Note on the Weierstrass Elliptic Functions, and their Applications.

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The object of the present Note is to establish, from fundamental principles, the formulæ in the Weierstrass notation, corresponding to those given in Jacobi's notation by Mr. Glaisher in his "Note on the Functions Zu, Θu, $\Pi(u, a)$," read before this Society, Feb. 11th, 1886, and to show how naturally the formulæ arise from the definitions of the functions employed; afterwards, to apply the formulæ to some well known physical problems.

1. Starting with Euler's differential relation,

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$

where

$$X = ax^4 + bx^3 + cx^2 + dx + e,$$
$$Y = ay^4 + by^3 + cy^2 + dy + e,$$

and with the integral relation obtained by Euler,

$$a(x+y)^2 + b(x+y) + C = \left(\frac{\sqrt{X} - \sqrt{Y}}{x-y}\right)^2,$$

which is the key-note of the Theory of Elliptic Functions; then, if X and Y are already of Weierstrass's canonical form, namely,

$$X = 4x^3 - g_2 x - g_3, \quad Y = 4y^3 - g_2 y - g_3,$$

this integral relation of Euler becomes, since now $a = 0$, $b = 4$,

$$4(x+y+z) = \left(\frac{\sqrt{X} - \sqrt{Y}}{x-y}\right)^2,$$

writing $4z$ for C; this relation is symmetrical in x, y, and z; and may be written

$$x+y+z = \tfrac{1}{4}\left(\frac{\sqrt{Y} - \sqrt{Z}}{y-z}\right)^2$$
$$= \tfrac{1}{4}\left(\frac{\sqrt{Z} - \sqrt{X}}{z-x}\right)^2$$
$$= \tfrac{1}{4}\left(\frac{\sqrt{X} - \sqrt{Y}}{x-y}\right)^2;$$

or

$$\begin{vmatrix} 1, & x, & \sqrt{X} \\ 1, & y, & \sqrt{Y} \\ 1, & z, & \sqrt{Z} \end{vmatrix} = 0;$$

leading to the symmetrical differential relation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} = 0;$$

so that, as shown by Professor Cayley, in *Crelle*, Vol. 87, p. 74, the integral of

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

may be considered as the particular case of

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} = 0;$$

where z is treated as an arbitrary constant, being the value of y corresponding to a certain value of x, in this case the infinite value.

2. If the general elliptic differential element $dx/\sqrt{U_x}$, where U_x denotes the general biquadratic expression in x, is not already of Weierstrass's canonical form, it may immediately be reduced to it by putting

$$s = -\frac{H_x}{U_x},$$

where H_x is the Hessian of the quartic U_x; and then (Cayley, *Elliptic Functions*, p. 346)

$$\frac{dx}{\sqrt{U_x}} = \tfrac{1}{2}\frac{ds}{\sqrt{(4s^3 - g_2 s - g_3)}},$$

where g_2 and g_3 are the quadrinvariant and the cubinvariant of the quartic X; also

$$4s^3 - g_2 s - g_3 = \frac{G_x^2}{U_x^3},$$

where G_x denotes the sextic covariant.

Euler's integral relation in the general case may therefore be put in the form,

$$\frac{H_x}{U_x} + \frac{H_y}{U_y} + \tfrac{1}{4}\left\{\frac{\dfrac{G_x}{U_x^{\frac{3}{2}}} - \dfrac{G_y}{U_y^{\frac{3}{2}}}}{\dfrac{H_x}{U_x} - \dfrac{H_y}{U_y}}\right\}^2 = \text{constant};$$

or symmetrically,

$$\begin{vmatrix} 1, & \dfrac{H_x}{U_x}, & \dfrac{G_x}{U_x^{\frac{3}{2}}} \\ 1, & \dfrac{H_y}{U_y}, & \dfrac{G_y}{U_y^{\frac{3}{2}}} \\ 1, & \dfrac{H_z}{U_z}, & \dfrac{G_z}{U_z^{\frac{3}{2}}} \end{vmatrix} = 0.$$

3. Introducing now the notation of Weierstrass, where if

$$u = \int_x^\infty \frac{dx}{\sqrt{(4x^3 - g_2 x - g_3)}},$$

the canonical elliptic integral of the first kind, then x is an elliptic function of u which is denoted by $\wp u$; so that

$$x = \wp u,$$

and

$$\frac{dx}{du} = \wp' u$$

$$= -\sqrt{(4x^3 - g_2 x - g_3)} = -\sqrt{X}.$$

Then, if $y = \wp v$, $z = \wp w$, we have

$$u + v + w = 0,$$

and Euler's integral relation becomes

$$\wp u + \wp v + \wp w = \tfrac{1}{4}\left(\frac{\wp' v - \wp' w}{\wp v - \wp w}\right)^2$$

$$= \tfrac{1}{4}\left(\frac{\wp' w - \wp' u}{\wp w - \wp u}\right)^2$$

$$= \tfrac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2;$$

or,

$$\begin{vmatrix} 1, & \wp u, & \wp' u \\ 1, & \wp v, & \wp' v \\ 1, & \wp w, & \wp' w \end{vmatrix} = 0.$$

4. Since $\wp u$ is an even function of u, therefore

$$\wp w = \wp(u+v),$$

so that

$$\wp(u+v) + \wp u + \wp v = \tfrac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2;$$

and, changing the sign of v, since $\wp' v$ is an odd function of v, therefore

$$\wp(u-v)+\wp u+\wp v = \tfrac{1}{4}\left(\frac{\wp' u+\wp' v}{\wp u-\wp v}\right)^2.$$

Subtracting these equations, we obtain

$$\wp(u-v)-\wp(u+v) = \frac{\wp' u\,\wp' v}{(\wp u-\wp v)^2} \qquad \text{(A)},$$

the equation analogous to Mr. Glaisher's equation (A) (p. 153), which he takes as his starting point.

5. Integrating our equation (A) with respect to v,

$$\zeta(u+v)+\zeta(u-v)+C = \frac{\wp' u}{\wp u-\wp v} \qquad (\mathrm{A}_1),$$

where C is the constant of integration, independent of v, and the function ζu is defined by

$$\zeta u = -\int \wp u\, du, \qquad \zeta' u = -\wp u,$$

(Halphen, *Traité des Fonctions Elliptiques et de leurs Applications*, p. 135, Paris, 1886); so that ζu is analogous to Jacobi's function Zu.

To determine the constant C in (A_1), put $v=0$; then $\wp v = \infty$, and $C = -2\zeta u$, so that

$$\zeta(u+v)+\zeta(u-v)-2\zeta u = \frac{\wp' u}{\wp u-\wp v} \qquad (\beta),$$

analogous to Mr. Glaisher's equation (β).

