## *Homographic and Circular Reciprocants.* By L. J. Rogers, B.A.

[*Read March* 11*th*, 1886.]

Contents of Paper.



In the following pages the following abbreviations will always be employed:

$$t, a, b, c, \ldots \text{ denote } \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots \text{ respectively,}$$

$n$ denotes the characteristic of a reciprocant, and $i$ the degree, in cases where the reciprocant is homogeneous.

As I shall have to refer to homogeneous and orthogonal reciprocants, it will not be out of place to state first the manner in which successive educts in each system are formed. In every case these unreduced educts will be employed as the protomorphs of the corresponding system. We shall begin with the simplest forms, viz., the mixed homogeneous, and pass on to pure, and then orthogonal forms.

§ 1. The formation of mixed homogeneous reciprocants has been explained by Prof. Sylvester in *Mess. of Math.*, Sept., 1885. The simplest absolute reciprocant of the system is $a/t^{\frac{3}{2}}$, and the successive educts are formed by operating with $\frac{1}{t^{\frac{1}{2}}}\frac{d}{dx}$, each operation giving a new absolute reciprocant.

Now, if we take such a function $\mu$ that

$$\left(\frac{d\mu}{dx}\right)^2 = \frac{dy}{dx},$$

this operation is $\frac{d}{d\mu}$, and, if we consider its effect on a reciprocant of characteristic $n$, and omit the denominator $t^{\frac{1}{2}}$, it is equivalent to $t\delta_x - \frac{1}{2}na$, raising the characteristic 3.

We write then $$\frac{d}{d\mu} = t\frac{d}{dx} - \tfrac{1}{2}na,$$

and define $M_1$, $M_2$ ... by the following equations:—

$$\left.\begin{aligned} M_1 &= a & (n = 3)\\ M_2 &= tb - \tfrac{3}{2}a^2 & (n = 6)\\ 2M_3 &= 2t^2c - 10tab + 9a^3 & (n = 9)\\ 4M_4 &= 4t^3d - 30t^2ac - 20t^2b^2 + 124ta^2b - 81a^4 & (n = 12) \end{aligned}\right\} \quad\ldots\ldots(1),$$

&c.,

so that $$\frac{dM_r}{d\mu} \equiv M_{r+1}.$$

In this system every $M$ is negative in character; and therefore every reciprocant function of the $M$'s must be either of an odd or an even degree in every term, though not necessarily homogeneous.

Thus $2M_3 - 9M_1^3$ is a reciprocant (the post-Schwarzian), the degree being odd throughout, but $M_4 + M_2^2$ is not; $n$ is given by the equation

$$n = 2t\delta_t + 3a\delta_a + 4b\delta_b + \ldots,$$

and in an irreducible function of the $M$'s, $n = 3w$, where $w$ is the common weight of each term; but, if the $M$-function contain a factor $t^\nu$ the characteristic of the remaining factor is $3w - 2\nu$, since the characteristic of $t$ is 2. As we only concern ourselves with this remaining factor, we shall write

$$\left.\begin{aligned} n &= 3w - 2\nu \\ i &= w - \nu \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

§ 2. *Pure Reciprocants.*

Pure reciprocants are formed by the successive operation of $a^{-\frac{1}{3}}\frac{d}{dx}$ on any absolute pure reciprocant.

If we take a function $\rho$ such that

$$\left(\frac{d\rho}{dx}\right)^3 = \frac{d^2y}{dx^2},$$

this operation is $\dfrac{d}{d\rho}$, and is equivalent to the operation of $a\delta_x - \dfrac{n}{3}b$ on a pure reciprocant of character $n$. Hence

$$\frac{d}{d\rho} = a\delta_x - \frac{n}{3}b \quad\text{......(1)},$$

omitting denominators as in § 1.

