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VIII. On a new Harmonic Analyser

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§ 19. We can make a comparison between the work done by a ring magnet when it is divided at one point with the work done when the ring is divided at two points. The reluctance data show that though the mean air-gap reluctance may be larger than that of the iron, it is not very greatly so in any practical case, and we can therefore obtain no information by supposing that one is much greater or less than the other; but must proceed by actual trial from the curves to find out which is the most efficient arrangement.

§ 20. In the case of a mechanism represented by a ring divided at one point only, we must remember that the closure of the induction curves involves a "sliding" magnetic contact, and if friction on the bearings is to be avoided this practically ties us down to iron of symmetrical form.

§ 21. Incidentally I had occasion to observe the change of reluctance caused by cutting a bar, and then grinding and polishing the ends. This was not done quite so well as in our most successful attempts. The reluctance corresponded to a separation of the bars by about 20 wave-lengths of sodium light, but I am certain that the bars could not have been half so far apart as this, so the surface reluctance is still unaccounted for.

Sydney, 13th July, 1893.

VIII. *On a new Harmonic Analyser.*

By Prof. O. HENRICI, *F.R.S.**

§ 1. **A**CCORDING to the theory of Fourier's Series any function y of x can, under certain restrictions, be expanded in a series progressing according to cosines and sines of multiples of x .

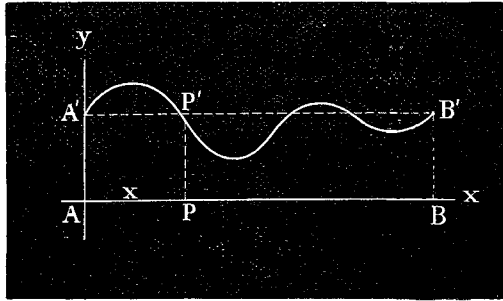
This function may be represented graphically by a curve, x and y being taken as rectangular co-ordinates, or it may be defined by aid of such a curve.

Anyhow, we shall suppose this curve given, and also that it extends from $x=0$ to $x=c$ (fig. 1). For this interval the curve may be drawn perfectly arbitrary as long as it gives for every x one single finite value of y . This implies that if a point moves along the curve the corresponding value of x always increases. The curve may, however, be discontinuous, so that for a particular value of x the ordinate changes suddenly from a value y_1 to a value y_2 , as from C to C' in

* Communicated by the Physical Society: read March 9, 1894.

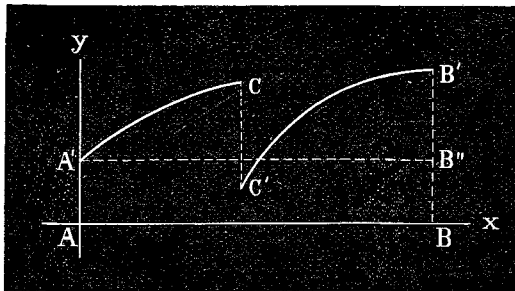
fig. 2. There may be any finite number of such discontinuities. For our purposes it is necessary to make the curve continuous by joining the two points C' and C by a straight

Fig. 1.



line. If the curve represents a periodic phenomenon with period c , then the ordinate for $x=c$ will, as a rule, equal the initial ordinate for $x=0$ (as in fig. 1). The curve when repeated along the axis of x will therefore be continuous.

Fig. 2.



Otherwise there will be a discontinuity as at B in fig. 2. In this case also the curve has to be continued from its end point B' along the last ordinate to a point B'' which has the same ordinate as the initial point A' , so that the line $A'B''$ is parallel to the axis of x .

We can now express the equation to the curve in the form

of a Fourier Series,

$$y = \frac{1}{2} A_0 + A_1 \cos \theta + A_2 \cos 2\theta + \dots + A_n \cos n\theta + \dots \\ + B_1 \sin \theta + B_2 \sin 2\theta + \dots + B_n \sin n\theta + \dots$$

where $\theta = \frac{2\pi x}{c}$.

