On the limits of certain infinite series and integrals.

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Four investigations*) have appeared recently which deal with various extensions of Abel's well known theorem on the continuity of $\sum a_n x^n$; the following note, although quite elementary, seems to include all the results hitherto obtained, and shews more clearly the reasons for the various conditions which have been given.

It will be seen that the conditions requisite to prove Theorems B, C (§§ 3, 4, below) are less stringent (at least in theory) than those given by the authors quoted: thus, instead of requiring that $\sum n^k |v_n|$ is convergent and imposing various further conditions**) on all the differences $\Delta^2 v_n, \Delta^3 v_n, \ldots, \Delta^{k+1} v_n$, we only require that $n^k |v_n|$ shall tend to zero as n tends to ∞, and that $\sum n^k |\Delta^{k+1} v_n|$ shall be less than a number K, independent of the variable x. In §§ 1, 2 we discuss the case $k = 1$ at some length, as the case most frequently used in applications; and it is proved that the conditions of Theorem A include those given in all the previous investigations on this subject.

§ 5 contains a discussion of a question suggested by a comparison of §§ 3, 4: *Are the mean-values of Cesàro and Hölder necessarily the same, for a given oscillatory series?* This question is answered in the affirmative for $k = 1, 2$: for higher values of k, we can only say that Cesàro's mean certainly exists whenever Hölder's mean exists, but the converse has not been proved to be true.

In § 6 we obtain two theorems on the limiting values of integrals, which correspond to Theorem A and a theorem due to Dedekind (and Cahen) referred to in § 1.

*) L. Fejér, Math. Annalen, Bd. 58, 1904, p. 51. G. H. Hardy, Proc. Lond. Math. Soc. (2), vol. 3, 1906, p. 247; and Math. Annalen, Bd. 64, 1907, p. 77. C. N. Moore, Trans. Amer. Math. Soc. vol. 8, 1907, p. 299.

**) That these conditions are really superfluous is clear from the Lemma proved in § 4.

§ 1.

Series summable by a single mean.

Suppose that the series $\sum a_n$ can be summed by taking a single mean, so that if

$$s_n = a_0 + a_1 + \cdots + a_n$$

then

$$\frac{s_0 + s_1 + \cdots + s_n}{n+1}$$

has a definite limit s as n tends to ∞.

It follows that we can find a constant C such that

$$|s_0 + s_1 + \cdots + s_n| < (n+1)\,C.$$

Then

$$|s_n| = |(s_0 + s_1 + \cdots s_n) - (s_0 + s_1 + \cdots + s_{n-1})| < (2n+1)\,C$$

and so

$$|a_n| = |s_n - s_{n-1}| < 4n\,C.$$

Again if we write

$$\sigma_n = s_0 + s_1 + \cdots + s_n$$

we find

$$s_n = \sigma_n - \sigma_{n-1}$$

and so

$$a_n = s_n - s_{n-1} = \sigma_n - 2\sigma_{n-1} + \sigma_{n-2}.$$

These preliminary results being established we proceed next to enunciate and prove the first theorem, which is a direct generalization of one due to Dedekind and Cahen*).

Theorem A. *Suppose*

1) *that the series* $\sum a_n$ *is summable by a single mean, as just explained;*

2) *that* v_n *is a function of* x *with the properties:*

$$\left.\begin{array}{ll}(\alpha) & \sum n\,|\Delta^2 v_n| < K^{**)} \\ (\beta) & \lim\limits_{n\to\infty} n v_n = 0\end{array}\right\} \quad \text{if } x > 0,$$

$$(\gamma) \quad \lim_{x\to 0} v_n = 1,$$

*) See E. Landau, Münchener Sitzungsberichte Bd. 36, 1906, pp. 157, 160.

**) Since all the terms in the series $\sum n\,|\Delta^2 v_n|$ are *positive*, this condition implies the convergence of the series.

where K is independent of x and n. Then the series $\sum a_n v_n$ *converges if x is positive, and*

$$\lim_{x\to 0} \sum a_n v_n = s.$$

For we have identically

$$\begin{aligned} & a_0 v_0 + a_1 v_1 + \cdots + a_n v_n \\ &= \sigma_0 v_0 + (\sigma_1 - 2\sigma_0) v_1 + (\sigma_2 - 2\sigma_1 + \sigma_0) v_2 + \cdots + (\sigma_n - 2\sigma_{n-1} + \sigma_{n-2}) v_n \\ &= \sigma_0 \Delta^2 v_0 + \sigma_1 \Delta^2 v_1 + \cdots + \sigma_{n-1} \Delta^2 v_{n-1} + \sigma_n v_n - \sigma_{n-1} v_{n+1} \end{aligned}$$

where

$$\Delta^2 v_n = v_n - 2v_{n+1} + v_{n+2}.$$

Now, since $|\sigma_n| < (n+1)C$, and since $\sum n |\Delta^2 v_n|$ is convergent, it follows that the series $\sum \sigma_n \Delta^2 v_n$ is absolutely convergent; so that

$$\sigma_0 \Delta^2 v_0 + \sigma_1 \Delta^2 v_1 + \cdots + \sigma_{n-1} \Delta^2 v_{n-1}$$

tends to a definite limit as n tends to infinity.