Operating successively with $\dfrac{d}{d\rho}$ on the well-known pure reciprocant $3ac-5b^2$, which we shall call $3R_2$, we get the following system of pure educts,

$$\left.\begin{aligned} 3R_2 &= 3ac-5b^2 && (n=8) \\ 9R_3 &= 9a^2d-45abc+40b^3 && (n=12) \\ 9R_4 &= 9a^3-9a^2(7bd+5c^2)+255ab^2c-160b^4 && (n=16) \end{aligned}\right\} \text{......(2)},$$

&c.,

so that
$$\frac{dR_r}{d\rho} \equiv R_{r+1}.$$

In this system any $R$ with an even suffix is positive, and any with an odd suffix is negative in character. Any isobaric function of the $R$'s is a reciprocant, and there is no restriction as in the case of mixed reciprocants.

As in the homogeneous mixed system, we have

$$n = 3a\delta_a + 4b\delta_b + 5c\delta_c + \ldots,$$

and in an irreducible $R$-function $n = 4w$; but, if $a^\nu$ occur as a factor,

$$\left.\begin{aligned} n &= 4w-3\nu \\ i &= w-\nu \end{aligned}\right\} \text{......(3)}.$$

Any pure reciprocant is annihilated by $V$ where

$$V = 3a^2\delta_b + 10ab\delta_c + (15ac+10b^2)\,\delta_d + \ldots,$$

in which $10ab = \delta_x(3a^2)+4ab$, $15ac+10b^2 = \delta_x(10ab)+5ac$;

the next coefficient $= \delta_x(15ac+10b^2)+6ad$, &c.

§ 3. *Orthogonal Reciprocants.*

In this system the generating operation is

$$(1+t^2)^{-\frac{1}{2}}\frac{d}{dx}, \quad\text{or}\quad \frac{d}{ds},$$

where
$$\left(\frac{ds}{dx}\right)^2 = 1+t^2.$$

Acting on the numerator of an absolute orthogonal, $\frac{d}{ds}$ is equivalent to $(1+t^2)\,\delta_x - nta$, and raises the characteristic by 3. The simplest absolute orthogonal is $a\,(1+t^2)^{-\frac{3}{2}}$, which we shall call $\phi_1$, being $\frac{d\phi}{ds}$, where $\phi$ and $s$ are intrinsic coordinates. Or rather, omitting denominators, we have, for the system of orthogonal protomorphs,

$$\left.\begin{aligned} \phi_1 &= a & (n=3) \\ \phi_2 &= (1+t^2)\,b - 3a^2t & (n=6) \\ \phi_3 &= (1+t^2)^2\,c - 10abt\,(1+t^2) + 3\,(5t^2-1)\,a^3 & (n=9) \\ &\text{\&c.} \end{aligned}\right\} \quad \ldots\ldots(1),$$

Here also, as in mixed homogeneous reciprocants, every $\phi$ is negative in character, and the same remarks apply to $\phi$-functions as to $M$-functions.

The characteristic of any $\phi$-function, containing $(1+t^2)^\nu$ as a factor, is just the same as what we had in $M$-functions,

$$n = 3w - 2\nu,$$

and
$$i = w - \nu.$$

The differential equation for determining $n$ is more complicated than in the previous cases. We have, in fact,

$$nt = (1+t^2)\,\delta_t + t\,(3a\delta_a + 4b\delta_b + 5c\delta_c + \ldots) + V \,\ldots\ldots\ldots\ldots(2),$$

where $V$ is the pure reciprocant annihilator.

I have hitherto been obliged, for the sake of explaining my notation, to dwell at some length on facts already known; but the chief difference of notation lies in the fact, that I have used as protomorphs in every system the unreduced educts of the first and simplest form. Unless this principle be adhered to, the $M$-, $R$-, and $\phi$-functions will not be isobaric, as is necessary that they should be. I have moreover always made the coefficient of the leading term in every protomorph in any system equal to unity.

One point of difference must be noticed as regards the weights of the letters $t, a, b, c \ldots$ in the $M$ and $R$ protomorphs.

In making $a = M_1$, $tb - \frac{3}{2}a^2 = M_2$, &c., we assume the weights of $t, a, b, \ldots$ to be 0, 1, 2, ... respectively; but, in writing $ac - \frac{5}{3}b^2 = R_2$, &c., we assume the weights of $a, b, c, \ldots$ to be 0, 1, 2, ... respectively. This is done in order to make the characteristic of each protomorph in either system a constant numerical multiple of the weight.