6. Putting $v=u$ in (A_1), then C assumes the indeterminate form $\infty-\infty$, and must be evaluated; we shall find, eventually,

$$\zeta(u+v)+\zeta(u-v)-\zeta 2u = -\tfrac{1}{2}\,\frac{\wp'(u+v)-\wp'(u-v)}{\wp(u+v)-\wp(u-v)} \qquad (\alpha),$$

or

$$= \frac{-\wp' u}{\wp u-\wp v}+\tfrac{1}{2}\frac{\wp'' u}{\wp' u},$$

analogous to Mr. Glaisher's equation (α). For, since

$$\zeta(u+v)+\zeta(u-v)-2\zeta u = \frac{\wp' u}{\wp u-\wp v},$$

therefore, interchanging u and v,

$$\zeta(u+v)-\zeta(u-v)-2\zeta v = \frac{-\wp' v}{\wp u-\wp v};$$

and therefore, by addition,

$$\zeta(u+v)-\zeta u-\zeta v = \tfrac{1}{2}\frac{\wp' u-\wp' v}{\wp u-\wp v},$$

or

$$= \sqrt{\{\wp(u+v)+\wp u+\wp v\}};$$

and, changing u and v into $u+v$ and $u-v$,

$$\zeta 2u-\zeta(u+v)-\zeta(u-v) = \tfrac{1}{2}\frac{\wp'(u+v)-\wp'(u-v)}{\wp(u+v)-\wp(u-v)}$$

$$= \tfrac{1}{2}\frac{\wp'' u}{\wp' u}-\frac{\wp' u}{\wp u-\wp v}.$$

7. Next, introducing Weierstrass's function σu, defined by

$$\log \sigma u = \int_0 \zeta u\, du,$$

or

$$\sigma u = \exp \int_0 \zeta u\, du,$$

so that Weierstrass's function σu is analogous to Jacobi's function Θu, or rather Hu; then

$$\zeta u = \frac{\sigma' u}{\sigma u} = \frac{d}{du}\log \sigma u,$$

and

$$\wp u = -\zeta' u = -\frac{d^2}{du^2}\log \sigma u.$$

Integrating equation (β) with respect to u between the limits 0 and u, we find

$$\log\frac{\sigma(v+u)}{\sigma v}+\log\frac{\sigma(v-u)}{\sigma v}-2\log\sigma u = \log(\wp u-\wp v),$$

so that

$$\wp u-\wp v = \frac{\sigma(v+u)\,\sigma(v-u)}{\sigma^2 v\,\sigma^2 u},$$

or

$$= -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v} \quad\ldots\ldots\ldots\ldots\ldots\ldots(\beta_1),$$

the fundamental formula in the use of Weierstrass's elliptic functions, analogous to Mr. Glaisher's equation (β_1).

8. By integrating (β) with respect to v instead of with respect to u, we find

$$\int \zeta(u+v)\,dv+\int \zeta(u-v)\,dv-2v\zeta u = \int \frac{\wp' u\,dv}{\wp u-\wp v},$$

or
$$\int_0 \frac{\wp' u\, dv}{\wp u - \wp v} = \log \frac{\sigma(u+v)}{\sigma(u-v)} - 2v\zeta u \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta_2),$$

the elliptic integral of the third kind in the Weierstrass notation, corresponding to Jacobi's Π (v, u).

By integrating the equation in § 6,

$$\tfrac{1}{2} \frac{\wp' u - \wp' v}{\wp u - \wp v} = \zeta\,(u+v) - \zeta u - \zeta v,$$

with respect to u and v respectively, we obtain

$$\int_0 \tfrac{1}{2} \frac{\wp' u - \wp' v}{\wp u - \wp v}\, du = \log \frac{\sigma(u+v)}{\sigma u} - u\zeta v,$$

$$\int_0 \tfrac{1}{2} \frac{\wp' u - \wp' v}{\wp u - \wp v}\, dv = \log \frac{\sigma(u+v)}{\sigma v} - v\zeta u,$$

either of which may be taken as Weierstrass's canonical form of the elliptic integral of the third kind; and these two integrals illustrate the theorem analogous to that obtained by the interchange of amplitude u and parameter v in Jacobi's elliptic integral of the third kind Π (u, v).

Or, interchanging u and v in (β_2),

$$\int \frac{\wp' v\, du}{\wp u - \wp v} = \log \frac{\sigma(u-v)}{\sigma(u+v)} + 2u\zeta v;$$

so that, by addition,

$$\int \frac{\wp' u\, dv + \wp' v\, du}{\wp u - \wp v} = 2u\,\zeta v - 2v\,\zeta u,$$

corresponding to interchange of *amplitude* and *parameter*.

9. Supposing $X = 4x^3 - g_2 x - g_3$ resolved into three real linear factors $4(x-e_1)(x-e_2)(x-e_3)$, where $e_1 > e_2 > e_3$, and denoting by ω_1 and ω_3 the real and imaginary half-periods of the elliptic functions, then

$$\omega_1 = \int_{e_1}^{\infty} \frac{dx}{\sqrt{X}}, \quad \omega_1 + \omega_3 = \int_{e_2}^{\infty} \frac{dx}{\sqrt{X}}, \quad 2\omega_1 + \omega_3 = \int_{e_3}^{\infty} \frac{dx}{\sqrt{X}},$$

$$2\omega_1 + 2\omega_3 = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{X}};$$

so that
$$\omega_1 = \int_{e_1}^{\infty} \frac{dx}{\sqrt{X}} = \int_{e_3}^{e_2} \frac{dx}{\sqrt{X}},$$
$$\omega_3 = \int_{e_2}^{e_1} \frac{dx}{\sqrt{X}} = \int_{-\infty}^{e_3} \frac{dx}{\sqrt{X}};$$

and, consequently,
$$\wp\omega_1 = e_1, \quad \wp(\omega_1+\omega_3) = e_2, \quad \wp\omega_3 = e_3;$$
while
$$\wp'\omega_1 = \wp'(\omega_1+\omega_3) = \wp'\omega_3 = 0.$$

10. Returning to the fundamental formula
$$\wp u - \wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v},$$

we may suppose the right-hand side resolved into the factors
$$\frac{\sigma(u+v)}{\sigma u\,\sigma v}\exp(-u\zeta v) \quad\text{and}\quad \frac{\sigma(u-v)}{\sigma u\,\sigma v}\exp(u\zeta v);$$

so that, denoting $\dfrac{\sigma(u+v)}{\sigma u\,\sigma v}\exp(-u\zeta v)$ by ϕu or $\phi(u, v)$, then
$$\phi(u, -v) = \phi(-u, v) = -\frac{\sigma(u-v)}{\sigma u\,\sigma v}\exp(u\zeta v),$$
and
$$\wp u - \wp v = \phi(u, v)\,\phi(u, -v).$$

Also, replacing v by the particular values ω_1, $\omega_1+\omega_3$, and ω_3, we obtain
$$\phi u = \frac{\sigma_1 u}{\sigma u},\ \frac{\sigma_2 u}{\sigma u},\ \frac{\sigma_3 u}{\sigma u},$$
or
$$\phi u = \sqrt{(\wp u - e_1)},\ \sqrt{(\wp u - e_2)},\ \sqrt{(\wp u - e_3)};$$
and then ϕu is a doubly periodic function.

But, for any other value of v, the function $\phi(u, v)$ is multiplied by a constant factor when the variable is increased by a complete period $2\omega_1$ or $2\omega_3$ (Halphen, p. 227); this function was introduced by Hermite, and called by him a doubly periodic function of the second kind (*fonction doublement périodique de second espace*).

We see, from § 8, that its logarithm is an elliptic integral of the third kind; for
$$\log\phi(u, v) = \log\frac{\sigma(u+v)}{\sigma u\,\sigma v} - u\zeta v$$
$$= \int \tfrac{1}{2}\,\frac{\wp' u - \wp' v}{\wp u - \wp v}\,du.$$

11. This function ϕu is shown by Halphen (*Fonctions elliptiques*, p. 235) to satisfy the equation

$$\frac{d^2\phi}{du^2} = (2\wp u + \wp v)\,\phi,$$

Lamé's differential equation for $n = 1$, with Weierstrass's notation.