Also

$$|\sigma_n v_n| < (n+1) C |v_n|, \qquad |\sigma_{n-1} v_{n+1}| < n C |v_{n+1}|,$$

so that $\sigma_n v_n$ and $\sigma_{n-1} v_{n+1}$, tend to zero as n tends to infinity, in virtue of condition (β). It follows that $\sum_0^\infty a_n v_n$ is convergent and that

$$\sum_0^\infty a_n v_n = \sum_0^\infty \sigma_n \Delta^2 v_n. \tag{1}$$

Taking the special case $a_0 = 1$, $a_1 = a_2 = a_3 = \cdots = 0$, we find $\sigma_n = n + 1$ and so (1) gives

$$v_0 = \sum_0^\infty (n+1) \Delta^2 v_n \tag{2}$$

Thus, combining (1) and (2), we find that

$$\sum_0^\infty a_n v_n - s v_0 = \sum_0^\infty \{\sigma_n - (n+1)s\} \Delta^2 v_n. \tag{3}$$

Now $\frac{\sigma_n}{n+1}$ has s as its limit, so that we can determine m in such a way as to satisfy the inequality

$$\left|\frac{\sigma_n}{n+1} - s\right| < \varepsilon, \quad \text{if } n \geqq m$$

or

$$|\sigma_n - (n+1)s| < (n+1)\varepsilon, \qquad \text{if } n \geqq m.$$

Also, for all values of n

$$|\sigma_n| < (n+1) C, \text{ and } |s| \leqq C.$$

Making use of these inequalities in (3) we have

$$\left|\sum_0^\infty a_n v_n - s v_0\right| < 2C\sum_0^{m-1}(n+1)\,|\Delta^2 v_n| + \varepsilon\sum_m^\infty (n+1)\,|\Delta^2 v_n|$$

$$(4)\qquad < 2C\sum_0^{m-1}(n+1)\,|\Delta^2 v_n| + \varepsilon K$$

in virtue of condition (α).

Now as x tends to 0, $\Delta^2 v_n$ tends to 0 in virtue of condition (γ); and, since m has now been fixed, it follows that

$$(5)\qquad \lim_{x\to 0}\sum_0^{m-1}(n+1)\,|\Delta^2 v_n| = 0\cdot$$

Thus from (4) and (5), we find that

$$\overline{\lim_{x\to 0}}\left|\sum_0^\infty a_n v_n - s v_0\right| \leqq \varepsilon K$$

where ε may be taken as small as we please, by proper choice of the index m: but this choice of m does not affect the limit on the left, and so this limit must be actually zero.

Hence

$$\lim_{x\to 0}\left(\sum_0^\infty a_n v_n - s v_0\right) = 0$$

or

$$\lim_{x\to 0}\sum_0 a_n v_n = s\,.$$

§ 2.

Comparison of the theorem of § 1 with earlier results.

Consider first C. N. Moore's Theorem I*); it is there assumed that for all positive values of x

$$|v_n| < A\,(nx)^{-(2+\varrho)},\quad |\Delta^2 v_n| < Bn^{-2}(nx)^{-\varrho},$$

and that

$$\Delta^2 v_n \geqq 0,\quad (0 \leqq nx \leqq c),$$

where A, B, ϱ, c are certain positive constants.

When these conditions hold good, (β) of § 1 is obviously satisfied.

To examine (α), suppose that x is such that c falls between νx and $(\nu+1)x$, so that

$$|\Delta^2 v_n| = \Delta^2 v_n,\quad 0 \leqq n \leqq \nu.$$

*) Trans. Amer. Math. Society, vol. 8, April 1907, p. 300.

Now, taking $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = 0, \cdots$ in (1) of § 1, we have

$$\sum n \Delta^2 v_n = v_1$$

so that

$$\sum n |\Delta^2 v_n| - v_1 = \sum_{\nu+1}^{\infty} n \{ |\Delta^2 v_n| - \Delta^2 v_n \} \leqq 2 \sum_{\nu+1}^{\infty} n |\Delta^2 v_n|.$$

Or, using Moore's condition,

$$\sum n |\Delta^2 v_n| - v_1 < 2B \sum_{\nu+1}^{\infty} n^{-(1+\varrho)} x^{-\varrho}$$

and by a well-known elementary theorem this again is less than

$$\frac{2B}{\varrho} \frac{1}{(\nu x)^{\varrho}} < \frac{2B}{\varrho (c-x)^{\varrho}}.$$

Consequently condition (α) is satisfied if $x \leqq c' < c$.

I note incidentally that in the same way Moore's Theorem II (l. c. p. 307) can be deduced from the theorem of Dedekind and Cahen already mentioned, which requires the condition $\sum |\Delta v_n| < k$.

We consider next Hardy's Theorem I*) which requires (for $k = 1$) that

$$v_n \geqq 0, \; \Delta v_n \geqq 0, \; \Delta^2 v_n \geqq 0$$

and that $\sum n v_n$ is convergent for $x > 0$.

Under these conditions we have

$$\sum n |\Delta^2 v_n| = \sum n \Delta^2 v_n = v_1$$

so that condition (α) of § 1 is satisfied; and $n v_n$ tends to zero, so that condition (β) is also satisfied. Thus our Theorem A includes Hardy's Theorem I (for $k = 1$).

As regards Hardy's Theorem II**), we then suppose that

$$v_n = \varphi(nx),$$

where

$$|\varphi''(\xi)| \leqq \frac{K}{\xi^{2+\varrho}}, \qquad \text{if } \xi \geqq 1,$$

and also

$$|\varphi''(\xi)| \leqq M, \qquad \text{if } 0 \leqq \xi \leqq 1.$$

Now we have at once

$$\Delta^2 \varphi(nx) = \varphi\{(n+2)x\} - 2\varphi\{(n+1)x\} + \varphi(nx)$$

$$= \int_0^x dt \int_0^x \varphi''(nx + t + v)\, dv$$

so that

$$|\Delta^2 v_n| \leqq M x^2, \qquad \text{if } 0 < nx \leqq 1,$$

*) Math. Annalen, Bd. 64, p. 78.

**) See p. 86 of the paper quoted.

and also

$$|\Delta^2 v_n| \leqq \frac{Kx^2}{(nx)^{2+\varrho}}, \qquad \text{if } nx \geqq 1.$$

Hence, since ϱ is positive, we find

$$\sum n\,|\Delta^2 v_n| \leqq Mx^2 \sum_1^{\nu} n + \sum_{\nu+1}^{\infty} \left\{ \frac{Kx^2}{(nx)^{2+\varrho}} \right\}$$

$$\leqq \frac{1}{2} Mx^2 \nu(\nu+1) + \left(\frac{K}{\nu(1+\varrho)}\right)(\nu x)^{-\varrho}$$

where ν is the integral part of $\frac{1}{x}$, and thus we get

$$\sum n\,|\Delta^2 v_n| \leqq \frac{1}{2} M(1+x) + 2^{\varrho} Kx$$

so that under Hardy's conditions, we can apply Theorem A of § 1 above. It is of course to be remembered that (for $k = 1$) Hardy's Theorem II is the same as Fejér's result*), which is therefore also included under § 1 above.