As an example of the utility of these protomorphs, we can deduce some interesting properties of the operator $V$.

It is easy to show that any $M$-function can be expressed in terms of $t$, $M_1$, $M_2$, $R_2$, $R_3$, ...

For, let $F$ be such a function, and let it contain letters as far as $f$. Then, since $M_6$ and $R_5$ are linear in $f$, we see that $M_6$ is a function of $t$, $a$, $b$, $c$, $d$, $e$, $R_5$. In the same way, $M_5$ and $e$ are functions of $t$, $a$, $b$, $c$, $d$, $R_4$. In this way we may eliminate $b$, $c$, $d$, $e$, $f$ from the $M$-function and replace them by functions of $R_5$, $R_4$, $R_3$, $R_2$, $M_2$.

The operation of $V$, therefore, on $F$ is

$$Vt\,.\,\delta_t + VM_1\,\delta_{M_1} + VM_2\,\delta_{M_2} + VR_2\,.\,\delta_{R_2} + \dots,$$

every term of which vanishes except the third, and

$$VF = VM_2\frac{dF}{dM_2} = 3a^2t\,\frac{dF}{dM_2} \quad \text{.....................(3)}.$$

Now $\dfrac{dF}{dM_2}$ will be a reciprocant, for, let $F$ be expanded in powers of $M_2$, viz.,

$$F = A + BM_2 + CM_2^2 + \dots,$$

where $A$, $B$, $C$ are reciprocants, then, if

| | | | | | |
|---|---|---|---|---|---|
| | $F$ | be of characteristic | $n$ | and character | $q$, |
| so will | $A$ | ,, | $n$ | ,, | $q$, |
| | $B$ | ,, | $n-6$ | ,, | $-q$, |
| | $C$ | ,, | $n-12$ | ,, | $q$, |
| | | &c. | | &c. | |

Moreover,

$$\frac{dF}{dM_2} = B + 2CM_2 + 3DM_2^2 + \dots,$$

every term of which is obviously of characteristic $n-6$, and character $-q$.

Hence $3a^2t\,\dfrac{dF}{dM_2}$ or $VF$ is a reciprocant of characteristic $n+2$ and character $-q$ .......................................................................(4).

The same reasoning applies to orthogonals, since all such can be expressed in terms of $1+t^2$, $\phi_1$, $\phi_2$, $R_2$, $R_3$, &c. $V$ therefore acting on an orthogonal yields another of opposite character, and of characteristic greater by 2 than the first.

Similarly, the operation $2t^2\delta_t - V$ is the same as

$$2t^2\delta_t + 2t^2 \frac{dM_2}{dt} \cdot \frac{d}{dM_2} - VM_2 \cdot \frac{d}{dM_2},$$

which is easily reduced to

$$2t^2\delta_t + 2tM_2 \frac{d}{dM_2} \quad \text{or} \quad 2t\left(t\delta_t + M_2 \frac{d}{dM_2}\right).$$

If therefore $V$ be expanded in forms of $t$ and $M_2$, the operation

$$t\delta_t + M_2 \frac{d}{dM_2}$$

will only alter the value of the numerical coefficients in the expansion, *i.e.*, it will give another reciprocant of the same kind.

Hence $2t^2\delta_t - V$ gives a new mixed homogeneous reciprocant, increasing the characteristic by 2, and not altering the character ...(5).

Now, $1-t^2$ is a negative reciprocant of characteristic 2, therefore

$$2t^2(1+t^2)\,\delta_t - (1+t^2)\,V$$

and

$$(1-t^2)\,V$$

both add 4 to $n$, and are positive operations, *i.e.*, do not alter the character.

Adding and dividing by $t^2$, whose $n$ is 4, we get the positive operator

$$(1+t^2)\,\delta_t - V \quad\text{..............................}(6),$$

which does not alter the character.

