*Read June 12th*, 1879.]

In a Note published in the Report of the British Association for 1878 (Dublin Meeting), I gave a short account of the coordinates (then described as 18 in number) of a conic in space, and of the equations of condition by which they were connected. A modification of the notation there used will perhaps place these quantities in a clearer light.

The six coordinates of a right line may be derived from the equations of the two planes of which it is the intersection, by eliminating each of the variables in succession. Thus, if the equations of the two planes be

$$\alpha x + \beta y + \gamma z + \delta t = 0,$$
$$\alpha' x + \beta' y + \gamma' z + \delta' t = 0;$$

then, by eliminating $t$, we obtain

$$(\alpha\delta' - \alpha'\delta)\, x + (\beta\delta' - \beta'\delta)\, y + (\gamma\delta' - \gamma'\delta)\, z = 0.$$

Or, adopting the usual notation, we have the following expression for the six coordinates in question, a, b, c, f, g, h; viz.,

$$\mathrm{a, b, c, f, g, h} = \begin{matrix} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta'; \end{matrix}$$

the foregoing equation may be written thus

$$\mathrm{f}x + \mathrm{g}y + \mathrm{h}z = 0.$$

And all the four forms derived by the elimination of $x, y, z, t$, in turn, from the equations of the planes, may be comprised in the formula

$$\begin{pmatrix} . & \mathrm{c}, & -\mathrm{b}, & \mathrm{f} \\ -\mathrm{c}, & . & \mathrm{a}, & \mathrm{g} \\ \mathrm{b}, & -\mathrm{a}, & . & \mathrm{h} \\ \mathrm{f}, & \mathrm{g}, & \mathrm{h}, & . \end{pmatrix} (x, y, z, t) = 0.$$

And the six coordinates a, ..., f, ..., are, as is well-known, connected by the following relation $\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0.$

In like manner, if a conic in space be determined by the intersection of the quadric surface

$$(a, b, c, d, f, g, h, l, m, n)(x, y, z, t)^2 = 0 \quad \text{..............(1)}$$

by the plane

$$\alpha x + \beta y + \gamma z + \delta t = 0 \quad \text{..........................(2),}$$

we may, by eliminating the variables in turn, obtain therefrom the following forms; viz.,

$$\left.\begin{array}{l} a(\beta y + \gamma z + \delta t)^2 - 2\alpha(\beta y + \gamma z + \delta t)(hy + gz + lt) + \alpha^2(b, c, d, n, m, f)(y, z, t)^2 = 0 \\ b(\alpha x + \gamma z + \delta t)^2 - 2\beta(\alpha x + \gamma z + \delta t)(hx + fz + mt) + \beta^2(a, c, d, n, l, g)(x, z, t)^2 = 0 \\ c(\alpha x + \beta y + \delta t)^2 - 2\gamma(\alpha x + \beta y + \delta t)(gx + fy + nt) + \gamma^2(a, b, d, m, l, h)(x, y, t)^2 = 0 \\ d(\alpha x + \beta y + \gamma z)^2 - 2\delta(\alpha x + \beta y + \gamma z)(lx + my + nz) + \delta^2(a, b, c, f, g, h)(x, y, z)^2 = 0 \end{array}\right\} \ldots(3).$$