Finally, if we use Hardy's Theorem III**), we must suppose that $\Delta^2 v_n$ changes sign only a finite number of times, say r times. We have then

$$\sum n\,|\Delta^2 v_n| \leqq |v_1| + 4rM$$

if M is the maximum of $n\,|v_n|$: and so condition (α) is again satisfied.

It is therefore clear that Theorem A includes all previous results which apply to the case $k = 1$.

§ 3.

Series which are k-times indeterminate.

Following Cesàro***) we say that *a series is k-times indeterminate* if the limit of $\frac{S_n^{(k)}}{A_n^{(k)}}$ exists and is finite as n tends to infinity, but the limit of $\frac{S_n^{(k-1)}}{A_n^{(k-1)}}$ does not exist; here we write for brevity

$$S_n^{(k)} = s_n + ks_{n-1} + \frac{k(k+1)}{2!} s_{n-2} + \cdots + \frac{k(k+1)\cdots(k+n-1)}{n!} s_0$$

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*) Math. Annalen, Bd. 58, p. 62.

**) See p. 88 of the paper quoted.

***) Bulletin des Sciences mathématiques (2), t. 14, 1890, p. 114; the definition of „*une série k fois indéterminée*“ is given on p. 119. See also Borel, Séries Divergentes p. 91, where, however, the precise definition is not elaborated for higher values than $k = 1$. The reader may also consult Cesàro, „Sulla determinazione assintotica delle serie di potenze“ Rendiconti della R. Accademia delle Scienze fisiche e matematiche di Napoli 28 ottobre 1893; and Bromwich, Infinite Series (London, 1908), Arts. 122—129.

and

$$A_n^{(k)} = \binom{n+k}{k} = \frac{(n+k)!}{n!\,k!}.$$

More briefly, we may define $S_n^{(k)}$ by means of the identities

$$\sum S_n^{(k)} x^n = (1-x)^{-k} \sum s_n x^n = (1-x)^{-(k+1)} \sum a_n x^n$$

from which it follows that

$$\sum s_n x^n \ = (1-x)^k \sum S_n^{(k)} x^n$$

and

$$\sum a_n x^n \ = (1-x)^{k+1} \sum S_n^{(k)} x^n.$$

It results at once from the last identity that

$$(1) \quad a_n = S_n^{(k)} - (k+1) S_n^{(k)} + \frac{(k+1)k}{2!} S_{n-2}^{(k)} - \cdots + (-1)^{k+1} S_{n-k-1}^{(k)}$$

where it is to be understood that when a negative suffix occurs in the formula, the corresponding $S^{(k)}$ is to be replaced by zero; so that, for example,

$$a_0 = S_0^{(k)}, \quad a_1 = S_1^{(k)} - (k+1) S_0^{(k)},$$

and so on.

If we now substitute for $a_0, a_1, \ldots, a_n$ their values given by these formulae, we find that

$$(2) \quad \begin{aligned} &a_0 v_0 + a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \\ &= S_0^{(k)} \Delta^{k+1} v_0 + S_1^{(k)} \Delta^{k+1} v_1 + \cdots + S_{n-k-1}^{(k)} \Delta^{k+1} v_n + R_n, \end{aligned}$$

where

$$(3) \quad \begin{aligned} R_n = {} & S_{n-k}^{(k)} \{ v_{n-k} - (k+1) v_{n-k+1} + \cdots + (-1)^k (k+1) v_n \} \\ & + S_{n-k+1}^{(k)} \left\{ v_{n-k+1} - (k+1) v_{n-k+2} + \cdots + (-1)^{k-1} \frac{k(k+1)}{2!} v_n \right\} \\ & + \cdots\cdots \\ & + S_n^{(k)} v_n. \end{aligned}$$

Now when the series is k-times indeterminate we can find a constant C, so that, for all values of n,

$$\left| \frac{S_n^{(k)}}{A_n^{(k)}} \right| < C$$

or

$$(4) \quad | S_n^{(k)} | < C' n^k, \quad \text{since } \lim_{n \to \infty} \left\{ \frac{A_n^{(k)}}{n^k} \right\} = \frac{1}{k!}.$$

Thus from (3) and (4) we find that

$$\begin{aligned} |R_n| < {} & C'(n-k)^k \{ |v_{n-k}| + (k+1) |v_{n-k+1}| + \cdots + (k+1) |v_n| \} \\ & + C'(n-k+1)^k \left\{ |v_{n-k+1}| + (k+1) |v_{n-k+2}| + \cdots + \frac{(k+1)k}{2!} |v_n| \right\} \\ & + \cdots + C' n^k |v_n|. \end{aligned}$$

Now remembering that

$$1+(k+1)+\frac{(k+1)k}{2!}+\cdots+(k+1)+1=2^{k+1},$$

we see that the last inequality gives

$$|R_n|<2^{k+1}C'\{(n-k)^k|v_{n-k}|+(n-k+1)^k|v_{n-k+1}|+\cdots+n^k|v_n|\}.$$

Thus R_n will tend to zero as n tends to ∞, provided that

(5) $$\lim_{n\to\infty}(n^k v_n)=0.$$

Under this condition, then, we see from (2) that the series $\sum_0^\infty a_n v_n$ converges or diverges with the series $\sum_0^\infty S_n^{(k)}\,\Delta^{k+1}v_n$.

Now, since $\lim_{n\to\infty}\left\{\frac{A_n^{(k)}}{n^k}\right\}=\frac{1}{k!}$ and $\lim_{n\to\infty}\left\{\frac{S_n^{(k)}}{A_n^{(k)}}\right\}$ is finite, it follows that the series $\sum_0^\infty S_n^{(k)}\,\Delta^{k+1}v_n$ is absolutely convergent provided that

(6) $$\sum_0^\infty n^k|\Delta^{k+1}v_n|$$

is convergent.