Acting on $tb - \frac{3}{2}a^2$, we thus get

$$(1+t^2)\,b - 3a^2.$$

Acting on $tc - 5ab$, we get

$$(1+t^2)\,c - 10abt + 15a^3.$$

The operator (6) seems to be analogous to the reciprocantive operator $\delta_t$ for orthogonals; for, operating on the last two obtained orthogonals, we come back to the reciprocants we started with. The same is easily found to be true generally, if we start with a mixed homogeneous reciprocant which only contains $t$ to the first power.

This analogy seems to foreshadow the results established in the next section.

§ 4. *Connection between Orthogonal and Mixed Homogeneous Reciprocants.*

Let $\xi$, $\eta$ be determined by the equations

$$\xi = x + yi, \qquad \eta = y + xi,$$

and let $\tau$, $\alpha$, $\beta$ represent $\quad \dfrac{d\eta}{d\xi},\ \dfrac{d^2\eta}{d\xi^2},\ \ldots$ .

We have then $\quad \tau = \dfrac{t+i}{1+ti} = i\,\dfrac{1+t^2}{(1+ti)^2},$

$$\frac{d\xi}{dx} = 1 + ti = i^{\frac{1}{2}}\sqrt{\frac{1+t^2}{\tau}},$$

$$\alpha = \frac{2a}{(1+ti)^2}\cdot\frac{dx}{d\xi} = \frac{2a}{(1+ti)^3},$$

or $\quad \dfrac{\alpha}{\tau^{\frac{3}{2}}} = \dfrac{2}{i^{\frac{3}{2}}}\cdot\dfrac{a}{(1+t^2)^{\frac{3}{2}}} = \dfrac{\kappa}{i^{\frac{1}{2}}}\cdot\dfrac{a}{(1+t^2)^{\frac{3}{2}}}$ say.

Differentiating again, we get

$$\frac{\tau\beta - \frac{3}{2}\alpha^2}{\tau^{\frac{5}{2}}} = \frac{\kappa}{i^{\frac{1}{2}}}\cdot\frac{(1+t^2)\,b - 3a^2t}{(1+t^2)^{\frac{5}{2}}}\cdot\frac{dx}{d\xi},$$

or $\quad \dfrac{\tau\beta - \frac{3}{2}\alpha^2}{\tau^3} = \dfrac{\kappa}{i}\cdot\dfrac{(1+t^2)\,b - 3a^2t}{(1+t^2)^3}.$

By this process we form the successive orthogonal protomorphs on the right hand, and the mixed homogeneous in $\xi$ and $\eta$ on the left. Calling, therefore, the latter $\mu_1$, $\mu_2$, ..., we get

$$\left.\begin{aligned}\kappa\phi_1 &= i^{\frac{1}{2}}\mu_1\\ \kappa\phi_2 &= i\mu_2\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and, generally, $\quad \kappa\phi_n = i^{\frac{1}{2}n}\mu_n,$

where $\quad \kappa = -2\sqrt{-1}.$

Any homogeneous isobaric $\phi$-function only differs by a constant from the corresponding $\mu$-function, so that if any $M$-function be equated to zero, and its complete primitive be $y = f(x)$, then the corresponding

$\phi$-function will have $\quad y + xi = f(x + yi) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$

for its complete primitive.

Moreover, any orthogonal can be converted into a mixed homogeneous reciprocant in $\xi$ and $\eta$. For, since a $\phi$-function must be isobaric, we may neglect the powers of $i^{\frac{1}{2}}$, as we see from § 1, and since the degrees of the terms are either all even or all odd, only even powers of $\kappa$ can occur, after neglecting some power of $\kappa$, if necessary, which runs through the whole. Hence the resulting $\mu$-function is always real. Thus $\phi_3+18\phi_1^3$ is the well-known orthogonal

$$(1+t^2)\,c-10abt+15a^3,$$

so that the corresponding $\mu$-function is

$$\kappa^3\mu_3+18\kappa\mu_1^3=\kappa\,(\kappa^2\mu_3+18\mu_1^3),$$

or, giving $\kappa$ its value $-2\sqrt{-1}$, we get

$$2\mu_3-9\mu_1^3,$$

which is easily seen to be the post-Schwarzian in $\xi$ and $\eta$.