If then, dropping for the moment, the coefficients $a, b, \ldots$, we put

| | | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ | $h$ | $l$ | $m$ | $n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A = (0, -\gamma, \beta, 0)^2$ | = | ... | $\gamma^2$ | $\beta^2$ | ... | $-2\beta\gamma$ | ... | ... | ... | ... | ... |
| $B = (\gamma, 0, -\alpha, 0)^2$ | = | $\gamma^2$ | ... | $\alpha^2$ | ... | ... | $-2\gamma\alpha$ | ... | ... | ... | ... |
| $C = (-\beta, \alpha, 0, 0)^2$ | = | $\beta^2$ | $\alpha^2$ | ... | ... | ... | ... | $-2\alpha\beta$ | ... | ... | ... |
| $F = (-\delta, 0, 0, \alpha)^2$ | = | $\delta^2$ | ... | ... | $\alpha^2$ | ... | ... | ... | $-2\alpha\delta$ | ... | ... |
| $G = (0, -\delta, 0, \beta)^2$ | = | ... | $\delta^2$ | ... | $\beta^2$ | ... | ... | ... | ... | $-2\beta\delta$ | ... |
| $H = (0, 0, -\delta, \gamma)^2$ | = | ... | ... | $\delta^2$ | $\gamma^2$ | ... | ... | ... | ... | ... | $-2\gamma\delta$ |
| $-A' = (\gamma, 0, -\alpha, 0)(-\beta, \alpha, 0, 0)$ | = | $-\beta\gamma$ | ... | ... | ... | $-\alpha^2$ | $\alpha\beta$ | $\alpha\gamma$ | ... | ... | ... |
| $-B' = (-\beta, \alpha, 0, 0)(0, -\gamma, \beta, 0)$ | = | ... | $-\alpha\gamma$ | ... | ... | $\alpha\beta$ | $-\beta^2$ | $\beta\gamma$ | ... | ... | ... |
| $-C' = (0, -\gamma, \beta, 0)(\gamma, 0, -\alpha, 0)$ | = | ... | ... | $-\alpha\beta$ | ... | $\alpha\gamma$ | $\beta\gamma$ | $-\gamma^2$ | ... | ... | ... |
| $F' = (0, -\delta, 0, \beta)(0, 0, -\delta, \gamma)$ | = | ... | ... | ... | $\beta\gamma$ | $\delta^2$ | ... | ... | ... | $-\delta\gamma$ | $-\delta\beta$ |
| $G' = (0, 0, -\delta, \gamma)(-\delta, 0, 0, \alpha)$ | = | ... | ... | ... | $\gamma\alpha$ | ... | $\delta^2$ | ... | $-\delta\gamma$ | ... | $-\delta\alpha$ |
| $H' = (-\delta, 0, 0, \alpha)(0, -\delta, 0, \beta)$ | = | ... | ... | ... | $\alpha\beta$ | ... | ... | $\delta^2$ | $-\delta\beta$ | $-\delta\alpha$ | ... |
| $P = (0, -\gamma, \beta, 0)(-\delta, 0, 0, \alpha)$ | = | ... | ... | ... | ... | ... | $-\beta\delta$ | $\gamma\delta$ | ... | $-\alpha\gamma$ | $\alpha\beta$ |
| $L = (-\beta, \alpha, 0, 0)(-\delta, 0, 0, \alpha)$ | = | $\beta\delta$ | ... | ... | ... | ... | ... | $-\alpha\delta$ | $-\alpha\beta$ | $\alpha^2$ | ... |
| $-L' = (\gamma, 0, -\alpha, 0)(-\delta, 0, 0, \alpha)$ | = | $-\gamma\delta$ | ... | ... | ... | ... | $\alpha\delta$ | ... | $\alpha\gamma$ | ... | $-\alpha^2$ |
| $Q = (\gamma, 0, -\alpha, 0)(0, -\delta, 0, \beta)$ | = | ... | ... | ... | ... | $\alpha\delta$ | ... | $-\gamma\delta$ | $\beta\gamma$ | ... | $-\alpha\beta$ |
| $M = (0, -\gamma, \beta, 0)(0, -\delta, 0, \beta)$ | = | ... | $\gamma\delta$ | ... | ... | $-\beta\delta$ | ... | ... | ... | $-\beta\gamma$ | $\beta^2$ |
| $-M' = (-\beta, \alpha, 0, 0)(0, -\delta, 0, \beta)$ | = | ... | $-\alpha\delta$ | ... | ... | ... | ... | $\beta\delta$ | $-\beta^2$ | $\alpha\beta$ | ... |
| $R = (-\beta, \alpha, 0, 0)(0, 0, -\delta, \gamma)$ | = | ... | ... | ... | ... | $-\alpha\delta$ | $\beta\delta$ | ... | $-\beta\gamma$ | $\alpha\gamma$ | ... |
| $N = (\gamma, 0, -\alpha, 0)(0, 0, -\delta, \gamma)$ | = | ... | ... | $\alpha\delta$ | ... | ... | $-\gamma\delta$ | ... | $\gamma^2$ | ... | $-\alpha\gamma$ |
| $-N = (0, -\gamma, \beta, 0)(0, 0, -\delta, \gamma)$ | = | ... | ... | $-\beta\delta$ | ... | $\gamma\delta$ | ... | ... | ... | $-\gamma^2$ | $\beta\gamma$ |

...... (4).