Thus, under the two conditions (5) and (6), we have the equation

(7) $$\sum_0^\infty a_n v_n=\sum_0^\infty S_n^{(k)}\,\Delta^{k+1}v_n,$$

in which both sides converge, and the right hand is *absolutely* convergent, although the series on the left may not converge absolutely.

In particular if we write

$$a_0=1,\ a_1=a_2=a_3=\cdots=0,$$

we find

$$s_0=s_1=s_2=\cdots=1$$

and so

$$S_n^{(k)}=A_n^{(k)}.$$

Thus the equation $\sum a_n v_n=\sum S_n^{(k)}\,\Delta^{k+1}v_n$ gives in this special case the identity

(8) $$v_0=\sum_0^\infty A_n^{(k)}\,\Delta^{k+1}v_n.$$

Combining (7) and (8) we find that

(9) $$\sum_0^\infty a_n v_n-s v_0=\sum_0^\infty\{S_n^{(k)}-sA_n^{(k)}\}\,\Delta^{k+1}v_n,$$

where s may be any number.

Suppose now that s is Cesàro's limit, so that

$$\lim_{n\to\infty}\left\{\frac{S_n^{(k)}}{A_n^{(k)}}\right\}=s,$$

while

$$\lim_{n\to\infty}\left\{\frac{A_n^{(k)}}{n^k}\right\}=\frac{1}{k!}.$$

Thus we can find m so that

$$|S_n^{(k)}-sA_n^{(k)}|<\varepsilon n^k,\quad \text{if } n\geqq m.$$

Further we can find a constant C such that

$$|S_n^{(k)}|<Cn^k,\quad |s|\,A_n^{(k)}<Cn^k$$

for all values of n. So we find on using these inequalities in (9)

$$(10)\qquad \left|\sum_0^\infty a_n v_n - s v_0\right| \leqq \sum_0^\infty |S_n^{(k)}-sA_n^{(k)}|\,|\Delta^{k+1}v_n|$$

$$<2C\sum_0^{m-1}(n+1)^k\,|\Delta^{k+1}v_n|+\varepsilon\sum_m^\infty n^k\,|\Delta^{k+1}v_n|.$$

Let us now introduce the further conditions that v_n is a function of x such that

$$(11)\qquad \lim_{x\to 0} v_n=1,\quad\text{and}\quad \sum_0^\infty n^k\,|\Delta^{k+1}v_n|<K,\quad \text{if } x>0$$

where K is independent of x and n.

Thus, as in § 1, since $\lim\limits_{x\to 0}\Delta^{k+1}v_n=0$, we find from (10) and (11) that

$$\overline{\lim_{x\to 0}}\left|\sum_0^\infty a_n v_n - s v_0\right|\leqq \varepsilon K,$$

and since ε may be made arbitrarily small by proper choice of m (which does not affect the limit on the left) we see that

$$\lim_{x\to 0}\left(\sum_0^\infty a_n v_n - s v_0\right)=0,$$

or

$$\lim_{x\to 0}\sum_0^\infty a_n v_n = s.$$

Thus we have proved:

Theorem B. *Suppose*

1) *that* $\sum a_n$ *is k-times indeterminate and has the sum s in Cesàro's sense,*

2) *that* v_n *is a function of* x *with the properties*:

$$\begin{array}{ll} (\alpha) & \sum n^k |\Delta^{k+1} v_n| < K^{*)} \\ (\beta) & \lim_{n \to \infty} n^k v_n = 0 \end{array} \Bigg\} \quad \text{if } x > 0,$$

$$(\gamma) \qquad \lim_{x \to 0} v_n = 1,$$

where K *is independent of* x *and* n. *Then the series* $\sum a_n v_n$ *converges if* x *is positive, and*

$$\lim_{x \to 0} \sum a_n v_n = s.$$

This is obviously the exact extension of Theorem A, given in § 1; we proceed next to consider the effect of using Hölder's mean instead of Cesàro's in this theorem.

§ 4.

Series for which Hölder's k-fold mean exists.

The results obtained in § 3 are not capable of being compared directly with the corresponding conclusions of Hardy's paper. Indeed Hardy does not use Cesàro's limit at all, but employs instead another kind of generalized mean, first introduced into analysis by Hölder**); for the sake of brevity, we shall refer to this mean as *Hölder's mean.*

Hölder repeats the process of taking the arithmetic mean as often as may prove necessary: thus if we write

$$(n+1) H_n^{(1)} = s_0 + s_1 + \cdots + s_n,$$

and

$$(n+1) H_n^{(r+1)} = H_0^{(r)} + H_1^{(r)} + \cdots + H_n^{(r)}, \quad (r = 1, 2, 3, \cdots),$$

it may happen that $H_n^{(k)}$ tends to a definite finite limit s as n tends to infinity. When this in the case, we say that *Hölder's k-fold mean exists and is equal to s.*

We shall now obtain the theorems, analogous to those of § 3, which apply when this k-fold mean exists.

If we apply the transformation used by Hardy (l. c., p. 81) to the sum

$$(1) \qquad a_0 v_0 + a_1 v_1 + \cdots + a_n v_n$$

taken up to n terms, instead of to infinity, it will be seen that the sum takes the form

$$s_0 \Delta v_0 + s_1 \Delta v_1 + \cdots + s_n \Delta v_n + s_n v_{n+1}.$$

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*) As we have already pointed out in connexion with Theorem A, this condition implies the convergence of the series $\sum n^k \left|\Delta^{k+1} v_n\right|$.