§ 5. A Homographic Reciprocant is a reciprocant that remains unaltered when $x$, $y$ are changed into $\dfrac{Lx+M}{x+N}$, $\dfrac{L'y+M'}{y+N'}$, respectively, where $LMNL'M'N'$ are constants.

Such reciprocants, when equated to zero, give of course complete primitives of the form $\dfrac{L'y+M'}{y+N'}=f\left(\dfrac{Lx+M}{x+N}\right)$ ......................(1).

They will, moreover, always be mixed and homogeneous, since we may put $\lambda x+\mu$ for $x$, and $\lambda'y+\mu'$ for $y$, without affecting the reciprocant, but we cannot put $\lambda x+\mu y+\nu$ for $x$, &c., so that this class cannot contain pure reciprocants.

Let us first consider differential expressions which remain unaltered when $x$ is changed into $\dfrac{Lx+M}{x+N}$. If any function of $t$, $a$, $b$, ... belong to this set, and is also a reciprocant, it will be an homographic reciprocant.

It is well known that if $u$, $y$ be functions of $x$, that

$$(tb-\tfrac{3}{2}a^2)\left(\frac{dx}{du}\right)^6$$

$$=\left\{\frac{d^3y}{du^3}\cdot\frac{dy}{du}-\frac{3}{2}\left(\frac{d^2y}{du^2}\right)^2\right\}\left(\frac{dx}{du}\right)^2-\left\{\frac{d^3x}{du^3}\cdot\frac{dx}{du}-\frac{3}{2}\left(\frac{d^2x}{du^2}\right)^2\left(\frac{dy}{du}\right)^2\right\} \quad \ldots(2).$$

Now, if
$$u=\frac{Lx+M}{x+N},$$

then
$$\frac{d^3x}{du^3}\cdot\frac{dx}{du}-\frac{3}{2}\left(\frac{d^2x}{du^2}\right)^2=0,$$

as is easily shown, and consequently

$$\frac{tb-\frac{3}{2}a^2}{t^4} = \text{the same expression with } u \text{ instead of } x,$$

and is therefore unaltered if $x$ be changed homographically. We may, moreover, differentiate as often as we please for $y$, *i.e.*, operate with $\frac{1}{t}\frac{d}{dx}$, and we shall get a series of functions which remain unaltered by this change in $x$. If $w$ be the power of $t$ in the denominator of any such function, the operation $\frac{1}{t}\frac{d}{dx}$ is evidently the same as operating with $t\delta_x - wa$ on the numerator and adding 2 to $w$, and $w$ will be given by the equation

$$w = t\delta_t + 2a\delta_a + 3b\delta_b + \dots .$$

The numerators of the successive educts are

$$\left.\begin{array}{ll} tb-\frac{3}{2}a^2 & (w=4) \\ t^2c-6tab+6a^3 & (w=6) \\ \&c. & \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots (3).$$

Assuming as an annihilator $A\delta_a + B\delta_b + \dots = H$ say, we see that

$$H\,(t\delta_x - wa) = 0,$$

omitting the expression to be operated upon. That is,

$$tH\delta_x = wA = t\,(H\delta_x - \delta_x H),$$

because $$\delta_x H = 0.$$

But $H\delta_x - \delta_x H$ contains no differential operators of the second order, and because $\delta_x \equiv a\delta_t + b\delta_a + \dots$ we get

$$t\,(H\delta_x - \delta_x H) = tA\delta_t + t\,(B-\delta_x A)\,\delta_a + t\,(C-\delta_x B)\,\delta_b + \dots$$
$$= A\,(t\delta_t + 2a\delta_a + 3b\delta_b + \dots),$$

therefore $$t\,(B-\delta_x A) = 2aA, \ \&c.$$

Testing for $tb - \frac{3}{2}a^2$, we must evidently have

$$tB = 3aA,$$

therefore $$t\delta_x A = aA.$$

Integrating, we get $$A = t,$$

and because $$B - \delta_x A = 2a,$$
$$C - \delta_x B = 3b, \ \&c.,$$

we evidently get $$H = t\delta_a + 3a\delta_b + 6b\delta_c + 10c\delta_d + \dots \quad \dots\dots\dots\dots\dots(4),$$

so that the particular differential functions in question are binariants, and can be converted in ordinary invariants by changing $t$ into $a$; $a$ into $2!\,b$; $b$ into $3!\,c$, &c.