The absolute term $\frac{1}{2}A_0$ equals the mean ordinate of the curve, and can therefore be determined by any planimeter. It is the object of the Harmonic Analyser to determine the other coefficients. Their well-known values are

$$A_n = \frac{1}{\pi} \int_0^{2\pi} y \cos n\theta \, d\theta; \quad B_n = \frac{1}{\pi} \int_0^{2\pi} y \sin n\theta \, d\theta.$$

The Analyser is therefore an integrator.

If the paper with the curve be wrapped round a cylinder, the ordinate y falling on the generating lines or edges, the axis of x along a circumference, then the curve will run back in itself and form one continuous line, provided the circumference of the cylinder equals the base c . That edge which passes through the initial point A' of the original curve may be called the zero-edge.

Suppose the cylinder to lie horizontal with the zero-edge at the top, then our angle θ will be the angle through which the cylinder has to be turned in order to bring that point P to the top which corresponds to any given x . Each edge contains one point on the curve, excepting in case of a discontinuity where a finite length of the edge belongs to the curve.

§ 2. The first instrument of this kind was constructed by Lord Kelvin (Proceedings Roy. Soc. vol. xxiv., 1876). Since then several others have been devised. With regard to these I may refer to my article "Ueber Instrumente zur harmonischen Analyse" in the Catalogue prepared by Prof. W. Dyck of Munich for the Mathematical Exhibition which was held last summer in Munich, and also to the descriptions in the Catalogue of the various instruments exhibited.

These instruments differ essentially either in the manner in which the trigonometrical factor is introduced, or in the arrangement by which the actual integration is performed. Lord Kelvin uses for the latter purpose his brother's disk-globe and cylinder integrator, whilst a simple harmonic motion introduces the trigonometrical factor. Sommerfeld and Wiechert* of Königsberg make the cylinder on which

* See above Catalogue, p. 274.

the curve is drawn rotate about an axis perpendicular to that of the cylinder, and thus avoid the simple harmonic motion, which is always a drawback, as it introduces a great deal of friction. Both instruments are also large and heavy, practically fixtures in the room where they are used.

§3. Clifford has given a beautiful graphical representation of Fourier's Series, which I knew more fully from personal communication than from the short paper published in vol. v. of the Proceedings of the Lond. Math. Soc.

His result may be stated thus:—"If the curve to be analysed be stretched out in the direction of the x to n times its base without altering the y , and then wrapped round a cylinder with circumference c so that it goes n times round, then the orthogonal projection of this curve on that meridian plane which passes through the zero-point of the curve will enclose an area which is proportional to the coefficient B_n . In the same way A_n is got by aid of a plane perpendicular to the first."

It was this theorem which led me to the construction of an Harmonic Analyser. It can easily be put in the following form. Suppose the cylinder placed with its axis horizontal and the tangent plane to its upper edge drawn. This edge cuts the curve in n points. Let P be one of them. If now the cylinder be turned, and if at the same time the tangent plane be moved in its own plane in a direction perpendicular to the edge of contact, the point P will trace a curve on it. This plane will be the same as Clifford's curve in case the motion of the tangent plane is simply harmonic, completing one period for each rotation of the cylinder. The curve will be completed after n rotations of the cylinder.

The same curve will be traced if the original, unstretched, curve is wrapped (once) round the cylinder, whilst the tangent plane completes n periods of its simple harmonic motions for one revolution of the cylinder.

We thus get in a fixed plane a curve whose area equals, in some unit, the coefficients A_n or B_n , and this area can be determined by an ordinary planimeter. The curve, of course, need not be drawn out, as long as the tracer of the planimeter is always at the point P it will describe the curve.

This can easily be realized. A flat board, whose upper surface forms a platform on which the planimeter can rest, is placed by the side of the cylinder so that its upper surface lies in the tangent plane. A straight-edge is fixed above the upper edge of the cylinder. The tracer of the planimeter is pressed against it and made to follow the point P on the curve. After a complete revolution of the cylinder,

Phil. Mag. S. 5. Vol. 38. No. 230. July 1894.

I

the planimeter will register a number proportional to the coefficient A_n or B_n .

I had an instrument of this kind made early in 1889, but it did not turn out quite as simple as its theory. It gives, of course, only one coefficient at a time, though it would not be difficult to construct it to give more terms if it were not for the mechanism required to produce the simple harmonic motion. This always introduces a certain amount of friction if it is to work accurately. I therefore tried to do away with this, and obtained my object in the manner now to be described.