**) Mathematische Annalen, Bd. 20, 1882, p. 535.

Now (as Hardy points out) $\lim\limits_{n\to\infty} \frac{s_n}{n^k} = 0$, so that $\lim (s_n v_{n+1}) = 0$, provided that we make use of the hypothesis $\lim (n^k v_n) = 0$, which is the same as condition (β) in § 3. Thus for our purpose we may replace the sum by

(2) $$s_0 \Delta v_0 + s_1 \Delta v_1 + \cdots + s_n \Delta v_n .$$

Similarly, writing

$$(n+1) H_n^{(1)} = s_0 + s_1 + \cdots + s_n ,$$

we find that the sum (1) is equal to

$$H_0^{(1)} \Delta^2 v_0 + 2 H_1^{(1)} \Delta^2 v_1 + \cdots + (n+1) H_n^{(1)} \Delta^2 v_n + (n+1) H_n^{(1)} \Delta v_{n+1}$$

and again the last term tends to zero as n tends to infinity, because $\lim (n^k v_n) = 0$, and $\lim \frac{H_n^{(1)}}{n^{k-1}} = 0$.

If we continue this process we get a set of expressions equivalent to (1), namely,

(3) $$H_0^{(1)} \Delta^2 v_0 + 2 H_1^{(1)} \Delta^2 v_1 + \cdots + (n+1) H_n^{(1)} \Delta^2 v_n ,$$

(4) $$H_0^{(2)} \Delta^3 v_0 + 2^2 H_1^{(2)} \Delta^3 v_1 + \cdots + (n+1)^2 H_n^{(2)} \Delta^3 v_n - H_0^{(2)} \Delta^2 v_1 - \cdots - (n+1) H_{n-1}^{(2)} \Delta^2 v_n ,$$

and so on. The general result is

(5) $$\sum_{\mu=0}^{k-1} (-1)^\mu \sum_{\nu=\mu}^{n} f_{k-\mu}^{(k)}(\nu)\, H_{\nu-\mu}^{(k)}\, \Delta^{k+1-\mu} v_\nu$$

where the polynomial $f_{k-\mu}^{(k)}(\nu)$ is of degree $(k-\mu)$ in ν and is the same polynomial as that used by Hardy (l. c. p. 82). It is then easy to see that the argument given in § 3 can be at once extended to shew that each of the series

$$\sum_{\nu=\mu}^{\infty} f_{k-\mu}^{(k)}(\nu)\, H_{\nu-\mu}^{(k)}\, \Delta^{k+1-\mu} v_\nu$$

is absolutely convergent provided that all the series $\sum n^\lambda |\Delta^{\lambda+1} v_n|$ are convergent, when λ takes the values $1, 2, \cdots, k$. Thus $\sum\limits_0^\infty a_n v_n$ is then convergent. And if in addition each of the series $\sum n^\lambda |\Delta^{\lambda+1} v_n|$ is less than a fixed number K, we can prove by a method similar to that of § 3 that

$$\lim_{x\to 0} \sum a_n v_n = s .$$

We have, however, *apparently* introduced k separate conditions as to the differences $\Delta^2 v_n, \cdots, \Delta^{k+1} v_n$, whereas in § 3 we only needed the *one* condition $\sum n^k |\Delta^{k+1} v_n| < K$. We now proceed to shew that this one condition includes all the others; and obviously this statement will be established by proving the following lemma.

Lemma. *If* $\sum n^{\lambda}|\Delta u_n| < K$, *then* $\sum n^{\lambda-1}|u_n| < K$ *provided that* $\lim u_n = 0$, λ *being any positive integer.*

For since $\sum n^{\lambda}|\Delta u_n|$ converges, so also does $\sum|\Delta u_n|$: let us write then

(6) $$U_n = |\Delta u_n| + |\Delta u_{n+1}| + |\Delta u_{n+2}| + \cdots.$$

Thus

(7) $$U_n - U_{n+1} = |\Delta u_n|$$

and

(8) $$\begin{aligned} U_n - U_p &\geqq |\Delta u_n + \Delta u_{n+1} + \cdots + \Delta u_{p-1}| \\ &\geqq |u_n - u_p|, \quad \text{if } p > n. \end{aligned}$$

But, since both u_p and U_p tend to 0 as p tends to ∞, the last inequality gives

(9) $$U_n \geqq |u_n|.$$

Now we see from (7) that

(10) $$\begin{aligned} \sum_1^{\nu} n^{\lambda}|\Delta u_n| &= \sum_1^{\nu} n^{\lambda}(U_n - U_{n+1}) \\ &= \sum_1^{\nu}\{n^{\lambda} - (n-1)^{\lambda}\}\,U_n - \nu^{\lambda}U_{\nu+1} \end{aligned}$$

and

(11) $$\begin{aligned} \nu^{\lambda}U_{\nu+1} &= \nu^{\lambda}\{|\Delta u_{\nu+1}| + |\Delta u_{\nu+2}| + \cdots\} \\ &< \sum_{\nu+1}^{\infty} n^{\lambda}|\Delta u_n|. \end{aligned}$$

Thus, using (10) and (11), we find

$$\sum_1^{\nu}\{n^{\lambda} - (n-1)^{\lambda}\}\,U_n = \sum_1^{\nu} n^{\lambda}|\Delta u_n| + \nu^{\lambda}U_{\nu+1} < \sum_1^{\infty} n^{\lambda}|\Delta u_n| < K.$$

Hence, letting ν tend to ∞, we obtain

(12) $$\sum_1^{\infty}\{n^{\lambda} - (n-1)^{\lambda}\}\,U_n \leqq K.$$

But

$$\lim_{n\to\infty}\frac{n^{\lambda} - (n-1)^{\lambda}}{\lambda n^{\lambda-1}} = 1$$

and so, since U_n is positive, we see from (12) that*)

$$\sum_1^{\infty} n^{\lambda-1}U_n \leqq K.$$

*) K not denoting the same number as before, but still a *fixed* value independent of x.

Now from (9) we have

$$U_n \geqq |u_n|$$

so that finally

$$\sum_1^\infty n^{\lambda-1} |u_n| < K,$$

which is the inequality stated in the lemma.