Then, the equations (3) take the forms

$$\left.\begin{array}{l}(C, B, F, L, L', A')\,(y, z, t)^2 = 0 \text{ or } X = 0\\ (C, A, G, M, M', B')\,(x, z, t)^2 = 0 \text{ or } Y = 0\\ (B, A, H, N, N', C')\,(x, y, t)^2 = 0 \text{ or } Z = 0\\ (F, G, H, F', G', H')(x, y, z)^2 = 0 \text{ or } T = 0\end{array}\right\} \quad \ldots\ldots\ldots(5).$$

And the twenty-one quantities $A, B, \ldots$, may be called the "Twenty-one Coordinates of a Conic in Space." Or if, as in the Note quoted at the outset, we omit the three quantities $P, Q, R$, which do not occur in the forms 5, the remaining eighteen quantities may be called the "Eighteen Coordinates of a Conic in Space."

These coordinates, whether twenty-one or eighteen, like the six coordinates of a straight line, are not independent. In fact, by an inspection of their forms, it is not difficult to see that they satisfy the following equations; viz.,

$$\left.\begin{array}{llll} \cdot & -H\beta & +F'\gamma + N'\delta & = 0\\ \cdot & F'\beta & -G\gamma + M\delta & = 0\\ \cdot & N\beta & +M\gamma - A\delta & = 0\\ -H\alpha & \cdot & +G'\gamma + N\delta & = 0\\ G'\alpha & \cdot & -F\gamma + L'\delta & = 0\\ N\alpha & \cdot & +L'\gamma - B\delta & = 0\\ -G\alpha + H'\beta & & \cdot \quad +M'\delta & = 0\\ H'\alpha - F\beta & & \cdot \quad +L\delta & = 0\\ M'\alpha + L\beta & & \cdot \quad -C\delta & = 0\\ -A\alpha + C'\beta + B'\gamma & & \cdot & = 0\\ C\alpha - B\beta + A'\gamma & & \cdot & = 0\\ B'\alpha + A'\beta - C\gamma & & \cdot & = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(6);$$

and, if from these we eliminate $\alpha, \beta, \gamma, \delta$, so far as they appear explicitly, we shall obtain the following relations between $A, B, \ldots$; viz.,

$$\left|\begin{array}{cccccccccccc} \cdot & \cdot & \cdot & -H, & G', & N, & -G, & H', & M', & -A, & C', & B' \\ -H, & F', & N', & \cdot & \cdot & \cdot & H', & -F, & L, & C', & -B, & A \\ F', & -G, & M, & G', & -F, & L', & \cdot & \cdot & \cdot & B', & A', & -C \\ N', & M, & -A, & N, & L', & -B, & M', & L, & -C, & \cdot & \cdot & \cdot \end{array}\right| = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

These relations are nine in number; and as all are concerned with the ratios, and not the absolute values of the quantities in question, the total number of independent coordinates will be $18 - 9 - 1 = 8$, as it should be. As regards the three quantities $P, Q, R$, which do not appear in (6) or (7); we have, in the first place,

$$P + Q + R = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8);$$

secondly,

$$\begin{aligned} \alpha P + \beta L' - \gamma L &= 0, \\ -\alpha M + \beta Q + \gamma M' &= 0, \\ \alpha N' - \beta N + \gamma R &= 0; \end{aligned}$$

whence

$$\begin{vmatrix} P, & L', & -L \\ -M, & Q, & M' \\ N', & -N, & R \end{vmatrix} = 0 \quad \text{.........................(9)};$$

and the equations (8) and (9) express two relations between $P$, $Q$, $R$, and the other eighteen coordinates. To find the third relation, it will be convenient to remark that the twenty-one coordinates are the twenty-one minors of the matrix

$$\begin{matrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & a, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & . \end{matrix}$$

formed by selecting two out of the first four rows and columns, and adding to them the fifth row and column. Thus,

$$A = \begin{matrix} b, & f, & \beta, \\ f, & c, & \gamma, \\ \beta, & \gamma, & . \end{matrix}$$

and, more generally,

$$\left.\begin{aligned}
\begin{matrix} h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ \alpha, & \beta, & \gamma, & \delta, & . \end{matrix} &= -A,\ C',\ B',\ P,\ -M,\ N' \\
\begin{matrix} a, & h, & g, & l, & \alpha \\ l, & m, & n, & d, & \gamma \\ \alpha, & \beta, & \gamma, & \delta, & . \end{matrix} &= P,\ L',\ -L,\ -F,\ -H',\ -G' \\
\begin{matrix} g, & f, & e, & n, & \gamma \\ a, & h, & g, & l, & \alpha \\ \alpha, & \beta, & \gamma, & \delta, & . \end{matrix} &= C',\ -B,\ A',\ L',\ Q,\ -N \\
\begin{matrix} h, & b, & f, & m, & \beta \\ l, & m, & n, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & . \end{matrix} &= -M,\ Q,\ M',\ -H',\ -G,\ -F' \\
\begin{matrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ \alpha, & \beta, & \gamma, & \delta, & . \end{matrix} &= B',\ A',\ -C,\ -L,\ M',\ R \\
\begin{matrix} g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & . \end{matrix} &= N',\ -N,\ R,\ -G',\ -F',\ -H
\end{aligned}\right\} \quad \text{.........(10)}.$$