Lamé's general differential equation with Jacobi's elliptic function notation

$$\frac{d^2y}{dx^2} = \{n\,(n+1)\,k^2\,\mathrm{sn}^2 x + h\}\,y$$

is considered by Hermite in *Sur quelques applications des fonctions elliptiques*, Paris, 1885, *Premier fascicule.* This is a reprint of Hermite's papers which appeared in the *Comptes Rendus*, beginning in 1877.

By the use of Weierstrass's notation, the work of Hermite is considerably simplified. The equation of Lamé then becomes

$$\frac{d^2y}{dx^2} = \{n\,(n+1)\,\wp x + h\}\,y,$$

and, in accordance with Hermite's results, the solution of this equation is

$$y = CF(x) + C'F(-x),$$

where C and C' are arbitrary constants, and

$$F(x) = \left(\frac{d}{dx}\right)^{n-1}\Phi x - A_1\left(\frac{d}{dx}\right)^{n-2}\Phi x + A_2\left(\frac{d}{dx}\right)^{n-3}\Phi x \ldots,$$

where $$\Phi x = \frac{\sigma(x+\omega)}{\sigma x\,\sigma\omega}\exp(\lambda - \zeta\omega)\,x = e^{\lambda x}\phi\,(x, \omega)\,;$$

also A_1, A_2, ..., λ, and $\wp\omega$ are certain definite constants, rational functions of h and the invariants g_2 and g_3.

12. A great part of Hermite's work is devoted to the determination of these constants for the successive integral values of n, the complexity of the work increasing very rapidly.

For $n = 1$ we have just seen that the solution of the corresponding equation of Lamé,

$$\frac{d^2y}{dx^2} = (2\wp x + h)\,y,$$

is obtained, putting $h = \wp v$, in the form

$$y = C\phi(x, v) + C'\phi(-x, v),$$

where $$\phi(x, v) = \frac{\sigma(x+v)}{\sigma x\, \sigma v} \exp(-x\zeta v), \text{ as above.}$$

For $n = 2$, the equation becomes

$$\frac{d^2y}{dx^2} = (6\wp x + h)\, y\,;$$

and, as we have seen in these *Proceedings*, June 10th, 1886, p. 374, the solution is then, putting $h = -3\wp(a-b)$,

$$y = C\frac{\sigma(x+a)\,\sigma(x+b)}{\sigma a\,\sigma b\,\sigma^2 x} \exp(-\zeta a - \zeta b)\, x$$

$$+ C'\frac{\sigma(x-a)\,\sigma(x-b)}{\sigma a\,\sigma b\,\sigma^2 x} \exp(\ \zeta a + \zeta b)\, x,$$

subject to the condition that

$$\wp' a = -\wp' b, \text{ and therefore } = \wp'(a-b)\,;$$

and this, by Halphen's equation (36) (*Fonctions elliptiques*, p. 230), can be thrown into the form

$$y = C\frac{d}{dx}\left\{\frac{\sigma(x+\omega)}{\sigma x\,\sigma\omega} \exp(-\zeta a - \zeta b)\, x\right\}$$

$$+ C'\frac{d}{dx}\left\{\frac{\sigma(x-\omega)}{\sigma x\,\sigma\omega} \exp(\ \zeta a - \zeta b)\, x\right\},$$

where $$\omega = a + b,$$

agreeing thus with Hermite's form of the result, if

$$\lambda = \zeta\omega - \zeta a - \zeta b = \tfrac{1}{2}\frac{\wp' a - \wp' b}{\wp a - \wp b} = \frac{\wp' a}{\wp a - \wp b}\,;$$

so that $$\zeta(a-b) = \ \zeta a - \zeta b,$$

also $$\wp(a-b) = -\wp a - \wp b.$$

13. It is interesting to verify that the above value of y is a solution of the differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = 6\wp x - 3\wp(a-b).$$

Taking the particular solution

$$y = \frac{\sigma(x+a)\,\sigma(x+b)}{\sigma a\,\sigma b\,\sigma^2 x}\exp(-\zeta a-\zeta b)\,x,$$

then
$$\frac{1}{y}\frac{dy}{dx} = \zeta(x+a)-\zeta x-\zeta a+\zeta(x+b)-\zeta x-\zeta b$$

$$= \tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a}+\tfrac{1}{2}\frac{\wp' x-\wp' b}{\wp x-\wp b};$$

and, differentiating again,

$$\frac{1}{y}\frac{d^2y}{dx^2} = \left(\frac{1}{y}\frac{dy}{dx}\right)^2+2\wp x-\wp(x+a)-\wp(x+b)$$

$$= \left(\tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a}+\tfrac{1}{2}\frac{\wp' x-\wp' b}{\wp x-\wp b}\right)^2+2\wp x-\wp(x+a)-\wp(x+b)$$

$$= 4\wp x+\wp a+\wp b+\tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a}\,\frac{\wp' x-\wp' b}{\wp x-\wp b}.$$

But, if
$$\wp' a = -\wp' b = \wp'(a-b),$$

$$\tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a}\,\frac{\wp' x-\wp' b}{\wp x-\wp b} = \tfrac{1}{2}\,\frac{\wp'^2 x-\wp'^2 a}{(\wp x-\wp a)(\wp x-\wp b)}$$

$$= 2(\wp x+\wp a+\wp b),$$

in consequence of the relations

$$\wp'^2 a = \wp'^2 b,$$

or
$$4\wp^3 a-g_2\wp a-g_3 = 4\wp^3 b-g_2\wp b-g_3;$$

or
$$4(\wp^2 a+\wp a\,\wp b+\wp^2 b) = g_2,$$

so that
$$\wp'^2 x-\wp'^2 a = (\wp x+\wp a)\left\{4(\wp^2 x+\wp a\,\wp x+\wp^2 a)-g_2\right\}$$

$$= 4(\wp x-\wp a)(\wp^2 x+\wp a\,\wp x-\wp a\,\wp b-\wp^2 b)$$

$$= 4(\wp x-\wp a)(\wp x-\wp b)(\wp x+\wp a+\wp b).$$

Therefore
$$\frac{1}{y}\frac{d^2y}{dx^2} = 6\wp x+3\wp a+3\wp b$$

$$= 6\wp x-3\wp(a-b),$$

since
$$\wp(a-b)+\wp a+\wp b = 0,$$

in consequence of the relation

$$\wp' a = -\wp' b = \wp'(a-b).$$

When $\wp'(a-b)=0$, the solution of the differential equation is an algebraical function of $\wp x$, and we obtain the particular solutions considered by Lamé.

14. If, however, we turn to p. 106 of Hermite's work, we must replace h by $6\wp a$, in order to obtain an analogous solution; and then Lamé's differential equation, in the form

$$\frac{1}{y}\frac{d^2y}{dx^2}=6\wp x+6\wp a,$$

has the solution

$$\begin{aligned} y= &\;\; C\frac{d}{dx}\left\{\frac{\sigma(x+\omega)}{\sigma x\,\sigma\omega}\exp\left(\;\;\lambda-\zeta\omega\right)x\right\} \\ &+C'\frac{d}{dx}\left\{\frac{\sigma(x-\omega)}{\sigma x\,\sigma\omega}\exp\left(-\lambda+\zeta\omega\right)x\right\} \\ = &\;\; C\frac{d}{dx}\left\{\phi(x,\omega)\,e^{\lambda x}\right\}+C'\frac{d}{dx}\left\{\phi(-x,\omega)\,e^{-\lambda x}\right\}; \end{aligned}$$

subject to the conditions

$$\lambda^2=\tfrac{1}{4}\frac{\wp'^2\omega}{(\wp\omega-\wp a)^2}=\wp\omega+2\wp a,$$

so that $$\lambda=-\tfrac{1}{2}\frac{\wp'\omega}{\wp\omega-\wp a}=-\tfrac{1}{2}\zeta(\omega+a)-\tfrac{1}{2}\zeta(\omega-a)+\zeta\omega,$$

or $$\lambda-\zeta\omega=-\tfrac{1}{2}\zeta(\omega+a)-\tfrac{1}{2}\zeta(\omega-a);$$

also $$\wp\omega=\frac{4\wp^3a+\frac{1}{2}g_3}{\wp''a},$$

or $$\wp\omega-\wp a=-\tfrac{1}{2}\frac{\wp'^2a}{\wp''a},$$

giving $\wp\omega$, and consequently λ, in terms of $\wp a$ or h.

