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*Isoscelian Hexagrams.** *By* R. TUCKER, M.A.

[*Read Nov.* 14*th*, 1889.]

I. *Positive Hexagrams.*†

1. At the point L in BC (Fig. i.), make the $\angle BLN'=B$, $\angle CLM'=C$; then make $\angle M'NA = A$, $\angle BNL'= B$, $\angle L'MC = C$, and join MN'. Now $\angle M'LN'= A = \angle M'NA$, therefore L, N', N, M' are concyclic; also $\angle CLM'= C = \angle CML'$, therefore L, L', M, M' are concyclic; and $\angle AM'N = 180°-2A = \angle ML'N$, $\therefore$ L', M, M', N are concyclic;

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* *Cf. Proc.*, Vol. XIX., p. 163.

† When λ corresponds to the point of departure of a positive isoscelian, I call the figure a positive hexagram; and when to that of a negative isoscelian, a negative hexagram.

hence L, L', M, M', N, N' are concyclic, and $\angle AMN' = \angle ANM' = A$.

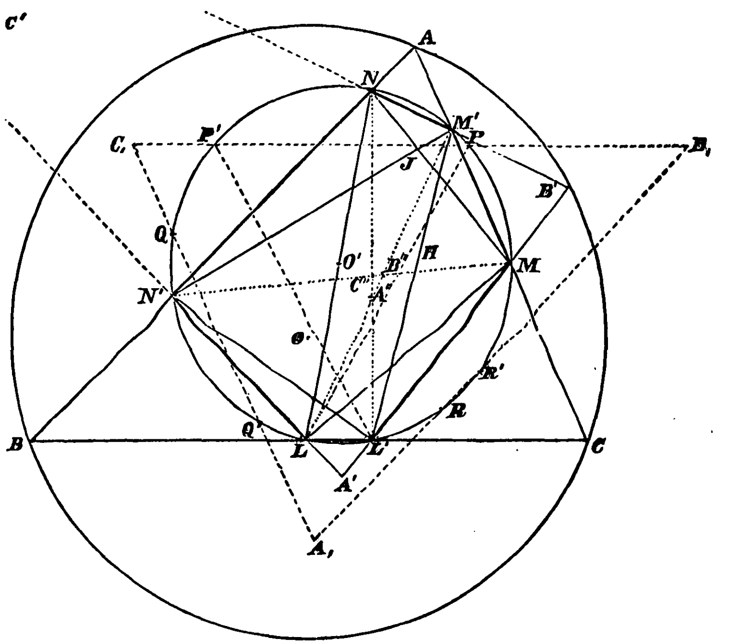

Fig. i.

2. The lines $N'L$, $L'M$, $M'N$ are positive isoscelians, and LM', NL MN' are negative isoscelians. Since

$$\left.\begin{aligned} NL' + L'M &= a, \\ LM' + M'N &= b, \\ MN' + N'L &= c, \end{aligned}\right\}$$

all such isoscelian hexagrams are *isoperimetrical*, and have their perimeters equal to the perimeter of the triangle ABC.

3. Write

$$\left.\begin{aligned} BL &= \lambda a, & CL &= \lambda' a, \\ CM &= \mu b, & AM &= \mu' b, \\ AN &= \nu c, & BN &= \nu' c, \end{aligned}\right\}$$

so that $$\lambda' = 1 = \mu + \mu' = \nu + \nu'.$$

Now $$2BL' \cos B = BN = \nu' c,$$

$$2CL' \cos C = CM = \mu b;$$

whence
$$\left.\begin{aligned} 2a &= \nu' c \sec B + \mu b \sec C, \\ 2b &= \lambda' a \sec C + \nu c \sec A, \\ 2c &= \mu' b \sec A + \lambda a \sec B; \end{aligned}\right\}$$

i.e., if we write

$$p = a\cos A, \quad q = b\cos B, \quad r = c\cos C,$$
$$\mu q = r - p + \lambda p, \quad \mu' q = p + q - r - \lambda p,$$
$$\nu r = r - q + \lambda p, \quad \nu' r = \quad q \quad - \lambda p.$$

4. The trilinear coordinates may be written

$$\left.\begin{matrix} L, & 0, & \lambda' c, & \lambda b \\ M, & \mu c, & 0, & \mu' a \\ N, & \nu' b, & \nu a, & 0 \end{matrix}\right\}, \quad \left.\begin{matrix} L', & 0, & \mu\cos B, & \nu'\cos C \\ M', & \lambda'\cos A, & 0, & \nu\cos C \\ N', & \lambda\cos A, & \mu'\cos B, & 0 \end{matrix}\right\}.$$

5. The equations may be written

$$\text{to } NL', \quad \nu a\cos C\alpha - \nu' b\cos C\beta + \mu b\cos B\gamma = 0,$$
$$\text{to } LM', \quad \nu c\cos C\alpha + \lambda b\cos A\beta - \lambda' c\cos A\gamma = 0;$$

these intersect, in A'', on $cr\gamma = bq\beta$; hence AA'', BB'', CC'' cointersect in

$$T\,(pa\alpha = qb\beta = cr\gamma);$$

and the triangle $A''B''C''$,

$$\left.\begin{aligned} A'' &= 2B - C, \\ B'' &= 2C - A, \\ C'' &= 2A - B, \end{aligned}\right\}$$

is similar to the negative in-isoscelian triangle.