And, consequently, from the identity

$$\begin{matrix} a, & h, & g, & l, & \alpha, & \alpha \\ h, & \beta, & f, & m, & \beta, & \beta \\ g, & f, & c, & n, & \gamma, & \gamma \\ l, & m, & n, & d, & \delta, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & . & . \\ \alpha, & \beta, & \gamma, & \delta, & . & . \end{matrix}$$

we may obtain any of the three following forms; viz.,

$$\left.\begin{aligned} AF - B'G' - CH' + P^2 \ - L'M - LN' &= 0 \\ -A'F' + BG - CH' - ML' + Q^2 \ - M'N &= 0 \\ -A'F - B'G' + CH - NL - NM' + R^2 \ &= 0 \end{aligned}\right\} \ldots\ldots(11),$$

any one of which may be taken as a third relation connecting $P$, $Q$, $R$ with the other eighteen quantities.

The equation of the quadric (1) admits of a peculiar transformation by means of the coordinates of the conic in question, as follows:— Adding together the equations (5), we find

$$\begin{aligned} & X+Y+Z+T \\ = \ & (B+C+F)\,x^2+2\,(A'+F')\,yz+2\,(M'+N)\,xt \\ & +(C+A+G)\,y^2+2\,(B'+G')\,zx+2\,(N'+L)\,yt \\ & +(A+B+H)\,z^2+2\,(C'+H')\,xy+2\,(L'+M)\,zt \\ & +(F+G+H)\,t^2 \\ = \ & (A+B+C+F+G+H)\,(x^2+y^2+z^2+t^2) \\ & -Hy^2-Gz^2-At^2+2\,(Mzt+N'yt+F'yz) \\ & -Hx^2-Fz^2-Bt^2+2\,(L'zt+Nxt\,+G'zx) \\ & -Gx^2-Fy^2-Ct^2+2\,(Lyt\,+M'xt+H'xy) \\ & -Ax^2-By^2-Cz^2+2\,(A'yz+B'zx+C'xy). \end{aligned}$$

But, in virtue of the conditions (7), each of the expressions forming the last four lines may be resolved into the product of two linear factors, say, $p, p'$; $q, q'$; $r, r'$; $s, s'$. Hence, putting

$$\begin{aligned} \rho^2 &= x^2+y^2+z^2+t^2, \\ S &= A+B+C+F+G+H \\ &= (a+b+c+d)(\alpha^2+\beta^2+\gamma^2+\delta^2) \\ &\qquad -(a,\ b,\ c,\ d,\ f,\ g,\ h,\ l,\ m,\ n)\,(\alpha,\ \beta,\ \gamma,\ \delta)^2, \end{aligned}$$

the equation in question takes the form

$$S\rho^2 - pp' - qq' - rr' - ss' = 0.$$

But, on the other hand,

$$(H, G, A, -M, -N', -F)(y, z, t)^2$$
$$= (b, c, d, m, n, f)(z\delta - t\gamma,\ t\beta - y\delta,\ y\gamma - z\beta)^2,$$
$$(H, F, B, -L', -N, -G)(x, z, t)^2$$
$$= (a, c, d, l, n, g)(z\delta - t\gamma,\ ta - x\delta,\ x\gamma - za)^2,$$
$$(G, F, C, -L, -M', -H)(x, y, t)^2$$
$$= (a, b, d, m, l, h)(y\delta - t\beta,\ ta - x\delta,\ x\beta - ya)^2,$$
$$(A, B, C, -A', -B', -C')(x, y, z)^2$$
$$= (a, b, c, f, g, h)(y\gamma - z\beta,\ za - x\gamma,\ x\beta - ya)^2.$$

And these expressions vanish respectively for the values

$$y : z : t = \beta : \gamma : \delta,$$
$$x : z : t = a : \gamma : \delta,$$
$$x : y : t = a : \beta : \delta,$$
$$x : y : z = a : \beta : \gamma;$$

which are, consequently, the equations to the lines in which the pairs of planes $p, p'$; $q, q'$; $r, r'$; $s, s'$, respectively intersect. It may also be added that the plane containing the two lines

$$q, q';\ r, r' \text{ is } \delta x - at = 0;$$
$$r, r';\ p, p' \ ,,\ \delta y - \beta t = 0;$$
$$p, p';\ q, q' \ ,,\ \delta z - \gamma t = 0;$$
$$p, p';\ s, s' \ ,,\ \beta z - \gamma y = 0;$$
$$q, q';\ s, s' \ ,,\ \gamma x - az = 0;$$
$$r, r';\ s, s' \ ,,\ ay - \beta x = 0.$$