§ 6. Besides the annihilator given in § 5, homographic reciprocants will also have another annihilator which bears a remarkable analogy to $V$.

It is easy to show, from § 5 (2), that the expression

$$\frac{tb - \frac{3}{2}a^2}{t^2} \text{ remains unaltered on changing } y \text{ into } \frac{L'y + M'}{y + N'},$$

and consequently we may differentiate as often as we please for $x$, and we get a series of expressions having the same property.

The power of $t$ in the denominator is the same as $i$, the degree of the numerator, therefore

$$i = t\delta_t + a\delta_a + b\delta_b + \dots,$$

and the operator for the numerator is

$$t\delta_x - ia.$$

Proceeding, as in § 5, and assuming

$$A\delta_a + B\delta_b + \dots$$

as an annihilator, we get $$t(B - \delta_x A) = Aa,$$

$$t(C - \delta_x B) = Ab, \text{ \&c.}$$

Testing for $tb - \frac{3}{2}a^2$, we have, as before,

$$tB = 3aA,$$

$$t\delta_x A = 2aA,$$

whence $$A = t^2,$$

and $$B - \delta A = at,$$

$$C - \delta B = bt,$$

$$D - \delta C = ct, \text{ \&c.},$$

giving the law of forming the successive coefficients. Hence the annihilator is

$$t^2\delta_a + 3at\delta_b + (4bt + 3a^2)\,\delta_c + (5ct + 10ab)\,\delta_d + \dots \quad \dots\dots\dots(1).$$

Homographic Reciprocants have therefore two annihilators, but neither are sufficient to prove the reciprocant property.

§ 7. Hitherto we have obtained only one homographic reciprocant, but if we can obtain one more we can deduce an infinite number. For we can then obtain one absolute homographic reciprocant, which by differentiation for $x$ will give another non-absolute reciprocant, also homographic. This, combined with the first to form another absolute reciprocant, will, by again differentiating for $x$, produce another; and the process may be carried on indefinitely.

We can easily show that

$$4M_2M_4-5M_3^2+a^2M_2^2 \quad \text{.........................} (1)$$

is homographic.

By actual calculation we easily find

$$HM_1 = t,$$

$$HM_2 = 0,$$

$$HM_3 = tM_2,$$

$$4HM_4 = 10tM_3-2atM_2.$$

Hence

$$H(4M_2M_4-5M_3^2+a^2M_2^2)$$

$$= M_2(10tM_3-2atM_2)-10tM_2M_3+2atM_2^2 = 0.$$

We see, therefore, that there will be an infinite number of homographic reciprocants.

§ 8. We now come to a class of reciprocants for which the most suitable name seems to be Circular Reciprocants.

We have seen, in § 4, that from any class of $M$-functions can be deduced a corresponding class of $\phi$-functions, the latter being real only if the former be reciprocants. Now, we have obtained such a class in Homographic Reciprocants, viz.,

$$M_2,\ 4M_2M_4-5M_3^2+a^2M_2,\ \&c.,$$

see § 7, whence we see that the class of $\phi$-functions

$$\phi_2,\ 4\phi_2\phi_4-5\phi_3^2-4a^2\phi_2 \dots \&c.$$

are not altered if we change

$$x+yi \text{ into } \frac{L(x+yi)+M}{x+yi+N},$$

and

$$y+xi \text{ into } \frac{L'(y+xi)+M'}{y+xi+N'}.$$

Since we may replace $i$ by $-i$ in either of these equations, we obtain the relations that

$$x^2+y^2 \text{ becomes } \frac{(Lx+M)^2+L^2y^2}{(x+N)^2+y^2},$$

$$x \quad ,, \quad \frac{(Lx+M)(x+N)+Ly^2}{(x+N)^2+y^2},$$

$$y \quad ,, \quad \frac{(LN-M)\,y}{(x+N)^2+y^2},$$

with a further transformation obtained by interchanging $x$ and $y$, and adding dashes to $L$, $M$, $N$.