Thus we now have proved

Theorem C. *Suppose*

1) *that* $\sum a_n$ *is summable by taking* k *arithmetic means in Hölder's manner, and that* s *is its sum;*

2) *that* v_n *is a function of* x *with the properties:*

$$\left.\begin{array}{ll} (\alpha) & \sum n^k |\Delta^{k+1} v_n| < K \\ (\beta) & \lim\limits_{n\to\infty} n^k v_n = 0 \end{array}\right\} \textit{ if } x > 0,$$

$$(\gamma) \qquad \lim_{x\to 0} v_n = 1.$$

Then the series $\sum a_n v_n$ *is convergent if* $x > 0$ *and*

$$\lim_{x\to 0} \sum a_n v_n = s.$$

The similarity between Theorems B, C leads to the conjecture that whenever either Cesàro's limit or Hölder's mean exists, the other must also exist. But so far, I have only completely proved this conjecture for $k = 2$; the proof will be found in § 5.

Let us now compare Theorem C with Hardy's Theorems*); in Theorem I, Hardy assumes that**) $\Delta^{k+1} v_n \geqq 0$, so that

$$\sum n^k |\Delta^{k+1} v_n| = \sum n^k \Delta^{k+1} v_n \leqq k! \sum A_n^{(k)} \Delta^{k+1} v_n.$$

Now we proved incidentally in § 3 (see equation (8) p. 357) that

$$\sum A_n^{(k)} \Delta^{k+1} v_n = v_0$$

so that condition (α) is satisfied when $\Delta^{k+1} v_n \geqq 0$.

In Hardy's Theorem II, we take

$$v_n = \varphi(nx)$$

where

$$|\varphi^{k+1}(\xi)| \leqq M, \quad \text{if } 0 \leqq \xi \leqq 1$$

and

$$|\varphi^{k+1}(\xi)| \leqq \frac{K}{\xi^{k+1+\varrho}}, \quad \text{if } \xi \geqq 1, \qquad \text{(where } \varrho > 0\text{)}$$

---

*) As already stated in § 2, the results of Fejér and Moore only apply to the case $k = 1$.

**) Hardy also supposes $v_n \geqq 0$, $\Delta v_n \geqq 0$, $\cdots$, $\Delta^k v_n \geqq 0$, but we do not make use of these inequalities here.

Thus, exactly as in § 2, we find that

$$|\Delta^{k+1}v_n| \leqq Mx^{k+1}, \quad \text{if } 0 < nx \leqq 1$$

and

$$|\Delta^{k+1}v_n| \leqq \frac{Kx^{k+1}}{(nx)^{k+1+\varrho}}, \quad \text{if } nx \geqq 1.$$

Then, taking ν to be the integral part of $\frac{1}{x}$, we have

$$\sum n^k |\Delta^{k+1}v_n| \leqq Mx^{k+1} \sum_1^{\nu} n^k + Kx^{-\varrho} \sum_{\nu+1}^{\infty} n^{-(1+\varrho)}$$

and

$$\sum_1^{\nu} n^k < \int_1^{\nu+1} t^k\,dt < \frac{(\nu+1)^{k+1}}{k+1}, \quad \sum_{\nu+1}^{\infty} n^{-(1+\varrho)} < \int_{\nu}^{\infty} t^{-(1+\varrho)}\,dt = \frac{1}{\varrho \nu^{\varrho}}.$$

Hence

$$\sum n^k |\Delta^{k+1}v_n| < \frac{M}{k+1}(1+x)^{k+1} + \frac{2^{\varrho}K}{\varrho}$$

and so condition (α) is again satisfied.

Under the conditions given in Hardy's Theorem III, $\Delta^{k+1}v_n$ changes sign only a finite number of times, say r times. Then

$$\sum n^k |\Delta^{k+1}v_n| \leqq |\Delta^k v_1| + 4rM$$

where M is the maximum of $n^k|\Delta^k v_n|$; and so condition (α) holds here also.

Thus Hardy's Theorems I—III are included under Theorem C above: but it ought to be observed that it often happens that Hardy's conditions are the easiest to use in practical applications

§ 5.

Connexion between Cesàro's limit and Hölder's mean.

Let us write $C_n^{(k)} = \frac{S_n^{(k)}}{A_n^{(k)}}$, where $S_n^{(k)}$, $A_n^{(k)}$ have the meanings assigned to these symbols in § 3; then $\lim\limits_{n\to\infty} C_n^{(k)}$ is Cesàro's limit. Also $H_n^{(k)}$ has the meaning explained in § 4, so that $\lim\limits_{n\to\infty} H_n^{(k)}$ is Hölder's mean.

We shall shew that *for $k = 1, 2$, when $H_n^{(k)}$ has a limit s, then $C_n^{(k)}$ also tends to the same limit s; and conversely, when $C_n^{(k)}$ has a limit s, $H_n^{(k)}$ tends to the limit s.*

The case $k = 1$ needs no proof beyond the remark that identically

$$H_n^{(1)} = C_n^{(1)}$$

which is obviously true on comparison of the two definitions.

Next, suppose that $H_n^{(2)}$ has a limit s; then consider $S_n^{(2)}$. We have

$$\begin{aligned} S_n^{(2)} &= (n+1)s_0 + ns_1 + \cdots + s_n \\ &= H_0^{(1)} + 2H_1^{(1)} + \cdots + (n+1)H_n^{(1)} \\ &= (n+1)^2 H_n^{(2)} - nH_{n-1}^{(2)} - (n-1)H_{n-2}^{(2)} - \cdots - H_0^{(2)}; \end{aligned}$$

also

$$A_n^{(2)} = \frac{1}{2}(n+1)(n+2).$$

Thus

$$C_n^{(2)} = 2\left[\frac{n+1}{n+2}H_n^{(2)} - \frac{nH_{n-1}^{(2)} + (n-1)H_{n-2}^{(2)} + \cdots + H_0^{(2)}}{(n+1)(n+2)}\right].$$