### *The condition that two conics in space may meet.*

The two conics may, in general, be considered as the sections of two quadric surfaces $S$, $S_1$, by two planes $L$, $L_1$. And if $S = 0$, $L = 0$ be the equations to the first conic, say the conic $(S, L)$; and $S_1 = 0$, $L_1 = 0$ those of the second conic, the condition required will be found by eliminating the variables from the four equations $S = 0$, $L = 0$, $S_1 = 0$, $L_1 = 0$. This may be effected in more than one way. First, we may eliminate from the equations as they stand, but the result would not be in a form suited to the present purpose. Secondly, we may eliminate one of the variables, say $t$, from the equations $S$, $L$; the same from the equations $S_1$, $L_1$; and also from the equations $L$, $L_1$. We shall then have three equations in $x$, $y$, $z$; and the elimination will give us a relation between the coordinates of the conics $(S, L)$, $(S_1, L_1)$

of the degree 2, and the coordinates of the line $(L, L_1)$ of the degree 4. Thirdly, we may eliminate the variables from two of the equations (5), and two of the corresponding equations of the second conic. The results of these three methods are, of course, substantially the same, but they differ in form.

By the second method, we shall have to eliminate $x, y, z$ from equations which may be written thus

$$(F,\ G,\ H,\ F',\ G',\ H')\,(x,\ y,\ z)^2 = 0,$$
$$(F_1,\ G_1,\ H_1,\ F_1',\ G_1',\ H_1')\,(x,\ y,\ z)^2 = 0,$$
$$\mathrm{f}x+\mathrm{g}y+\mathrm{h}z = 0.$$

Where $F_1, G_1, \ldots$ represent the coordinates of the conic $(S_1, L_1)$. Multiplying the last of these three equations by $x, y, z$, respectively, we shall have five equations in $x^2, y^2, z^2, yz, zx, xy$. These give the following values

$$x^2 : y^2 : z^2 : yz : zx : xy = \begin{array}{cccccc} F, & G, & H, & F', & G', & H', \\ F_1, & G_1, & H_1, & F_1', & G_1', & H_1', \\ \mathrm{f}, & \cdot & \cdot & \cdot & \mathrm{h}, & \mathrm{g}, \\ \cdot & \mathrm{g}, & \cdot & \mathrm{h}, & \cdot & \mathrm{f}, \\ \cdot & \cdot & \mathrm{h}, & \mathrm{g}, & \mathrm{f}, & \cdot \end{array}$$

and if, for brevity, we write $FG_1 - F_1G = (F,\ G)$, &c., it will be found *à une facteur près*, that

$$\begin{aligned}
\tfrac{1}{2}x^2 = &-(H, G')\,\mathrm{g}^3 + (G, H')\,\mathrm{h}^3 \\
&+[2\,(G', H') + (H, H')]\,\mathrm{g}^2\mathrm{h} - (G, F')\,\mathrm{h}^2\mathrm{f} \\
&+[2\,(H', F') - (G, G')]\,\mathrm{gh}^2 \qquad + (H, F')\,\mathrm{fg}^2 + (G, H)\,\mathrm{fgh},
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}y^2 = &-(F, H')\,\mathrm{h}^3 + (H, F')\,\mathrm{f}^3 \\
&+ \qquad +[2\,(G', H') + (F\ F')]\,\mathrm{h}^2\mathrm{f} - (H, G')\,\mathrm{f}^2\mathrm{g} \\
&+(F, G')\,\mathrm{gh}^2 + [2\,(F', G') - (H, H')]\,\mathrm{hf}^2 \qquad + (H, F)\,\mathrm{fgh},
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}z^2 = &-(G, F')\,\mathrm{f}^3 + (F, G')\,\mathrm{g}^3 \\
&-(F, H')\,\mathrm{g}^2\mathrm{h} \qquad + [2\,(H', F') + (G, G')]\,\mathrm{f}^2\mathrm{g} \\
&\qquad + \ G, H')\,\mathrm{f}^2\mathrm{h} + [2\,(G', H') - (F, F')]\,\mathrm{fg}^2 + (F, G)\,\mathrm{fgh},
\end{aligned}$$