§ 4. If the definite integrals which determine the coefficients A_n and B_n be integrated by parts, we get for the former

$$n\pi A_n = [y \sin n\theta]_0^{2\pi} - \int_0^{2\pi} \sin n\theta \, dy,$$

the limits relating to θ .

If the original curve is continuous, the integrated part vanishes. This is not the case if there is a discontinuity, at least not if θ is retained as the independent variable.

To prove that in this case also the integrated part can be neglected, let us consider the curve in fig. 2. Let θ' be the value of θ for which the discontinuity $C'C'$ occurs, and let y_1' be the ordinate of C , and y_2' that of C' .

The integral with regard to θ has to be broken up into two, the first going from 0 to θ' , the second from θ' to 2π . The integrated part, therefore, gives

$$y_1' \sin n\theta' - y_2' \sin n\theta',$$

and this, in general, does not vanish.

The remaining integral has to be taken for the two parts of the curve from A' to C and from C' to B' , if the curve is not made continuous. But if the curve is made continuous, we have also to take the integral for the intervals from C to C' , and from B' to B'' . For these $d\theta$ vanishes, but not dy . This gives in addition the integrals

$$-\int_C^{C'} \sin n\theta \, dy = -\sin n\theta' \int_C^{C'} dy = -\sin n\theta' (y_2' - y_1');$$

hence just the terms obtained before from the integrated part.

The second integral for the interval $B'B''$ vanishes because it is multiplied by $\sin 2n\pi$. In case of the coefficient B_n this is not the case, but then the integrated part also contains more terms which equal it. Hence :—

If the integration is performed with regard to y we get

$$n A_n = \frac{-1}{\pi} \int \sin n\theta \, dy, \quad n B_n = \frac{1}{\pi} \int \cos n\theta \, dy,$$

both taken over the whole continuous curve from A' to B'' . If the integration be continued from B'' to A' on the line parallel to the axis of x nothing is added to the integral, because here dy vanishes.

For the Analysers now to be described this extension of the integration should always be made in order to eliminate certain errors of the instrument.

These new integrals are of a very different form from the old ones, and require accordingly a different mechanism. As the tracer of the instrument follows the curve, each dy has to be multiplied by $\sin n\theta$ or $\cos n\theta$. In other words, we have to decompose the dy for each element of the curve into two components at right angles to each other, of which the one makes an angle θ with the axis of x , and then add all components of each kind to get A_n and B_n .

Originally I did this by aid of a pair of registering-wheels such as are used in Amsler's well-known planimeter, the axes of the two being at right angles. If such a wheel moves along a straight line of length p , making an angle $n\theta$ with its own axis, it will register not p but $p \cos n\theta$, whilst the second wheel at right angles to it gives $p \sin n\theta$.

A model of this instrument was made in 1889.

The curve is wrapped round a horizontal cylinder. Parallel to this a carriage runs on a rail carrying the tracer which moves along the upper edge of the cylinder. It also carries a vertical spindle which has the two registering-wheels attached to it. These roll on a horizontal platform. If, now, this spindle is made to turn through an angle $n\theta$ when the cylinder has turned through an angle θ , and if the tracer is made to follow the curve, then the two registering-wheels will give the coefficients A_n and B_n . For the details of the construction I must refer to Prof. Dyck's Catalogue, p. 213.

§ 5. The next improvement is due to Mr. A. Sharp, of the Teaching Staff in the Guilds' Central Technical College. Having used my model, he brought me a design in which the principles explained were realized in a different manner. Among the alterations introduced one struck me as being of importance. It consisted in an inversion of the motion, the curve being drawn on the drawing-board and the instrument made to move over it whilst the registering-wheels rolled on the paper.

It seemed to me that we had now all the elements needed

for a really good instrument, and only wanted a practised instrument-maker to realize it. I therefore called in 1892 on Coradi in Zürich, well known for his planimeters and integrators. He set to work at once and sent me in a short time a drawing of his construction, and it is due to his skill that the instrument has, at last, reached a high degree of perfection. One Analyser has been made for the Guilds' Central Technical College, which I shall describe. But I must mention at once that Herr Coradi has since greatly improved it, so much so that it is now one of the most perfect integrators made.