Now, since

$$\lim H_n^{(2)} = s$$

we have also

$$\lim \frac{n+1}{n+2} H_n^{(2)} = s.$$

Further, by an extension (due to Stolz*)) of Cauchy's theorem on the limits of quotients, we have

$$\lim \frac{nH_{n-1}^{(2)} + (n-1)H_{n-2}^{(2)} + \cdots + H_0^{(2)}}{(n+1)(n+2)} = \lim \frac{(n+1)H_n^{(2)}}{2(n+2)} = \frac{1}{2}s.$$

Hence

$$\lim_{n\to\infty} C_n^{(2)} = s, \quad \text{if} \lim_{n\to\infty} H_n^{(2)} = s.$$

On the other hand, if it is known that $C_n^{(2)}$ tends to a definite limit s, we remark that

$$S_n^{(2)} - S_{n-1}^{(2)} = (n+1)H_n^{(1)}$$

so that

$$\begin{aligned} (n+1)H_n^{(2)} &= H_0^{(1)} + H_1^{(1)} + \cdots + H_n^{(1)} \\ &= S_0^{(2)} + \frac{1}{2}\{S_1^{(2)} - S_0^{(2)}\} + \cdots + \frac{1}{n+1}\{S_n^{(2)} - S_{n-1}^{(2)}\}. \end{aligned}$$

*) Stolz, Mathematische Annalen, Bd. 14, 1879, p. 234; Allgemeine Arithmetik, Bd. 1, p. 173. Bromwich, Infinite Series, p. 378. The theorem states that

$$\lim_{n\to\infty} \frac{f(n)}{\varphi(n)} = \lim_{n\to\infty} \frac{f(n+1) - f(n)}{\varphi(n+1) - \varphi(n)}$$

provided that the right-hand limit exists, and that $\varphi(n)$ tends *steadily* to ∞ with n (so that $\varphi(n+1) > \varphi(n)$). Cauchy's theorem is given by taking $\varphi(n) = n$.

Thus

$$(n+1)\,H_n^{(2)} = \frac{S_0^{(2)}}{1\cdot 2} + \frac{S_1^{(2)}}{2\cdot 3} + \cdots + \frac{S_{n-1}^{(2)}}{n(n+1)} + \frac{S_n^{(2)}}{n+1}$$

$$= \frac{1}{2}\left\{C_0^{(2)} + C_1^{(2)} + \cdots + C_{n-1}^{(2)}\right\} + \frac{1}{2}(n+2)\,C_n^{(2)}$$

or

$$H_n^{(2)} = \frac{C_0^{(2)} + C_1^{(2)} + \cdots + C_n^{(2)}}{2(n+1)} + \frac{1}{2}\,C_n^{(2)}.$$

If we apply Cauchy's theorem to the fraction above we get

$$\lim_{n\to\infty} \frac{C_0^{(2)} + C_1^{(2)} + \cdots + C_n^{(2)}}{2(n+1)} = \lim_{n\to\infty} \frac{C_n^{(2)}}{2} = \frac{1}{2}s$$

and so

$$\lim_{n\to\infty} H_n^{(2)} = s, \quad \text{if } \lim_{n\to\infty} C_n^{(2)} = s.$$

Thus *Hölder's mean and Cesàro's limit are certainly equivalent for* $k = 1, 2$.

It seems probable (as already remarked at the end of § 4) that these limits may be always equivalent, in the sense that the existence of either implies that of the other and also the equality of the two limits. And in fact Dr. K. Knopp has proved*) that when $H_n^{(k)}$ has a definite limit s, then also $\lim C_n^{(k)} = s$; but the proof of the converse theorem**) appears to present algebraical complications which I have not so far succeeded in surmounting.

It is, however, clear from Knopp's result that Cesàro's limit is at least as general as Hölder's mean; and, on account of the greater simplicity of the algebra (compare §§ 3, 4) it seems preferable to use the former rather than the latter, as a definition of the „sum" of an oscillatory series.

§ 6.

Corresponding theorems for integrals.

In view of the results obtained by Moore (l c. p. 311—325) it would seem at first sight likely that a set of conditions (for convergence factors in summable integrals) could be formulated which would be exactly parallel

*) Inauguraldissertation, Berlin 1907, p. 19—23; this dissertation reached me during the investigations given here. I had already obtained Dr. Knopp's result for $k = 3$ but had not established it in general.

**) If true, this theorem would be: When $C_n^{(k)}$ tends to a definite limit, then $H_n^{(k)}$ tends to the same limit.

to those of §§ 1, 3 above for series. But this expectation is not quite fulfilled in the case corresponding to that of § 1, and so far I have not carried the investigations further.

Suppose that the function $\varphi(t)$ is uniformly continuous for all values of $t \geqq a > 0$, and that the integral

$$(1) \qquad \int_a^\infty \varphi(t)\, dt$$

is summable*) and has the sum s. Suppose further that the function $f(x, t)$ has the property

$$(2) \qquad \lim_{x \to 0} f(x, t) = 1$$

then we wish to determine conditions corresponding to (α), (β) of § 1 which will justify the equation

$$(3) \qquad \lim_{x \to 0} \int_a^\infty f(x, t)\, \varphi(t)\, dt = s.$$

Moore proves (l. c. p. 315) that when the integral (1) is summable, and $\varphi(t)$ is uniformly continuous, then

$$(4) \qquad \lim_{t \to \infty} \frac{\varphi_1(t)}{t^2} = 0.$$

The equation (4) should be contrasted with the result used in § 1, that when a series is summable (by a single mean) a constant C can be found so that

$$|s_n| < (2n + 1)\, C;$$

but it appears that when $\varphi(t)$ has a summable integral, there is no reason for the existence of a constant C such that

$$|\varphi(t)| < Ct;$$

just as a function may have a convergent integral (to ∞) and yet need not be always less than a fixed value.