$$\begin{aligned}
yz = &(G, H)\,\mathrm{f}^3 - 2\,(F, G')\,\mathrm{g}^2\mathrm{h} - 2\,(F, G)\,\mathrm{h}^2\mathrm{f} + 2\,(H, H')\,\mathrm{f}^2\mathrm{g} \\
&+ 2\,(F, H')\,\mathrm{gh}^2 - 2\,(G, G')\,\mathrm{hf}^2 - (H, F)\,\mathrm{fg}^2 - 4\,(G', H')\,\mathrm{fgh},
\end{aligned}$$

$$\begin{aligned}
zx = &(H, F)\,\mathrm{g}^3 + 2\,(F, F')\,\mathrm{g}^2\mathrm{h} - 2\,(G, H')\,\mathrm{h}^2\mathrm{f} - (G, H)\,\mathrm{f}^2\mathrm{g} \\
&- (F, G)\,\mathrm{gh}^2 + 2\,(G, F')\,\mathrm{hf}^2 - 2\,(H, H')\,\mathrm{fg}^2 - 4\,(H', F')\,\mathrm{fgh},
\end{aligned}$$

$$\begin{aligned}
xy = &(F, G)\,\mathrm{h}^3 - (H, F)\,\mathrm{g}^2\mathrm{h} + 2\,(G, G')\,\mathrm{h}^2\mathrm{f} - 2\,(H, F')\,\mathrm{f}^2\mathrm{g} \\
&- (F, F')\,\mathrm{gh}^2 - (G, H)\,\mathrm{hf}^2 + 2\,(H, G')\,\mathrm{fg}^2 - 4\,(F', G')\,\mathrm{fgh}.
\end{aligned}$$

If, from these equations, we form the identical expression

$$y^2 \,.\, z^2 - (yz)^2 = 0,$$

it will be easily seen that the term independent of $\mathrm{f}$ will vanish. The coefficient of $\mathrm{f}$ in the expression for $y^2 \,.\, z^2$ will be

$$4\,\{(FG')\,\mathrm{g} - (FH')\,\mathrm{h}\}\,\{(HF)\,\mathrm{g}^2 + 4\,(G'H')\,\mathrm{gh} + (FG)\,\mathrm{h}^2\}\,\mathrm{gh},$$

while that in the expression for $(yz)^2$ will be the same. Hence the whole expression will be divisible by $\mathrm{f}^2$. And, after various reductions, the final result will be found to be

$$\begin{aligned}
& -\mathrm{f}^4\,\{(GH)^2 + 4\,(GF')\,(HF')\} \\
& -\mathrm{g}^4\,\{(HF)^2 + 4\,(HG')\,(FG')\} \\
& -\mathrm{h}^4\,\{(FG)^2 + 4\,(FH')\,(GH')\} \\
& +4\mathrm{g}^3\mathrm{h}\,\{-(HF)\,(FF') + (HG')\,(FH') + (FG')\,(HH') + 4\,(FG')\,(F'G')\} \\
& +4\mathrm{gh}^3\,\{\ \ (FG)\,(FF') + (FG')\,(GH') + (FH')\,(GG') - 4\,(FH')\,(H'F')\} \\
& +4\mathrm{h}^3\mathrm{f}\,\{-(FG)\,(GG') + (FH')\,(GF') + (FF')\,(GH') + 4\,(GH')\,(G'H')\} \\
& +4\mathrm{hf}^3\,\{\ \ (GH)\,(GG') + (GH')\,(HF') + (GF')\,(HH') - 4\,(GF')\,(F'G')\} \\
& +4\mathrm{f}^3\mathrm{g}\,\{-(GH)(HH') + (GF')\,(HG') + (GG')\,(HF') + 4(HF')(H'F')\} \\
& +4\mathrm{fg}^3\,\{\ \ (HF)(HH') + (HF')\,(FG') + (FF')\,(HG') - 4\,(HG')\,(G'H')\} \\
& +2\mathrm{g}^2\mathrm{h}^2\,\{(HF)\,(FG) - 2\,(FF')^2 - 2\,(FG')\,(GG') - 2\,(FH')\,(HH') \\
& \qquad\qquad + 4\,(FG')\,(H'F') - 4\,(FH')\,(F'G')\} \\
& +2\mathrm{h}^2\mathrm{f}^2\,\{(FG)\,(GH) - 2\,(GG')^2 - 2\,(GH')(HH') - 2\,(GF')\,(FF') \\
& \qquad\qquad + 4\,(GH')(F'G') - 4\,(GF')\,(G'H')\} \\
& +2\mathrm{f}^2\mathrm{g}^2\,\{(GH)(HF) - 2\,(HH')^2 - 2\,(HF')\,(FF') - 2\,(HG')\,(GG') \\
& \qquad\qquad + 4(HF')(G'H') - 4\,(HG')\,(H'F')\} \\
& +4\mathrm{f}^2\mathrm{gh}\,\{2\,(GH)\,(G'H') - (HF)\,(GF') + (FG)\,(HF') \\
& \qquad + (GG')(HH') - (GH')\,(HG') \\
& \qquad + 2\,(GG')(F'G') - 2\,(HH')\,(H'F') + 4\,(H'F')\,(F'G)\} \\
& +4\mathrm{fg}^2\mathrm{h}\,\{(GH)\,(FG') + 2\,(HF)(H'F') - (FG)(HG') \\
& \qquad + (HH')\,(FF') - (HF')\,(FH') \\
& \qquad + 2\,(HH)\,(G'H') - 2(FF')(F'G') + 4(G'H')\,(F'G')\} \\
& +4\mathrm{fgh}^2\,\{-(GH)\,(FH') + (HF)\,(GH') + 2\,(FG)\,(F'G') \\
& \qquad + (FF')\,(GG') - (FG')(GF') \\
& \qquad + 2\,(FF')\,(H'F) - 2\,(GG')\,(G'H') + 4\,(H'F')\,(G'H')\}.
\end{aligned}$$