15. For $n=3$, the identification with Hermite's results (*Sur quelques applications, &c.*, pages 120 to 129) is obtained by writing Lamé's corresponding equation

$$\frac{1}{y}\frac{d^2y}{dx^2}=12\wp x+h,$$

and then $$h=-5l=15\wp a,$$

or $$l=-3\wp a,$$

so that the differential equation is

$$\frac{1}{y}\frac{d^2y}{dx^2} = 12\wp x + 15\wp a.$$

This suggests, by analogy, that the generalised form of Lamé's should be written, following Brioschi, in the form

$$\frac{1}{y}\frac{d^2y}{dx^2} = n(n+1)\wp x + n(2n-1)\wp a,$$

agreeing with the above forms for $n = 1, 2, 3$.

Then (Hermite, p. 120),

$$s_0 = -e_3, \quad s_1 = \frac{g_2}{20}, \quad s_2 = \frac{g_3}{28}, \quad s_3 = \frac{g_2^2}{600}, \ \dots;$$

and (Hermite, p. 124)

$$\Omega = \wp\omega, \quad \Omega_1 = \tfrac{1}{2}\wp'\omega, \quad \Omega_2 = \wp^2\omega - \tfrac{1}{8}g_2, \quad \Omega_3 = \tfrac{1}{2}\wp\omega\,\wp'\omega, \ \dots,$$

the same as the coefficients given by Halphen (p. 231) in the expansion in ascending powers of u of $\phi(u, \omega)$.

Then, when $n = 3$ (Hermite, p. 126), the identification of results is made by means of the equations

$$\lambda^3 - 3\wp\omega - \frac{\wp'\omega}{\lambda} = -3l = 9\wp a = \tfrac{3}{5}h,$$

$$h_1 = -\tfrac{1}{5}h, \quad a = \tfrac{3}{4}g_2, \quad b = \tfrac{27}{4}g_3, \quad 4a^3 - b^2 = \tfrac{27}{16}\Delta,$$

$$D = l^2 - a = \tfrac{3}{2}\wp''a, \quad S = -108\wp a, \ \dots,$$

$$\wp\omega = -\frac{\psi(l)}{36b(b^2-a)^2}, \quad \wp\omega - e_3 = -\frac{PA^2}{SD^2}, \quad \wp\omega - e_2 = \frac{QB^2}{SD^2},$$

$$\wp\omega - e_1 = \frac{RC^2}{SD^2}, \quad \wp'\omega = -\frac{2ABC\lambda}{SD^2}, \ \dots.$$

The solution of the differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = 12\wp x + h$$

is, therefore,

$$y = CF(x) + C'F(-x),$$

where

$$F(x) = \frac{d^2}{dx^2}\Phi x - \tfrac{1}{5}h\Phi x,$$

and $$\Phi x = \frac{\sigma(x+\omega)}{\sigma x\, \sigma\omega} \exp(\lambda - \zeta\omega)\, x$$

$$= \phi(x, \omega) \exp \lambda x,$$

where $$h = \tfrac{5}{3}\left(\lambda^2 - 3\wp\omega - \frac{\wp'\omega}{\lambda}\right), \;\ldots\,.$$

The direct verification that Fx is a particular solution of this differential equation is an interesting piece of analysis, but it is omitted here, as the work is rather long.

16. For higher integral values of n, the determination of $\wp\omega$ and λ in the solution of Lamé's differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = n(n+1)\wp x + h,$$

in the form $$y = CF(x) + C'F(-x),$$

and $$F(x) = \left(\frac{d}{dx}\right)^{n-1}\Phi x - A_1\left(\frac{d}{dx}\right)^{n-3}\Phi x + \ldots,$$

where $$\Phi x = \frac{\sigma(x+\omega)}{\sigma x\, \sigma\omega} \exp(\lambda - \zeta\omega)\, x,$$

increases rapidly in complication, and the general solution has not yet been obtained by Hermite; although he attempts the solution from the consideration of the product $F(x)\,F(-x)$ of two particular solutions, this product being in all cases a doubly periodic function.

17. The physical origin of Lamé's differential equation is explained in Maxwell's *Electricity and Magnetism*, Todhunter's *Functions of Laplace, Lamé, and Bessel*, Ferrers's *Spherical Harmonics*, and Heine's *Kugelfunctionen*.

Transforming Poisson's equation to ellipsoidal coordinates (Maxwell, I., p. 181), and employing Weierstrass's notation as explained in these *Proceedings*, June 10, 1886 (Vol. XVII., p. 378), the equation becomes

$$(\mu - \nu)\frac{d^2\phi}{du^2} + (\nu - \lambda)\frac{d^2\phi}{dv^2} + (\lambda - \mu)\frac{d^2\phi}{dw^2} = 0,$$

(*cf.* Klein, *Ueber Lamé'sche Functionen*, *Math. Ann.*, XVIII.).

Here λ, μ, ν are the three roots of the cubic equation

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1,$$

defining the three confocal quadric surfaces through a point; and then, as has been shown, we should put

$$a^2+\lambda = \wp u - e_1, \quad b^2+\lambda = \wp u - e_2, \quad c^2+\lambda = \wp u - e_3,$$

$$a^2+\mu = \wp v - e_1, \quad b^2+\mu = \wp v - e_2, \quad c^2+\mu = \wp v - e_3,$$

$$a^2+\nu = \wp w - e_1, \quad b^2+\nu = \wp w - e_2, \quad c^2+\nu = \wp w - e_3;$$

and, in going round three sides of the boundary of the *period rectangle*,

$u = r\omega_1$, for the confocal ellipsoids,

$v = \omega_1 + s\omega_3$, for the hyperboloids of one sheet,

$w = t\omega_1 + \omega_3$, for the hyperboloids of two sheets;

r, s, t denoting real proper fractions, the fourth side of the period rectangle giving imaginary surfaces; so that at two corners of the period rectangle $r = 1$ or $s = 0$ gives the focal ellipse, $s = 1$ or $t = 1$ the focal hyperbola.

Then Poisson's equation becomes of the symmetrical form

$$(\wp v - \wp w)\frac{d^2\phi}{du^2} + (\wp w - \wp u)\frac{d^2\phi}{dv^2} + (\wp u - \wp v)\frac{d^2\phi}{dw^2} = 0.$$

18. Lamé supposes that ϕ is the product of three functions U, V, W, such that U is a function of u only, V of v only, and W of w only; and then the equation becomes

$$(\wp v - \wp w)\frac{d^2U}{U\,du^2} + (\wp w - \wp u)\frac{d^2V}{V\,dv^2} + (\wp u - \wp v)\frac{d^2W}{W\,dw^2} = 0,$$

equivalent to

$$\frac{d^2U}{U\,du^2} = g\wp u + h,$$

$$\frac{d^2V}{V\,dv^2} = g\wp v + h,$$

$$\frac{d^2W}{W\,dw^2} = g\wp w + h,$$

where g and h are arbitrary constants; and g is replaced by $n(n+1)$, where n is an integer, by analogy with Laplace's equation, in order that the integral should be a *uniform* function; and we thus obtain three of Lamé's differential equations of the same form.