If now we apply the process of integration by parts, we find that

*) That is if

$$\int_a^t \varphi(t)\, dt = \varphi_1(t),$$

and

$$\int_a^t \varphi_1(t)\, dt = \varphi_2(t),$$

then

$$\lim_{t \to \infty} \left\{ \frac{\varphi_2(t)}{t} \right\} = s.$$

$$(5)\qquad \int_a^t \varphi(t) f(x,t)\,dt = \varphi_1(t) f(x,t) - \int_a^t \varphi_1(t) \frac{\partial f}{\partial t}\,dt$$
$$= \varphi_1(t) f(x,t) - \varphi_2(t) \frac{\partial f}{\partial t}$$
$$+ \int_a^t \varphi_2(t) \frac{\partial^2 f}{\partial t^2}\,dt.$$

This transformation suggests the conditions which here must correspond to (α), (β) of § 1. In fact we shall assume that $f(x, t)$ satisfies the conditions

$$\left.\begin{array}{ll}(\alpha) & \displaystyle\int_a^\infty t \left|\frac{\partial^2 f}{\partial t^2}\right| dt < K \\ (\beta) & t^2 |f| < X\end{array}\right\} \text{ if } x > 0$$

where K is independent of x and t; but X may depend on x, though not on t.

Then in virtue of (α) $\int_a^\infty (t-a) \frac{\partial^2 f}{\partial t^2} dt$ is a convergent integral; and on integration we find that

$$(6)\qquad \int_a^t (t-a) \frac{\partial^2 f}{\partial t^2}\,dt = (t-a) \frac{\partial f}{\partial t} - f(x,t) + f(x,a).$$

Thus as t tends to ∞, it is clear from (6) that $t \frac{\partial f}{\partial t}$ must tend to some definite limit; otherwise $\int_a^\infty (t-a) \frac{\partial^2 f}{\partial t^2} dt$ could not be convergent.

But, in virtue of (β), tf tends to zero, and so $t \frac{\partial f}{\partial t}$ cannot have any other limit than zero*) and so we have

$$(7)\qquad \lim_{t\to\infty} t \frac{\partial f}{\partial t} = 0.$$

Now it follows from (4) that

$$(8)\qquad \lim_{t\to\infty} \varphi_1(t) f(x,t) = 0$$

as a consequence of condition (β).

Also, since

$$\lim_{t\to\infty} \left\{\frac{\varphi_2(t)}{t}\right\} = s$$

*) In general if $F(t)$ tends to zero, it is easy to see that

$$\underline{\lim}_{t\to\infty} F'(t) \leqq 0 \leqq \overline{\lim}_{t\to\infty} F'(t).$$

it follows from (7) that

(9) $$\lim_{t\to\infty}\left\{\varphi_2(t)\frac{\partial f}{\partial t}\right\}=0.$$

Finally, we see that

(10) $$\int_a^\infty \varphi_2(t)\,\frac{\partial^2 f}{\partial t^2}\,dt$$

is an absolutely convergent integral, because

$$\lim\left\{\frac{\varphi_2(t)}{t}\right\}=s \quad\text{and}\quad \int_a^\infty t\left|\frac{\partial^2 f}{\partial t^2}\right|dt$$

converges in virtue of (α).

Thus we see from (5), (8), (9) and (10) that *the integral*

$$\int_a^\infty \varphi(t)\,f(x,t)\,dt$$

is convergent for all positive values of x.

Further the same four equations shew that

(11) $$\int_a^\infty \varphi(t)\,f(x,t)\,dt=\int_a^\infty \varphi_2(t)\,\frac{\partial^2 f}{\partial t^2}\,dt.$$

In particular from (6) and (7) we find the special result

(12) $$\int_a^\infty (t-a)\,\frac{\partial^2 f}{\partial t^2}\,dt=f(x,a).$$

Thus, combining (11) and (12) we obtain

(13) $$\int_a^\infty \varphi(t)\,f(x,t)\,dt-s\,f(x,a)$$
$$=\int_a^\infty \{\varphi_2(t)-s\,(t-a)\}\,\frac{\partial^2 f}{\partial t^2}\,dt.$$

The equation (13) exactly corresponds to equation (3) in § 1; and by repeating the argument given there, certain obvious changes being made, and using condition (α), it is easy to prove that

$$\lim_{x\to 0}\int_a^\infty \varphi(t)\,f(x,t)\,dt=\lim_{x\to 0} s\,f(x,a)=s.$$

Thus we obtain the result:

Theorem D. *Suppose that* $\varphi(t)$ *is uniformly continuous for* $t\geqq a>0$, *and that the integral*

$$\int_a^\infty \varphi(t)\,dt$$

is summable and has the sum s. Then

$$\lim_{x \to 0} \int_a^\infty \varphi(t) f(x,t)\, dt = s$$

provided that

$$\left.\begin{array}{ll} (\alpha) & \displaystyle\int_a^\infty t \left|\frac{\partial^2 f}{\partial t^2}\right| dt < K \\ (\beta) & \qquad t^2 |f| < X \end{array}\right\} \quad \text{if } x > 0,$$

$$(\gamma) \quad \lim_{x \to 0} f(x,t) = 1$$

where K is independent of x and t; while X is independent of t, though not necessarily independent of x.

It is also easy to establish the following theorem:

Theorem E. *Suppose that the integral*

$$\int_a^\infty \varphi(t)\, dt$$

converges to the value s. Then

$$\lim_{x \to 0} \int_a^\infty \varphi(t) f(x,t)\, dt = s$$

provided that

$$\left.\begin{array}{ll} (\alpha) & \displaystyle\int_a^\infty \left|\frac{\partial f}{\partial t}\right| dt < K \\ (\beta) & \displaystyle\lim_{t \to \infty} f(x,t) = 0 \end{array}\right\} \quad \text{if } x > 0,$$

$$(\gamma) \quad \lim_{x \to 0} f(x,t) = 1,$$

where K is independent of x and t.

It is easy to modify the method used at the beginning of § 2 to prove that the conditions of Theorems D, E include those given by Moore (l. c. pp. 318, 325).

Save for the form of the condition (β), Theorem D corresponds precisely to Theorem A for series; while Theorem E is the exact analogue of the theorem of Dedekind and Cahen already referred to in § 1.