The results due to eliminating the variables from the other forms of (5) will be obtained by writing

| | | | |
|---|---|---|---|
| in turn | c, −b, f; | and | $C, B, F, L, L', A'$; |
| | −c, a, g; | | $C, A, G, M, M', B'$; |
| | b, −a, h; | | $B, A, H, N, N', C'$; |
| | f, g, h; | | $F, G, H, F', G', H'$. |

For the third method we may proceed as follows. If $X_1=0$, $Y_1=0$, $Z_1=0$, $T_1=0$, be the equations corresponding to (5) for the second conic, we may eliminate $z$ from the two equations $Y=0$, $Y_1=0$; and $y$ from the two equations $Z=0$, $Z_1=0$: these will give two equations of the degree 4 in $x:t$, and of the degree 2 in the coordinates of each of the conics. The elimination of $x:t$ from these two equations will give a resultant of the degree $4\times2\times2=16$ in the coordinates of each of the conics. But it seems doubtful whether it would be worth while actually to effect the elimination.

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*Addition by Prof. Cayley.*

Write
$$U = (a, b, c, d, f, g, h, l, m, n \between x, y, z, t)^2,$$
$$U_0 = (\quad\quad ,, \quad\quad \between \xi, \eta, \zeta, \omega)^2,$$
$$W = (\quad\quad ,, \quad\quad \between x, y, z, t \between \xi, \eta, \zeta, \omega),$$
$$P = (\alpha, \beta, \gamma, \delta \between x, y, z, t),$$
$$P_0 = (\alpha, \beta, \gamma, \delta \between \xi, \eta, \zeta, \omega).$$

Then the equation of the cone having for its vertex the arbitrary point $(\xi, \eta, \zeta, \omega)$, and passing through the conic $U=0$, $P=0$, is

$$UP_0^2 - 2WPP_0 + U_0P^2 = 0.$$

Or, if to put the coefficients $\xi, \eta, \zeta, \omega$ in evidence, we write for a moment

$$A = (a, h, g, l \between x, y, z, t),$$
$$B = (h, b, f, m \between \quad ,, \quad ),$$
$$C = (g, f, c, n \between \quad ,, \quad ),$$
$$D = (l, m, n, d \between \quad ,, \quad ),$$

and therefore
$$W = A\xi + B\eta + C\zeta + D\omega;$$

then the equation is

$$U(\alpha\xi+\beta\eta+\gamma\zeta+\delta\omega)^2 - 2P(\alpha\xi+\beta\eta+\gamma\zeta+\delta\omega)(A\xi+B\eta+C\zeta+D\omega)$$
$$+P^2(a, b, c, d, f, g, h, l, m, n \between \xi, \eta, \zeta, \omega)^2 = 0.$$