19. The application of Lamé's equation for $n = 2$ to the dynamical problem of the motion of the spherical pendulum is given by Hermite (p. 109), and adapted to Weierstrass's notation in these *Proceedings*, June 10, 1886 (Vol. XVII., p. 374).

The equation of the projection of the motion of the bob of the pendulum on a horizontal plane is there given in the form

$$x+iy = 2il\,\frac{\sigma(u+a)\,\sigma(u+b)}{\sigma a\,\sigma b\,\sigma^2 u}\exp\left(-\zeta a-\zeta b\right)u,$$

where, $2r$ denoting a complete period of the motion, we must put

$$u = t\omega_1/r+\omega_3;$$

also
$$a = \omega_1+r\omega_3,\quad b = s\omega_3,$$

r and s denoting real proper fractions.

In the spherical pendulum the additional condition $\wp' a = -\wp' b$ is required, but in the representation of the general motion of a point on the axis of a top, projected on a horizontal plane, this restriction is not required, while the form of the equation remains unaltered; but it is no longer the solution of Lamé's equation, unless referred to moving axes rotating with a certain constant angular velocity about the vertical.

20. It is unfortunate, at first sight, that the parameters like a and b required in the solution of these and similar dynamical problems are always imaginary; but this inconvenience disappears when we employ the above form for $x+iy$ as a function of t.

As dynamical illustrations of Lamé's equation for $n=1$, Hermite considers (i.) the problem of the motion of a body under no forces, including the equation of Poinsot's herpolhode; (ii.) the equation of the tortuous Elastica in equilibrium under balancing forces and couples at its ends.

21. Let us consider, first, the equations of the Elastica (Hermite, *Sur quelques applications, &c.*, p. 93). These equations may be written

$$y'z''-y''z' = \alpha x'+\beta y,$$
$$z'x''-z''x' = \alpha y'-\beta x,$$
$$x'y''-x''y' = \alpha z'+\gamma,$$

where the accents denote differentiation with respect to the arc s, and α, β, γ denote constants depending on the flexibility of the wire, and the impressed forces; also the axis of z is taken along the central axis of the applied wrench at any point of the curve.

These equations are due in this form to Binet and Wantzel (*Comptes Rendus*, 1844), and may be established as follows: denoting by B the flexural rigidity of the wire, by Z and N the impressed force and couple, and by G the tangential torsional couple of the wire, then the equations of equilibrium are, taking component moments at any point xyz of the wire,

$$B(y'z''-y''z') = Gx'+Zy \quad \text{..........} (1),$$

$$B(z'x''-z''x') = Gy'-Zx \quad \text{..........} (2),$$

$$B(x'y''-x''y') = Gz'+N \quad \text{..........} (3),$$

equations of the above form.

Differentiating with respect to s, multiplying respectively by x', y', z', and adding, gives $G'=0$; so that G is constant.

Multiplying (1) by x', (2) by y', and (3) by z', and adding, gives

$$G-Z(xy'-x'y) = 0,$$

so that $$xy'-x'y = r^2\frac{d\theta}{ds} = h, \text{ a constant};$$

and therefore $$xy''-x''y = 0.$$

Again, multiplying (1) by x, and (2) by y, and adding, gives

$$Bz''(xy'-x'y)-Bz'(xy''-x''y) = G(xx'+yy'),$$

or $$Bz'' = Z(xx'+yy'),$$

so that $$Bz' = \tfrac{1}{2}Z(x^2+y^2)+H.$$

Then $$B^2z'^2 = Z^2\left\{(x^2+y^2)(x'^2+y'^2)-(xy'-x'y)^2\right\}$$
$$= 2Z(Bz'-H)(1-z'^2)-Z^2h^2,$$

so that z' is an elliptic function of the arc s.

We may, if we like, modify Hermite's formulæ to the Weierstrass notation, and obtain from these equations

$$x+iy = C\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp(\lambda-\zeta\omega)u,$$

$$z = 2\zeta u+\gamma u,$$

where $$u = s\omega_1/c+\omega_3,$$

$2c$ denoting the length of a complete wave of the elastica; or we may obtain these equations immediately from the previous solution for the top (§ 19), by means of Kirchhoff's *Kinetic Analogue*, which asserts that the tangent of the elastica, if properly orientated, can be made to keep always parallel to the axis of the top, provided that the point of contact on the elastica moves with a certain definite constant velocity, and provided that certain initial conditions are satisfied.

Then, in the Elastica, in consequence of the Kinetic Analogue,

$$\frac{d}{du}(x+iy) = C\frac{d}{du}\frac{\sigma(u+a+b)}{\sigma u\,\sigma(a+b)}\exp(-\zeta a-\zeta b)\,u,$$

$$\frac{dz}{du} = -2\wp u+\gamma,$$

(*Proc. Lond. Math. Soc.*, June 10th, 1886); and, integrating,

$$x+iy = C\frac{\sigma(u+a+b)}{\sigma u\,\sigma(a+b)}\exp(-\zeta a-\zeta b)\,u,$$

$$z = 2\zeta u+\gamma u,$$

where du/ds is constant, and therefore

$$u = s\omega_1/c+\omega_3;$$

the imaginary constant ω_3 being added in order that dz/ds should oscillate between finite limits.

22. In the motion of a rigid body about a fixed point under no forces, the solution of Euler's equations of motion,

$$A\frac{dp}{dt}-(B-C)\,qr = 0,\quad B\frac{dq}{dt}-(C-A)\,rp = 0,$$

$$C\frac{dr}{dt}-(A-B)\,pq = 0,$$

has already been given in these *Proceedings*, June 10th, 1886 (Vol. XVII., p. 366), in the form

$$Ap^2 = -m^2(B-C)(\wp u-e_1),$$

$$Bq^2 = -m^2(C-A)(\wp u-e_2),$$

$$Cr^2 = -m^2(A-B)(\wp u-e_3),$$

where the factor m^2 is now introduced for homogeneity; and then

$$\frac{du^2}{dt^2} = -m^2\frac{(B-C)(C-A)(A-B)}{ABC} = \mu^2,$$

suppose; so that

$$u = \mu t+\omega_3$$

$$= t\omega_1/\tau+\omega_3,$$

2τ denoting the complete period of the motion, and the imaginary constant ω_3 being added in order to make $\wp u$ oscillate between e_2 and e_3, and the polhode consequently enclose the principal axis A.

Also, e_1, e_2, e_3, g_2, and g_3 are determined, as before, from the relations

$$Ap^2+Bq^2+Cr^2 = T, \quad A^2p^2+B^2q^2+C^2r^2 = G^2,$$

where $\frac{1}{2}T$ and G denote the constant kinetic energy and resultant angular momentum of the system.

