the types (i.) and (ii.) coincide, giving rise to developments of the type (iv.), and all the remaining developments are of the type (iii.)

It is known that, when D is an uneven power of an uneven prime of the form $4n+1$, the equations (T) are always resoluble. But when D has any other value of either of the forms P or 2P, there is no known criterion for deciding whether these equations are or are not resoluble.

On the Value of a Certain Arithmetical Determinant. By HENRY J. STEPHEN SMITH, Savilian Professor of Geometry in the University of Oxford.

[*Read May* 11*th*, 1876.]

Let (m, n) denote the greatest common divisor of the integral numbers m and n; and let $\psi(m)$ be the number of numbers not surpassing m and prime to m; the symmetrical determinant

$$\Delta_m = \Sigma \pm (1, 1)(2, 2) \ldots\ldots (m, m)$$

is equal to
$$\psi(1) \times \psi(2) \times \ldots\ldots \times \psi(m).$$

This theorem may be established as follows. Let $p_1, p_2, \ldots$ be all the different primes dividing m, and consider the columns (P) of which the indices are

$$m, \ \frac{m}{p_1}, \ \frac{m}{p_2}, \ldots, \ \frac{m}{p_1p_2}, \ldots, \ \frac{m}{p_1p_2p_3}, \ldots\ldots.$$

Take these columns with the signs of the corresponding terms in the product
$$\psi(m) = m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\ldots;$$

and, attending to these signs, replace the terms of the last column of Δ_m by the sum of the corresponding terms in the columns (P). The value of Δ_m is not changed: the term (m, m) is evidently replaced by $\psi(m)$; and we shall now show that every other term (m, k) in the last column is replaced by zero; *i.e.*, that $\Delta_m = \psi(m) \times \Delta_{m-1}$, which is the theorem to be proved.

First, let k be prime to m; then

$$(m, k) = 1, \quad \left(\frac{m}{p}, \ k\right) = 1, \quad \&c.;$$

and (m, k) has to be replaced by a sum of units, of which as many are negative as positive; *i.e.*, by zero.

Secondly, let k be a divisor of m, other than unity or m itself; and let us separate the primes p into two classes, q and r, in such a manner that k does not divide any quotient of the form $\frac{m}{q}$, but does divide every quotient of the form $\frac{m}{r}$. There may or may not be any primes q, but there must be at least one prime r, or we should have $k = m$: we further observe that

$$(m, k) = \left(\frac{m}{r}, k\right) = \left(\frac{m}{r_1 r_2}, k\right) = \dots\dots = k;$$

$$\left(\frac{m}{q}, k\right) = \frac{k}{q}, \left(\frac{m}{q_1 q_2}, k\right) = \frac{k}{q_1 q_2}, \dots\dots$$

Thus, if we were to attend only to those columns of which the indices are $m, \frac{m}{q}, \dots\dots, \frac{m}{q_1 q_2}, \dots\dots$, we should have to replace (m, k), or k, by $k\Pi\left(1-\frac{1}{q}\right)$, just as before we replaced m by $\psi(m)$. But we have to attend to the complete series of columns (P); and thus we have to replace (m, k), not by $k\Pi\left(1-\frac{1}{q}\right)$ taken once, but by $k\Pi\left(1-\frac{1}{q}\right)$ taken as often as there are terms in the product $\Pi\left(1-\frac{1}{r}\right)$, and taken each time with the sign proper to the corresponding term of that product; *i.e.*, (m, k) is replaced by zero.

Lastly, let $k = h\delta$, δ being the greatest common divisor of k and m; so that $(m, k) = (m, \delta) = \delta$. If d is any divisor of m, we have the elementary theorem $\left(\frac{m}{d}, \delta\right) = \left(\frac{m}{d}, h\delta\right)$. For, if $\left(\frac{m}{d}, \delta\right) = \delta'$, we have also $(m, d\delta) = d\delta'$; and hence δ, which is a common divisor of m and $d\delta$, divides $d\delta'$, which is the greatest common divisor of those two numbers. But h is prime to $\frac{m}{\delta}$; therefore, *à fortiori*, h is prime to $\frac{m}{d\delta'}$; *i.e.*, $\left(\frac{m}{d\delta'}, h \cdot \frac{\delta}{\delta'}\right) = 1$, for $\frac{m}{d\delta'}$ is prime to $\frac{\delta}{\delta'}$ as well as to h; or, which is the same thing, $\left(\frac{m}{d}, h\delta\right) = \delta' = \left(\frac{m}{d}, \delta\right)$. It appears from this that in the columns (P) the terms which lie in the row of which the index is $h\delta$, are precisely the same as the terms which lie in the row of which the index is δ; and hence $(m, h\delta)$ is replaced by terms of which the sum is zero, because (m, δ) is replaced by terms of which the sum is zero.