§ 6. Fig. 3 shows an instrument of Coradi's second design. This will help to explain the first.

There is first of all a solid frame whose base is a long rectangle. It rests with three wheels on the drawing-board. One of these, D, in the middle of the front, serves merely as a support. The other two, E, E, are fixed to the ends of a long axle which runs along the back of the frame. This may be called the "shaft." It is placed parallel to the axis of x . The instrument can, therefore, roll over the paper in the direction of the ordinates y .

If thus moved through a distance dy , the shaft will turn through an angle proportional to dy . The shaft carries any required number of short "cylinders." In the figure there is one marked C situated in the middle of the shaft.

Above each of these cylinders is a vertical "spindle" S, whose geometrical axis cuts that of the shaft. In the new instrument each spindle carries one or two disks, H₃, H₄, in fig. 3; but in the old construction one crown wheel with its teeth pointing upwards, by aid of which the spindle is turned. At the lower end the integrating apparatus proper is attached, which is quite different in the two designs. But before explaining this let me describe how the spindle is turned.

Along the front of the frame runs a carriage W, to which the tracer F is fixed. This can be moved through a distance equal to the base c to which the curve is drawn. To the carriage a silver wire is also attached, which in the new design is stretched along the front of the frame and then by aid of guide-pulleys l, l over one of the disks H on top of the spindle S (see fig. 3). By giving the disk H a suitable diameter the spindle can be made to turn n times round, whilst the tracer describes the whole base. In the old instrument the wire only drives an extra spindle in the middle of the frame, which by aid of wheelwork drives all the working spindles. If the tracer on following the curve has reached a

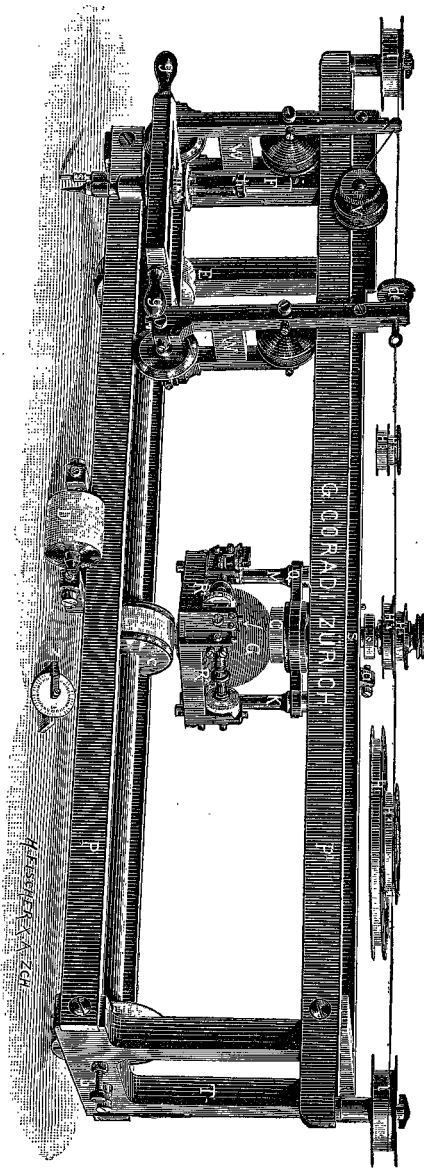


Fig. 3.

point P, then the spindle will have turned through an angle $n\theta$, where θ corresponds to the x of P.

If, now, the spindle had at its lower end two registering-wheels at right angles to each other rolling on the drawing-paper, we should have in principle my old model (§ 4) with Sharp's inversion. Instead of this Coradi gave each spindle one registering-wheel and made this roll on the cylinder C. This requires for each registering-wheel a separate spindle, hence two for each pair of coefficients A_n and B_n . It substitutes, however, the rolling on a smooth surface for that on the rough surface of the paper. The instrument made according to this design for the Guilds' Central Technical College has five such pairs, so that on going once over the curve the first five pairs A_n and B_n are obtained. The extra spindle which is driven by the silver wire contains, however, three extra disks, making four in all. If the wire is stretched over the top disk we get, as stated, the coefficients for $n=1, 2, 3, 4, 5$. The second pulley has half the diameter, so that the spindles turn twice as fast if the wire is stretched round it. Thus in going over the curve a second time we get the new coefficients for $n=6, 8, 10$. The remaining two disks give similarly the coefficients for $n=7$ and 9 respectively. Hence on going four times over the curve we get ten pairs of coefficients. In most cases the five pairs obtained at once will be amply sufficient.