23. For the herpolhode, take the equation of § 532 (Routh's *Rigid Dynamics*, 3rd edition),

$$\frac{d\phi}{dt} = \frac{T}{G} + \frac{(AT-G^2)(BT-G^2)(CT-G^2)}{ABCGT^2}\cot^2\zeta,$$

and we shall find (§ 25) that this can easily be thrown into the form

$$\frac{di\phi}{du} = \lambda - \tfrac{1}{2}\frac{\wp'\omega}{\wp u - \wp\omega}$$

$$= \lambda - \zeta\omega + \tfrac{1}{2}\zeta(u+\omega) - \tfrac{1}{2}\zeta(u-\omega),$$

so that $$i\phi = (\lambda-\zeta\omega)\,u + \tfrac{1}{2}\log\frac{\sigma(u+\omega)}{\sigma(u-\omega)},$$

or $$e^{i\phi} = \sqrt{\left\{\frac{\sigma(u+\omega)}{\sigma(u-\omega)}\right\}}\exp(\lambda-\zeta\omega)\,u.$$

Then $$\lambda^2 = -\frac{T^2}{G^2}\frac{dt^2}{du^2} = \frac{ABCT^2}{m^2G^2(B-C)(C-A)(A-B)},$$

and ω is the (imaginary) value of u which makes x^2+y^2 in the herpolhode vanish. So that

$$x^2+y^2 = k^2(\wp\omega - \wp u) = k^2\frac{\sigma(u+\omega)\,\sigma(u-\omega)}{\sigma^2 u\,\sigma^2\omega},$$

and therefore $$x+iy = k\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp(\ \lambda-\zeta\omega)\,u,$$

or $$x-iy = k\frac{\sigma(u-\omega)}{\sigma u\,\sigma\omega}\exp(-\lambda+\zeta\omega)\,u,$$

where $$u = t\omega_1/\tau + \omega_3,$$

is the equation of the herpolhode.

24. Referred to axes rotating with constant angular velocity T/G, the equation of the herpolhode will be

$$x+iy = k\phi(u,\omega),$$

or $$x-iy = k\phi(-u,\omega),$$

so that now x and y satisfy Lamé's equation for $n = 1$,

$$\frac{1}{x}\frac{d^2x}{du^2} = \frac{1}{y}\frac{d^2y}{du^2} = 2\wp u + \wp\omega,$$

and this relative herpolhode will be described therefore exactly as by a particle properly projected under a central attraction proportional to

$$r(2\wp u + \wp\omega) = 3r\wp\omega - 2r^3/k^2$$

(compare an article by Pinczon in the *Comptes Rendus*, 12th April, 1887, on the "Herpolhode").

25. In order to prove the equation

$$\frac{di\phi}{du} = \lambda - \tfrac{1}{2}\frac{\wp'\omega}{\wp u - \wp\omega},$$

we notice that we must put

$$\lambda = -i\frac{T}{G}\frac{dt}{du} = -i\frac{T}{Gm}\sqrt{\left\{\frac{ABC}{-(B-C)(C-A)(A-B)}\right\}};$$

also, in the herpolhode (Routh, p. 409),

$$\begin{aligned} x^2+y^2 = r^2-p^2 &= \frac{M\epsilon^4}{T}\left(\omega^2 - \frac{T^2}{G^2}\right) \\ &= -\frac{M\epsilon^4}{T}\left\{m^2\frac{B-C}{A}(\wp u - e_1) + m^2\frac{C-A}{B}(\wp u - e_2)\right. \\ &\qquad\qquad \left. + m^2\frac{A-B}{C}(\wp u - e_3) + \frac{T^2}{G^2}\right\} \\ &= \frac{M\epsilon^4\mu^2}{T}(\wp\omega - \wp u), \end{aligned}$$

provided that

$$\wp\omega = \frac{\dfrac{B-C}{A}e_1 + \dfrac{C-A}{B}e_2 + \dfrac{A-B}{C}e_3 - \dfrac{T^2}{m^2G^2}}{\dfrac{B-C}{A} + \dfrac{C-A}{B} + \dfrac{A-B}{C}},$$

so that

$$\begin{aligned} \wp\omega - e_1 &= -\frac{1}{\mu^2}\left(\frac{G}{B} - \frac{T}{G}\right)\left(\frac{G}{C} - \frac{T}{G}\right) \\ &= -\frac{1}{\mu^2}(\beta-\delta)(\gamma-\delta), \end{aligned}$$

with Hermite's notation, *Sur quelques applications*, &c., page 24.

Similarly, by symmetry,

$$\wp\omega - e_2 = -\frac{1}{\mu^2}\left(\frac{G}{C} - \frac{T}{G}\right)\left(\frac{G}{A} - \frac{T}{G}\right)$$

$$= -\frac{1}{\mu^2}(\gamma-\delta)(\alpha-\delta),$$

$$\wp\omega - e_3 = -\frac{1}{\mu^2}\left(\frac{G}{A} - \frac{T}{G}\right)\left(\frac{G}{B} - \frac{T}{G}\right)$$

$$= -\frac{1}{\mu^2}(\alpha-\delta)(\beta-\delta).$$

Our ω here is therefore practically the same as the ω employed by Hermite, p. 27; also

$$\wp'^2\omega = -4\frac{(AT-G^2)^2(BT-G^2)^2(CT-G^2)^2}{A^2B^2C^2G^6\mu^6},$$

so that, now (Routh, *Rigid Dynamics*, § 532),

$$\cot^2\zeta = \frac{\dfrac{M\epsilon^4T}{G^2}}{\dfrac{M\epsilon^4\mu^2}{T}(\wp\omega-\wp v)} = \frac{T^2}{G^2\mu^2}\frac{1}{\wp\omega-\wp u},$$

and $$\frac{di\phi}{du} = \lambda + i\frac{(AT-G^2)(BT-G^2)(CT-G^2)}{ABCG^3\mu^3}\frac{1}{\wp\omega-\wp u}$$

$$= \lambda - \tfrac{1}{2}\frac{\wp'\omega}{\wp u - \wp\omega},$$

on putting $$\wp'\omega = 2i\frac{(AT-G^2)(BT-G^2)(CT-G^2)}{ABCG^3\mu^3}.$$

26. Since the product $(AT-G^2)(BT-G^2)(CT-G^2)$ is negative, $\wp'\omega$ is a negative imaginary, and therefore $\omega = s\omega_3$, where s is a real proper fraction; and then, by comparison with Hermite's v, p. 27, $s = v/K'$; also v is the same as the quantity denoted by a in the article "Solution of Euler's equations of motion by means of Elliptic Functions" (*Quarterly Journal of Mathematics*, Vol. XIV., p. 267), and a slight consideration will easily show how the angles λ, μ, ν of that article can be expressed, by means of Weierstrass's notation, in as simple a manner as the angle ϕ of the herpolhode. The projection on the invariable plane of the spherical curve described by a point fixed on one of the principal axes will then be found to be given by an

equation of the form

$$x+iy = k\,\frac{\sigma(u+v)}{\sigma u\,\sigma v}\exp\,(\lambda-\zeta v)\,u$$

where $\omega+v=\omega_1$, $\omega_1+\omega_3$, or ω_3, according as the point is on the principal axis of A, B, or C.

27. De Sparre's theorem (*Comptes Rendus*, t. 99 and 101), which asserts that the herpolhode has *no* points of inflexion, can now be proved; for, writing the equation of the herpolhode

$$x+iy = \phi u = \frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp\,(\ \ \lambda-\zeta\omega)\,u,$$

$$x-iy = \phi_1 u = \frac{\sigma(u-\omega)}{\sigma u\,\sigma\omega}\exp\,(-\lambda+\zeta\omega)\,u,$$

then $$\frac{d^2\phi}{du^2} = x''+iy'',\quad \frac{d\phi_1}{du} = x'-iy;$$

and therefore $$\frac{d^2\phi}{du^2}\,\frac{d\phi_1}{du} = x'x''+y'y''+i\,(x'y''-x''y'),$$

and therefore the imaginary part is zero at a point of inflexion.