The following remarks are suggested by the preceding theorem, or by its demonstration.

(1.) If we denote by $\mathrm{I}a$ the greatest integer not surpassing the positive quantity a, the theorem may be expressed by the equation

$$\text{(A.)} \quad \frac{\Delta_m}{1\,.\,2\,.\,3\,\ldots\,m} = \Pi\left(1-\frac{1}{p}\right)^{\mathrm{I}\frac{m}{p}},$$

the sign of multiplication extending to all primes not surpassing m.

(2.) Instead of the greatest common divisors themselves, we may consider their powers of exponent s; writing $\Psi_s(m) = m^s\Pi\left(1-\frac{1}{p^s}\right)$, and following the same course of demonstration, we obtain the theorem

$$\Delta_{m,s} = \Sigma \pm (1, 1)^s\,(2, 2)^s \ldots\ldots (m, m)^s.$$

$$\text{(B.)} \quad = \Psi_s(1) \times \Psi_s(2) \times \Psi_s(3) \times \ldots\ldots \times \Psi_s(m),$$

or

$$\frac{\Delta_{m,s}}{(1\,.\,2\,.\,3\,\ldots\,m)^s} = \Pi\left(1-\frac{1}{p^s}\right)^{\mathrm{I}\frac{m}{p}};$$

from which we infer, as a particular result,

$$\frac{\Delta_{m,2}}{\Delta_{m,1}} = 1\,.\,2\,.\,3\,\ldots\,m \times \Pi\left(1+\frac{1}{p}\right)^{\mathrm{I}\frac{m}{p}}.$$

(3.) The equation (B) is an identity with respect to the exponent s, which may have any value whatever: the case in which $s=-1$ is especially interesting. Let $[m, n]$ be the least common multiple of m and n, so that $[m, n] = \frac{m \times n}{(m, n)}$; we find

$$\nabla_m = \Sigma\pm[1, 1]\,[2, 2] \ldots\ldots [m, m]$$
$$= 1\,.\,2\,.\,3\,\ldots\,m\,.\,\Pi\,(1-p)^{\mathrm{I}\frac{m}{p}},$$

whence

$$\frac{\nabla_m}{\Delta_m} = \pm\Pi\,.\,p^{\mathrm{I}\frac{m}{p}};$$

and, in general, if $\nabla_{m,s} = \Sigma\pm[1, 1]^s \ldots\ldots [m, m]^s$,

$$\frac{\nabla_{m,s}}{\Delta_{m,s}} = \pm(\Pi\,.\,p^{\mathrm{I}\frac{m}{p}})^s:$$

the sign being that of $(-1)^{\Sigma\mathrm{I}\frac{m}{p}}$.

(4.) If, for the greatest common divisor δ of m and n, we substitute any function whatever $\phi(\delta)$ of δ, and denote by $\Phi(m)$ the function

$$\phi(m) - \Sigma\phi\left(\frac{m}{p}\right) + \Sigma\phi\left(\frac{m}{p_1p_2}\right) - \ldots\ldots,$$

we arrive at the identity

$$\Sigma\pm\phi(1, 1)\,\phi(2, 2) \ldots\ldots \phi(m, m)$$
$$= \Phi(1) \times \Phi(2) \times \Phi(3) \times \ldots\ldots \times \Phi(m).$$

Two particular cases are worth attention.