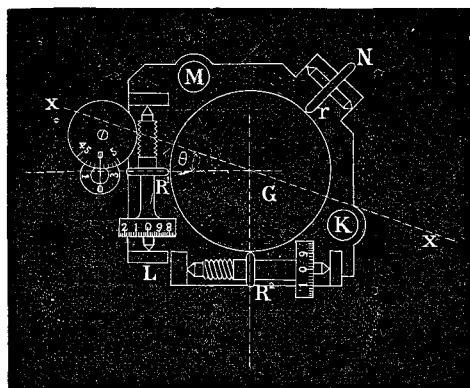
For the details of the construction I must again refer to Prof. Dyck's Catalogue (*Nachtrag*, p. 34) and only mention a few points. The axis of a registering-wheel lies in the diameter of a horizontal ring which is attached to the lower end of the spindle by aid of an elastic vertical steel plate. This presses the wheel against the cylinder, securing contact. On testing the instrument it was found that this plate was liable to slight torsion which affected the readings. It showed a number of other drawbacks of more or less importance. One is that the registering-wheel not only rolls but also slips. This slipping is absent in the Analyser of Lord Kelvin, who has dwelt strongly on the importance of avoiding it.

There was also a serious difficulty in taking the readings. The instrument registers up to 20 centim. If the zero-point has passed the index which gives the reading, 20 centim. have to be added or subtracted. Every one who has used a planimeter is accustomed to this, and knows how to take account of it, for he can either estimate the area sufficiently to see which correction is necessary, or he can go rapidly over the curve again, watching the zero-point. Neither method is possible with an Analyser which gives a large

number of readings at once. The new instrument is therefore constructed to record up to 200 centim.

§ 7. Last summer at the Munich Exhibition Herr Coradi submitted a new arrangement to me to obviate some of the imperfections of the instrument described, and this he has since carried out with an ingenuity which I cannot enough admire. He has practically got rid of all the imperfections of the old Analyser, and has now produced an instrument which, I fancy, leaves nothing to be desired. He himself says it is the best instrument of any kind he has yet made. The chief alteration is this, that he interposes between the registering-wheel at the lower end of the spindle and the cylinder a perfectly free glass sphere.

Fig. 4.



The "spindle" has now *firmly* attached to its lower end a square frame K L M N (comp. figs. 3 and 4) by aid of two solid rods K and M, instead of carrying the ring connected by aid of an elastic spring. This frame holds two registering-wheels R^1 and R^2 , whose axes K L and L M are at right angles. Between these lies the glass sphere G, resting with its lowest point on the cylinder belonging to the spindle. A third wheel r at N is by aid of a spring pressed against the sphere to secure contact between the latter and the registering-wheels. If, now, the tracer follows the curve this frame will turn with the spindle, the three wheels will carry the sphere with it, which will turn pivot-like on its lowest point. If, as in fig. 4, the plane of one wheel R^1 makes with the axis of x an angle $n\theta$, and if in this position the tracer, and with it the whole instrument, is moved through the distance dy , the "shaft" will turn proportionally to dy . This will set the sphere turning about its horizontal diameter xx parallel to the

shaft, and this motion will be communicated to each of the registering-wheels. It will be seen at once, if q denotes the radius of the sphere, the point of contact of the sphere and the wheel R^1 is at a distance $q \sin n\theta$ from the axis of the sphere, that therefore the turning communicated to this wheel will be proportional to $dy \sin n\theta$. Similarly the other wheel will turn proportionally to $dy \cos n\theta$. If the tracer moves through the whole curve, these two wheels will therefore register numbers proportional to A_n and B_n . The dimensions are so chosen that the readings give nA_n and nB_n in centimetres.

It will be seen that now one spindle does the work of two in the old instrument. There is, further, no slipping of any kind in the integrating apparatus.