But $$\frac{d\phi}{du} = \{\zeta\,(u+\omega)-\zeta u-\zeta\omega+\lambda\}\,\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp\,(\lambda-\zeta\omega)\,u$$

$$= \left(\tfrac{1}{2}\,\frac{\wp' u-\wp'\omega}{\wp u-\wp\omega}+\lambda\right)\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp\,(\lambda-\zeta\omega)\,u,$$

$$\frac{d^2\phi}{du^2} = \left(2\wp u+\wp\omega+\frac{\wp' u-\wp'\omega}{\wp u-\wp\omega}\lambda+\lambda^2\right)\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp\,(\lambda-\zeta\omega)\,u,$$

$$\frac{d\phi_1}{du} = \left(\tfrac{1}{2}\,\frac{\wp' u-\wp'\omega}{\wp u-\wp\omega}-\lambda\right)\frac{\sigma(u-\omega)}{\sigma u\,\sigma\omega}\exp\,(-\lambda+\zeta\omega)\,u;$$

and therefore

$$\frac{d^2\phi}{du^2}\,\frac{d\phi_1}{du} = -\left(2\wp u+\wp\omega+\frac{\wp' u-\wp'\omega}{\wp u-\wp\omega}\,\lambda+\lambda^2\right)\{\tfrac{1}{2}(\wp' u+\wp'\omega)-\lambda(\wp u-\wp\omega)\};$$

and the imaginary part of this expression, since $\wp u$, $\wp' u$, and $\wp\omega$ are real, $\wp'\omega$ is negative imaginary, and λ is positive imaginary, is

$$\left(2\wp u+\wp\omega-\frac{\lambda\,\wp'\omega}{\wp u-\wp\omega}+\lambda^2\right)\{\tfrac{1}{2}\wp'\omega-\lambda\,(\wp u-\wp\omega)\}+\frac{\lambda\,\wp'^2 u}{\wp u-\wp\omega};$$

and this, equated to zero, gives

$$(\wp u + \tfrac{1}{2}\wp\omega)\,\wp'\omega - \lambda\,(2\wp^2 u - \wp u\,\wp\omega - \wp^2\omega)$$
$$+\tfrac{1}{2}\frac{\wp'^2 u - \wp'^2\omega}{\wp u - \wp\omega}\lambda + \tfrac{3}{2}\lambda^2\wp'\omega - \lambda^3(\wp u - \wp\omega) = 0,$$

reducing to a simple equation for $\wp u$, giving

$$\wp u = \frac{3\lambda^2\wp'\omega + \lambda\,(6\wp^2\omega - g_2) + \wp\omega\,\wp'\omega}{2\lambda^3 - 6\lambda\wp\omega - 2\wp'\omega},$$

and, since $\wp u - e_2$ will be found to be negative, $\wp u - e_3$ positive, it follows that the points of inflexion are imaginary, although they may be real on Sylvester's generalized herpolhodes, described by the point of contact of a confocal ellipsoid rolling on a parallel plane.

28. Let us consider a third dynamical problem, the determination of the curve assumed by a uniform chain fixed at two points of a rotating body, when in relative equilibrium.

The equations of relative equilibrium, with the usual notation, are

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + m\omega^2 x = 0,$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + m\omega^2 y = 0,$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right) \qquad = 0,$$

the axis of z being the axis of rotation.

Three first integrals of these equations are immediately obtainable,

namely, $$T\frac{dz}{ds} = T_0,$$

$$Tr^2\frac{d\theta}{ds} = h,$$

$$T + \tfrac{1}{2}m\omega^2 r^2 = \lambda,$$

r and θ denoting polar coordinates in a plane perpendicular to the axis of rotation.

Therefore $$\frac{ds}{dz} = \frac{T}{T_0},$$

$$r^2\frac{d\theta}{dz} = \frac{h}{T_0},$$

and therefore $$1+\frac{dr^2}{dz^2}+r^2\frac{d\theta^2}{dz^2}=\frac{T^2}{T_0^2},$$

or $$1+\frac{dr^2}{dz^2}+\frac{h^2}{T_0^2 r^2}=\frac{T^2}{T_0^2},$$

or $$\frac{r^2 dr^2}{dz^2}=\frac{T^2}{T_0^2}r^2-r^2-\frac{h^2}{T_0^2},$$

or $$\left(\frac{dr^2}{dz}\right)^2=\frac{4r^2}{T_0^2}(\tfrac{1}{2}m\omega^2r^2-\lambda)^2-4r^2-\frac{4h^2}{T_0^2},$$

a cubic function of r^2; and therefore r^2 is an elliptic function of z.

29. If the chain is fixed to two points lying in a plane through the axis of rotation, the chain will lie altogether in this plane, and the equation of its curve takes the simple form with Jacobi's notation

$$r/a=\operatorname{sn} zK/c.$$

In the general case, however, the chain sweeps out a surface of revolution, whose equation is

$$x^2+y^2=a^2\operatorname{cn}^2 zK/c+b^2\operatorname{sn}^2 zK/c,$$

$2a$ and $2b$ being the maximum and minimum diameters of the surface.

30. Employing Weierstrass's notation, we must put

$$r^2=k^2(\wp u-\wp v),$$

so that $$k^4\wp'^2u\left(\frac{du}{dz}\right)^2=\frac{m^2\omega^4k^6}{T_0^2}(\wp u-\wp v)^3$$

$$-\frac{4\lambda m\omega^2k^4}{T_0^2}(\wp u-\wp v)^2+\left(\frac{4\lambda^2}{T_0^2}-4\right)(\wp u-\wp v)-\frac{4h^2}{T_0^2},$$

$$=\tfrac{1}{4}\frac{m^2\omega^4k^6}{T^2}(4\wp^3u-g_2\wp u-g_3),$$

provided that $$\frac{du^2}{dz^2}=\tfrac{1}{4}\frac{m^2\omega^4k^2}{T_0^2},$$

and $\wp v$, g_2, and g_3 are suitably chosen.

Then, since $u=v$ make $r^2=0$, therefore

$$k^4\wp'^2v\left(\frac{du}{dz}\right)^2=-\frac{4h^2}{T_0^2},\quad\text{or}\quad \wp'^2v=-\frac{16h^2}{m^2\omega^4k^6},$$

and $\wp' v$ is consequently imaginary.

31. From the equation $$r^2\frac{d\theta}{dz}=\frac{h}{T_0},$$

we obtain $$\frac{d\theta}{du}=\frac{h}{T_0r^2}\frac{dz}{du}=\frac{2h}{m\omega^2k^3}\frac{1}{\wp u-\wp v},$$

or $$\frac{di\theta}{dz}=-\tfrac{1}{2}\frac{\wp' v}{\wp u-\wp v},$$

if $\wp' v$ is taken as negative imaginary, and consequently $v=s\omega_3$, where s is a real proper fraction.

Then $$i\theta=\tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}-u\zeta v,$$

or $$e^{i\theta}=\sqrt{\left\{\frac{\sigma(u+v)}{\sigma(u-v)}\right\}}\,e^{-u\zeta v}.$$

But $$r^2=k^2(\wp u-\wp v),$$

and therefore
$$x+iy=re^{i\theta}=ik\frac{\sigma(u+v)}{\sigma u\,\sigma v}\exp(-u\zeta v)=ik\,\phi(u,v),$$
$$x-iy=ik\,\phi(-u,v),$$

give the equations of the curve assumed by the chain, the projection of which on a plane perpendicular to the axis of revolution is consequently similar to a herpolhode or the projection of the motion of the bob of a spherical pendulum; also, du/dz being constant, we must put
$$u=z\omega_1/c+\omega_3,$$
where $2c$ denotes the whole length of a complete wave of the chain, and the constant ω_3 is added in order to make $\wp u$ oscillate in value between e_2 and e_3.