On account of the numerators and denominators of the fractions presenting circular forms, we shall call such $\phi$-functions that are annihilated by (1) Circular Reciprocants. The simplest is $\phi_2$, or $(1+t^2)\,b-3a^2t$, giving the general equation to a circle as complete primitive when equated to zero.

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*Thursday, May 13th*, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. F. W. Watkin was admitted into the Society.

The following communications were made:—

On Cremonian Congruences contained in Linear Complexes: Dr. Hirst, F.R.S.

Solution of the Cubic and Biquadratic Equation by means of Weierstrass's Elliptic Functions: Prof. Greenhill.

On the Complex of Lines which meet a Unicursal Quartic Curve: Prof. Cayley, F.R.S.

On Airy's Solution of the Equations of Equilibrium of an Isotropic Elastic Solid under conservative forces: W. J. Ibbetson, B.A.

Conic Note: H. M. Taylor, M.A.

On the Converse of Stereographic Projection and on Contangential and Coaxal Spherical Circles: H. M. Jeffery, F.R.S.

The following presents were received:—

"Royal Society, Proceedings," Vol. XL., No. 242.

"Educational Times," for May.

"Physical Society—Proceedings," Vol. VII., Pt. 4; April, 1886.

"Solid Geometry," by Percival Frost. 3rd ed., 8vo; London, 1886.

"Annals of Mathematics," Vol. I., No. 6, Jan. 1885; Vol. II., No. 1, Sept. 1885; Charlottesville, Va.

"Bulletin de la Société Mathématique de France," T. XIV., No. 2.

"Journal de l'École Polytechnique." 55 cahier; 1885.

"Bulletin des Sciences Mathématiques," T. X.; Mai, 1886.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. II., F. 7 and 8.

"Archiv for Mathematik og Naturvidenskab," B. 10, H. 1, 2, 3, 4.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 3me Serie, T. 1, 1884, T. 2 (1re cahier), 1885.

"Mitteilungen der Mathematischen Gesellschaft," in Hamburg, No. 6; Marz, 1886.

"Tidsskrift for Mathematik," V. Raekke; 3 Aargang, 1—6 Hefte.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VI., No. 6.

"Observations pluviométriques et thermométriques de Juin, 1883, à Mai, 1884; rapport sur les Orages de 1883." 8vo, Bordeaux, 1884, par M. Lespiault.

"Observations pluviométriques et thermométriques de Juin, 1884, à Mai, 1885; rapport sur les Orages de 1884." 8vo, Bordeaux, 1885.

C. Neumann—"Über die Kugelfunctionen $P_n$ und $Q_n$, insbesondere über die Entwicklung der Ausdrücke

$$P_n(zz_1+\sqrt{1-z^2}\,.\,\sqrt{1-z_1^2}\cos\phi) \text{ und } Q_n(zz_1+\sqrt{1-z^2}\,.\,\sqrt{1-z_1^2}\cos\phi),$$

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"Theory and Practice of the Slide Rule; with a short explanation of the properties of Logarithms," by Lt.-Col. J. R. Campbell, F.G.S.; Spon, 1886 (from the Author).

"Annali di Matematica," T. XIV., F. 1.

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## *On the Complex of Lines which meet a Unicursal Quartic Curve.*

*By* Prof. CAYLEY.

[*Read May 13th,* 1886.]

The curve is taken to be that determined by the equations

$$x : y : z : w = 1 : \theta : \theta^3 : \theta^4,$$

viz., it is the common intersection of the quadric surface $\Theta = 0$, and the cubic surfaces $P = 0$, $Q = 0$, $R = 0$, where

$$\Theta = xw - yz,$$
$$P = x^2z - y^3,$$
$$Q = xz^2 - y^2w,$$
$$R = z^3 - yw^2.$$