And if we expand first in $\xi$, $\eta$, $\zeta$, $\omega$, and then in $x$, $y$, $z$, $t$, the final result is

| | $x^2$ | $y^2$ | $z^2$ | $t^2$ | $yz$ | $zx$ | $xy$ | $xt$ | $yt$ | $zt$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\xi^2$ | | $C$ | $B$ | $F$ | $2A'$ | | | | $2L$ | $2L'$ |
| $+\eta^2$ | $C$ | | $A$ | $G$ | | $2B'$ | | $2M'$ | | $2M$ |
| $+\zeta^2$ | $B$ | $A$ | | $H$ | | | $2C'$ | $2N$ | $2N'$ | |
| $+\omega^2$ | $F$ | $G$ | $H$ | | $2F'$ | $2G'$ | $2H'$ | | | |
| $+\eta\zeta$ | $2A'$ | | | $2F'$ | $-2A$ | $-2C'$ | $-2B'$ | $2(Q-R)$ | $-2M$ | $-2N'$ |
| $+\zeta\xi$ | | $2B'$ | | $2G'$ | $-2C'$ | $-2B$ | $-2A'$ | $-2L'$ | $2(R-P)$ | $-2N$ |
| $+\xi\eta$ | | | $2C'$ | $2H'$ | $-2B'$ | $-2A'$ | $-2C$ | $-2L$ | $-2M'$ | $2(P-Q)$ |
| $+\xi\omega$ | | $2M'$ | $2N$ | | $2(Q-R)$ | $-2L'$ | $-2L$ | $-2F$ | $-2H'$ | $-2G'$ |
| $+\eta\omega$ | $2L$ | | $2N'$ | | $-2M$ | $-2(R-P)$ | $-2M'$ | $-2H'$ | $-2G$ | $-2F'$ |
| $+\zeta\omega$ | $2L'$ | $2M$ | | | $-2N'$ | $-2N$ | $2(P-Q)$ | $-2G'$ | $-2F'$ | $-2H$ |

$= 0.$

In particular, if $\eta=0$, $\zeta=0$, $\omega=0$, then we have the foregoing equation $X=0$; and the like for the equations $Y=0$, $Z=0$, and $W=0$ respectively.

Take a, b, c, f, g, h for the six coordinates of the line through the point

$$\begin{vmatrix} x, & y, & z, & t \\ \xi, & \eta, & \zeta, & \omega \end{vmatrix}$$

that is, write

$$\begin{aligned} \mathrm{a} &= y\zeta - z\eta, & \mathrm{f} &= x\omega - t\xi, \\ \mathrm{b} &= z\xi - x\eta, & \mathrm{g} &= y\omega - t\eta, \\ \mathrm{c} &= x\eta - y\xi, & \mathrm{h} &= z\omega - t\zeta, \end{aligned}$$

where, of course,

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch} = 0.$$

Then the foregoing equation of the cone is

$$\left.\begin{array}{l} A\mathrm{a}^2 + B\mathrm{b}^2 + C\mathrm{c}^2 + F\mathrm{f}^2 + G\mathrm{g}^2 + H\mathrm{h}^2 \\ -2A'\mathrm{bc} - 2B'\mathrm{ca} - 2C'\mathrm{ab} + 2F'\mathrm{gh} + 2G'\mathrm{hf} + 2H'\mathrm{fg} \\ +2P\mathrm{af} + 2M\mathrm{ag} - 2N'\mathrm{ah} \\ -2L'\mathrm{bf} + 2Q\mathrm{bg} + 2N\mathrm{bh} \\ +2L\mathrm{cf} - 2M'\mathrm{cg} + 2R\mathrm{ch} \end{array}\right\} = 0.$$

And this may be regarded as the equation of the *conic* in terms of the twenty-one coordinates of the conic, and of the six coordinates of an arbitrary line meeting the conic. It is, in fact, the general form of the equation given in the paper—Cayley "On a new Analytical Representation of a Curve in Space," Quart. Math. Jour., t. iii. (1860), see p. 233.

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# *Déduction de différents Théorèmes géométriques d'un seul Principe Algébrique.* *Par* H. G. ZEUTHEN.*

Le principe algébrique dont nous allons exposer ici plusieurs applications géométriques s'exprime par le Théorème suivant :

*Soit donnée une forme algébrique entière, homogène et du second degré par rapport à chacune de deux couples de variables* $x_1$, $x_2$ *et* $\xi_1$, $\xi_2$ *; formons les deux discriminants de cette forme, qui seront des formes quartiques contenant la couple des variables regardées comme constantes pendant la formation du discriminant : je dis que ces deux discriminants auront les mêmes invariants.*

J'ai déduit ce théorème général d'une de ses applications géomé-

* Received from the Author too late for communication at the Meeting on June 12.