(α.) Let s be an integral number, and let

$$\phi_s(m) = 1^s+2^s+3^s+ \ldots\ldots (m-1)^s+m^s,$$

so that, when $s>1$,

$$\phi_s(m) = \frac{m^{s+1}}{s+1}+\tfrac{1}{2}m^s+\mathrm{B}_1\frac{s}{1\,.\,2}\,m^{s-1}-\mathrm{B}_3\frac{s\,(s-1)(s-2)}{1\,.\,2\,.\,3\,.\,4}\,m^{s-3}+\ldots\ldots,$$

B_1, B_3, ... being the fractions of Bernoulli, and the last term being $(-1)^{\frac{s-3}{2}}\mathrm{B}_{s-2}\frac{s}{2}\,m^2$, or $(-1)^{\frac{s-2}{2}}\mathrm{B}_{s-1}m$, according as s is uneven or even. Let also $\psi_s(m)$ be the sum of the powers s of the numbers prime to m and not surpassing m; we shall have

$$\Sigma\pm\phi_s(1,\,1)\ \ \phi_s(2,\,2)\ \ldots\ldots\ \phi_s(m,\,m)$$
$$= \psi_s(1)\times\psi_s(2)\times\psi_s(3)\times\ \ldots\ldots\ \times\psi_s(m,\,m).$$

The forms of the functions $\psi_s(m)$ are deducible from the expression for $\phi_s(m)$ (see a paper by Mr. Thacker in *Crelle*, Vol. XL., p. 89); we thus find

$$\psi_1(m) = \tfrac{1}{2}m^2\,\Pi\left(1-\frac{1}{p}\right),\quad \psi_2(m) = \tfrac{1}{3}m^3\,\Pi\left(1-\frac{1}{p}\right)+\tfrac{1}{6}m\Pi\,(1-a),$$

the general rule being that, in order to obtain $\psi_s(m)$, we are to substitute $m^k\Pi\,(1-p^{s-k})$ for m^k in the expression for $\phi_s(m)$.

(β.) Let $\sigma_s(m)$ be the sum of the powers s of the divisors of m; it will be found that

$$\sigma_s(m)-\Sigma\sigma_s\left(\frac{m}{p}\right)+\Sigma\sigma_s\left(\frac{m}{p_1p_2}\right)-\ \ldots\ldots = m^s.$$

For, if $m = p_1^{a_1}p_2^{a_2}\ldots\ldots$, and if we put

$$\mathrm{P} = (1+p^s+\ldots\ldots+p^{as}),$$
$$\mathrm{P}' = (1+p^s+\ldots\ldots+p^{(a-1)s}),$$

we have

$$\sigma_s(m) = \mathrm{P}_1\times\mathrm{P}_2\times\ldots\ldots,$$
$$\sigma_s\left(\frac{m}{p_1}\right) = \mathrm{P}_1'\times\mathrm{P}_2\times\ldots\ldots,$$
$$\sigma_s\left(\frac{m}{p_1p_2}\right) = \mathrm{P}_1'\times\mathrm{P}_2'\times\mathrm{P}_3\ \ldots\ldots,$$

whence

$$\sigma_s(m)-\Sigma\sigma_s\left(\frac{m}{p}\right)+\Sigma\sigma_s\left(\frac{m}{p_1p_2}\right)-\ \ldots\ldots = (\mathrm{P}_1-\mathrm{P}_1')\,(\mathrm{P}_2-\mathrm{P}_2')\ \ldots\ldots = m^s.$$

We thus obtain the equation

$$\Sigma\pm\sigma_s(1,\,1)\ \ \sigma_s(2,\,2)\ \ \sigma_s(3,\,3)\ \ldots\ldots\ \sigma_s(m,\,m) = (1\,.\,2\,.\,3\ \ldots\ldots\ m)^s,$$

in which s may be any quantity whatever: the cases in which $s=0$, $s=-1$, $s=+1$, are especially remarkable.

(5.) Returning to the equation

$$\begin{aligned}\Delta_m &= \Sigma \pm (1, 1)(2, 2) \ldots\ldots (m, m) \\ &= \psi(1) \times \psi(2) \times \ldots\ldots \times \psi(m),\end{aligned}$$

we may observe that it is by no means necessary that the numbers 1, 2, 3 ...... m should be the natural series of numbers. We may, in fact, take any different numbers $\mu_1, \mu_2 \ldots\ldots \mu_m$ that we please, subject only to the condition that, if μ be any one of these numbers, every divisor of μ must also appear among them, a condition which implies that unity is always one of the numbers μ. Subject to this condition, we have always