When the chain is fixed at two points in a plane perpendicular to the axis of rotation, the curve formed by the chain will be a plane curve, given by an equation of the form
$$x+iy=\frac{\sigma(u+v)}{\sigma u\,\sigma v}\exp(-u\zeta v)=\phi(u,v),$$
the general equation of a catenary under a central force varying as the distance.

A particular case of these catenaries is the curve
$$r\cosh m\theta=a,$$
a Cotes's spiral, the separating herpolhode.

When this catenary is a free orbit under a central force P, the intensity must vary as Ar^3+Br, partly as the third and partly as the first power of the distance, and then we shall find that du/dt is constant; in fact,

$$\frac{\ddot{x}}{x}=\frac{\ddot{y}}{y}=\frac{\ddot{x}+i\ddot{y}}{x+iy}=\frac{P}{r}=2\wp u+\wp v=2r^3+3\wp v;$$

so that
$$P=2r^3+3r\wp v.$$

32. These three dynamical illustrations show that the elliptic integral of the third kind must be treated as the logarithm of the vector of the corresponding azimuthal motion.

[The solutions of other well-known dynamical problems can be exhibited in a similar form.

Thus, in the motion of a particle inside a smooth paraboloid of revolution with vertical axis, the projection of the path on a horizontal plane is given by

$$x+iy=k\phi\,(u,v)\,e^{\lambda u},$$

where
$$\frac{dt}{du}=e_1-\wp u,$$

and
$$x^2+y^2=4mz=k^2(\wp v-\wp u).$$

In the motion of a particle on a smooth vertical cone of semivertical angle α,

$$x+iy=k(\wp v-\wp u)\left\{\frac{\sigma(u+v)}{\sigma(u-v)}\exp(-2u\zeta v)\right\}^{\operatorname{cosec}\alpha},$$

$$r=k(\wp v-\wp u),$$

and
$$\frac{dt}{du}=\wp v-\wp u;$$

and the motion of a sphere rolling on a rough vertical cone will be given by similar equations.

The path of a particle on a smooth horizontal table, attached to one end of a string passing through a hole in the table and supporting a particle hanging vertically, is of a similar nature; and the particle may be supposed to move on a smooth vertical cone, the string passing through a hole at the vertex and supporting another particle, without material alteration of the preceding results.

The formulæ obtained by M. Halphen in the *Journal de l'Ecole Polytechnique*, 1884, "Sur une courbe élastique," for the solution of

M. Maurice Lévy's problem of the curve formed by a plate, originally cylindrically circular, due to a uniform pressure on one side (for instance, a circular tube exposed to an external collapsing pressure), can now be slightly modified in order to be expressed in a similar form; for, writing $2v$ and $u-v$ for his v and u, then

$$x+iy = a\phi(u+v, 2v)\,\phi(u-v, 2v),$$

$$r^2 = a^2\{\wp(u+v)-\wp 2v\}\{\wp(u-v)-\wp 2v\},$$

$$Ar^4+Br^2+C = \tfrac{1}{2}a\,\{\wp(u+v)+\wp(u-v)-2\wp 2v\},$$

$$(Ar^4+Br^2+C)^2-r^2 = \tfrac{1}{4}a^2\{\wp(u-v)-\wp(u+v)\}^2.$$

These equations are derived from M. Lévy's relation

$$\frac{1}{\rho} = 4Ar^2+2B,$$

connecting ρ the radius of the curvature, and r the distance from a fixed point of any point on the curve; and the relation is immediately obtained by taking moments about any point on the curve,

$$\frac{a}{\rho} = -\frac{1}{\wp' 2v}\,\frac{\wp'(u-v)-\wp'(u+v)}{\wp(u-v)-\wp(u+v)},$$

and then
$$\frac{ds}{du} = ia\wp' 2v.$$

De Sparre's theorem (§ 27) has been considered recently by numerous writers; for instance, Darboux, Mannheim, Resal in the *Journal de l'Ecole Polytechnique*, 1885, and Hess, in the *Math. Annalen.*]

April 7th, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Messrs. Joseph Edwards, M.A., Fellow of Sidney Sussex College, Cambridge; Robert Russell, M.A., Trinity College, Dublin; and A. N. Whitehead, B.A., Fellow and Lecturer of Trinity College, Cambridge, were elected members.

The following communications were read (or taken as read):

On the Intersections of a Circle and a Plane Curve: Prof. Genese.

A new Theory of Harmonic Polygons: Rev. T. C. Simmons.

On some Properties of Simplicissima, with especial regard to the related spherical Loci: W. J. C. Sharp.

On Briot and Bouquet's Theory of the Differential Equation $F\left(u, \frac{du}{dz}\right) = 0$: Prof. Cayley, F.R.S.

The Cosine Orthocentres of a Plane Triangle and a Cubic through them: R. Tucker.

Note on a Tetrahedron: Dr. Wolstenholme.

The following presents were received:

"Mathematical Questions and their Solutions, from the 'Educational Times,'" Vol. XLVI.

"Proceedings of the Royal Society," Vol. XLII., Nos. 251 and 252.

"Educational Times," for April.

"Transactions of the Cambridge Philosophical Society," Vol. XIV., Part II., 1887.

"Johns Hopkins University Circulars," Vol. VI., No. 56, March, 1887.

"Smithsonian Report," 1884, Part II.; Washington, 1885.

"Transactions of the Connecticut Academy," Vol. VII., Part I.; Newhaven, 1886.

"Bulletin des Sciences Mathématiques," T. XI., March and April, 1887.

"Bulletin de la Société Mathématique de France," T. XV., No. 1.

"Beiblätter zu den Annalen der Physik und Chemie," B. XI., St. 2 and 3.

"Acta Mathematica," 9: 3.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., F. 3 and 4; Febbrajo, 1887.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Num. 29 and 30; Firenze.

"Actes de la Société Helvétique des Sciences Naturelles," réunie à Genève les 10, 11, et 12 Août 1886, 8vo; compte-rendu 1885—86; Genève, 1886.

"Compte-rendu des Travaux présentés à la Soixante-neuvième Session de la Société Helvétique des Sciences Naturelles," réunie à Genève les 10, 11, et 12 Août 1886, 8vo; Genève, 1886.

"Mittheilungen der Naturforschenden Gesellschaft in Bern aus dem Jahre 1886," Nr. 1143—1168, 8vo; Bern, 1887.

"Jahrbuch über die Fortschritte der Mathematik," B. XVI., H. 2; Jahrgang 1884.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VII., No. 4; Coimbra, 1886.

"Ueber die Integration der m-ten Wurzel aus einer rationalen Function," von F. Klitzkowski, 8vo pamphlet; Greifswald, 1887. Inaugural dissertation.)

A new Method for the Investigation of Harmonic Polygons. *By* Rev. T. C. Simmons, M.A.

[*Read April 7th*, 1887.]

1. If $P_1P_2P_3P_4$ ... be a cyclic polygon such that the perpendiculars drawn on the sides from some internal point K are all respectively proportional to those sides, then $P_1P_2P_3P_4$ ... is said to be a *harmonic polygon* whose *Lemoine point* is K.