Another improvement is that the wheelwork for turning the spindles is done away with. Each spindle is turned directly by the silver wire, and thus any slackness in the wheels is done away with.

It has also been possible to introduce an arrangement to set all spindles to zero after the wire has been tightened.

Lastly, the readings are taken with much greater ease as the registering apparatus is well exposed to the eye.

In order that the instrument may work accurately it is necessary that the point of contact of the sphere with its cylinder should lie in the geometrical axis of the spindle. But it is practically impossible to secure this. This point will therefore describe a small circle on the cylinder and this will turn the sphere about some horizontal diameter, and therefore also the registering-wheels. It is of importance to eliminate the error thus introduced. This is done by bringing the tracer back to the starting-point A on the curve by moving it from B to A (figs. 1, 2) parallel to the axis of x . The sphere will hereby repeat the motion which produced the error, but in the opposite sense, and therefore completely cancel it.

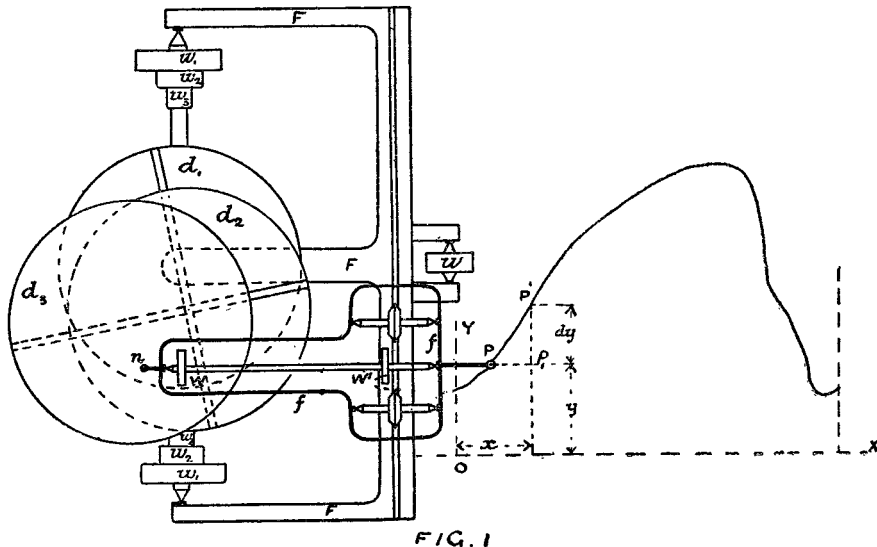
§ 8. The first instrument of this kind has been made for Prof. Klein at Göttingen. It contains one spindle, as in fig. 3. Going once over the curves it gives therefore one pair of coefficients. To get more, disks of different diameter have to be used to drive the spindle. Of these six are provided. Since then two further instruments have been finished; one with five spindles, which goes to Moscow, the other, with three spindles, for Prof. Weber in Zürich. The experience gained in the making of the Göttingen instrument has enabled Coradi to introduce a number of small improvements, with the result that the carriage runs in the Moscow instrument, where it has to drive five spindles, as easily as in the one for Göttingen with only one spindle.

He has also introduced a celluloid ring below the sphere, which on being raised presses the sphere against a similar ring above, thus preventing any damage to the integrating apparatus when the instrument is not being used.

Note.—At the request of Herr Coradi I add the statement that the idea of the new integrating apparatus, consisting of a sphere with two recording-wheels at right angles to each other, is not his own, but is due to Herr Max Küntzel, of Charlottenhof, near Königshütte in Silesia. Herr Küntzel invented the arrangement for an instrument designed to determine the coordinates of the vertices of a polygon, and submitted his design to Herr Coradi for the construction of such an instrument.

IX. *Harmonic Analyser, giving Direct Readings of the Amplitude and Epoch of the various constituent Simple Harmonic Terms.* By ARCHIBALD SHARP, B.Sc., Wh.Sc., A.M.I.C.E.*

LET the curve (fig. 1) be that represented by the equation $y=f(x)$, the scale of abscissæ being such that the period is 2π . Suppose a wheel W to roll on the paper



(fig. 2), and to be connected with a tracing-point P (fig. 1) in such a manner that as P moves uniformly in the direction

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