$$\begin{aligned}&\Sigma \pm (\mu_1, \mu_1)(\mu_2, \mu_2) \ldots\ldots (\mu_m, \mu_m) \\ &= \psi(\mu_1) \times \psi(\mu_2) \ldots\ldots \psi(\mu_m),\end{aligned}$$

or, more generally (see 4 *suprà*),

$$\begin{aligned}&\Sigma \pm \phi(\mu_1, \mu_1)\, \phi(\mu_2, \mu_2) \ldots\ldots \phi(\mu_m, \mu_m) \\ &= \Phi(\mu_1, \mu_1)\, \Phi(\mu_2, \mu_2) \ldots\ldots \Phi(\mu_m, \mu_m).\end{aligned}$$

The most obvious cases are—(α) when we reject the multiples of given primes; *e.g.*, when the numbers μ are the uneven numbers in their natural order; (β) when we consider only numbers composed with given primes, *e.g.*, when the numbers are all the divisors of one of them μ; (γ) when we consider only *linear* numbers, *i.e.*, numbers not divisible by any square. In all these cases the results are immediately obtained by the methods which we have already used, and which it is unnecessary to exemplify further.

(6.) Lastly, the symbols μ need not represent integral numbers at all, but may be any quantities which admit of resolution into factors in a definite manner. If, for example, $a^{ij} = x^{i-1}$ or x^{j-1} according as $i<j$, or $j<i$, we have

$$\begin{aligned}&\Sigma \pm a_{11}, a_{22} \ldots\ldots a_{mm} \\ &= (x^{m-1} - x^{m-2})(x^{m-2} - x^{m-3}) \ldots\ldots (x-1) \\ &= (x-1)^{m-1} x^{\frac{1}{2}(m-1)(m-2)}.\end{aligned}$$

June 8th, 1876.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Dr. Lindemann and Mr. W. S. W. Vaux were present as visitors.

Mr. Kempe spoke on "A general Method of Describing Plane Curves of the *n*th degree by Linkwork." Prof. Cayley and Mr. Roberts also spoke on the subject. Mr. Roberts then gave an account of a "Further Note on the Motion of a Plane under certain Conditions." Mr. Walker communicated a short Note "On the reduction of the Equation (U = 0) of the Plane Nodal Cubic to its canonical form $ax^3+by^3+6mxyz$." It was shown that the equation to the three lines of reference is $TU+8SH = 0$, and that the equation to (xy) is $L^2-MN = 0$, where L, M, N are the factors of $TU-24SH$. The quotient of $TU+8SH$ by L^2-MN will therefore give the third line z, which is the real axis of inflexion, the other two being the nodal tangents.

Prof. Cayley described a surface, depending upon the sinusoid, which was being constructed for him at Cambridge. The Chairman made a few remarks in connection with M. Hermite's recent Note on a Theorem of Eisenstein's.

On a General Method of describing Plane Curves of the n^{th} degree by Linkwork. *By* A. B. KEMPE, B.A.

[*Read June 9th*, 1876.]

LEMMA I.—*The Reversor.*—Let $O\xi\beta\alpha$ (Fig. 1) be the linkage known as the contra-parallelogram, $O\xi$ being equal to $\beta\alpha$, and $O\alpha$ to $\xi\beta$.

Make $\alpha\gamma$ a third proportional to $O\xi$ and $O\alpha$, and add the links $O\delta$, equal to $\alpha\gamma$, and $\delta\gamma$, equal to $O\alpha$.

Then the figure $O\alpha\gamma\delta$ is a contra-parallelogram similar to $O\xi\beta\alpha$; and the angle $\xi O\alpha$ is equal to the angle $\delta O\alpha$.

Thus, if $O\xi$ be made to make any angle with $O\alpha$, $O\delta$ will make the same angle with $O\alpha$ on the other side of it.*

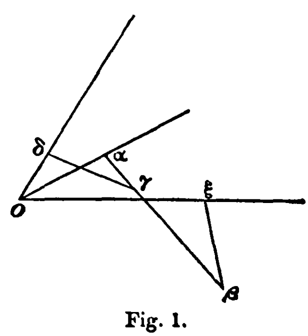

Fig. 1.

* This linkage, and the one next described, were first given by me in the "Messenger of Mathematics," Vol. IV., pp. 122, 123, in a paper "On some new Linkages," §§ 4 and 8.