6. The equations to LN', ML' are, respectively,

$$\left.\begin{aligned} \mu' b\cos B\alpha - \lambda b\cos A\beta + \lambda' c\cos A\gamma &= 0, \\ \mu' a\cos B\alpha + \nu' c\cos C\beta - \mu c\cos B\gamma &= 0, \end{aligned}\right\}$$

and these intersect, in A', on $bq\beta = cr\gamma$; hence AA', BB', CC' cointersect in T, and the triangle $A'B'C'$,

$$\left.\begin{aligned} A' &= 2C - B, \\ B' &= 2A - C, \\ C' &= 2B - A, \end{aligned}\right\}$$

is similar to the positive in-isoscelian triangle.

The triangles ABC, $A'B'C'$, $A''B''C''$ are then in perspective, with T for their centre of perspective. The point T is the isotomic of the circumcentre.

7. The respective axes of perspective are

$$(ABC,\ A'B'C'),\ \mu'q\,.\,a\alpha+\nu'r\,.\,b\beta+\lambda'p\,.\,c\gamma=0,$$

$$(ABC,\ A''B''C''),\ \nu r\,.\,a\alpha+\lambda p\,.\,b\beta+\mu q\,.\,c\gamma=0,$$

$$(A'B'C',\ A''B''C'')\ (\mu'q-\nu r)\,a\alpha+(\nu'r-\lambda p)\,b\beta+(\lambda'p-\mu q)\,c\gamma=0.$$

8. From § 3, we have the relation

$$0=\begin{vmatrix} -2, & \mu\sec C, & \nu'\sec B \\ \lambda'\sec C, & -2, & \nu\sec A \\ \lambda\sec B, & \mu'\sec A, & -2 \end{vmatrix},$$

whence

$$(\lambda\mu\nu+\lambda'\mu'\nu')\sec A\sec B\sec C+2\mu'\nu\sec^2 A+2\nu'\lambda\sec^2 B+2\lambda'\mu\sec^2 C=8.$$

Now the triangle $L'M'N'$, which is similar to ABC,

$$=\Delta\left[1-\frac{\lambda'\mu}{4\cos^2 C}-\frac{\mu'\nu}{4\cos^2 A}-\frac{\nu'\lambda}{4\cos^2 B}\right]$$

$$=\Delta\,(\lambda\mu\nu+\lambda'\mu'\nu')\sec A\sec B\sec C/8.$$

But, if ρ be the hexagram radius, then triangle $L'M'N'=\rho^2\Delta/R^2$;

therefore $\qquad 8\rho^2\cos A\cos B\cos C=R^2\,(\lambda\mu\nu+\lambda'\mu'\nu')$.

The triangle $LMN=\Delta\left\{1-\mu'\nu-\nu'\lambda-\lambda'\mu\right\}=\Delta\,(\lambda\mu\nu+\lambda'\mu'\nu')$, and it is similar to the pedal triangle of ABC.

9. Let the bisectors of angles $AM'N$, $AN'M$ meet in P; then, since these lines are perpendicular to AB, AC, respectively,

$$\angle M'PN'=\angle A,$$

therefore P is a point on the "H." circle.

The equations to PN', PM' are

$$\mu'a\cos B\alpha-\lambda a\cos A\beta-(\mu+1)\,c\cos B\gamma=0,$$

$$-\nu a\cos C\alpha+(\nu'+1)\,b\cos C\beta+\lambda'a\cos A\gamma=0;$$

these intersect in

$$\frac{a\cos A}{\cos B\cos C\cos(B-C)}=\frac{\beta}{\cos B\cos C+\lambda\sin C\sin A}$$

$$=\frac{\gamma}{\cos B\cos(B-C)-\lambda\sin A\sin B}=\frac{b\beta+c\gamma}{a\cos B\cos C},$$

therefore $$\frac{a\alpha}{\cos(B-C)} = \frac{b\beta + c\gamma}{\cos A} = \frac{\Delta}{\sin B \sin C},$$

i.e., the locus is a straight line B_1C_1, parallel to BC, and bisecting the distance between A and the orthocentre (H).

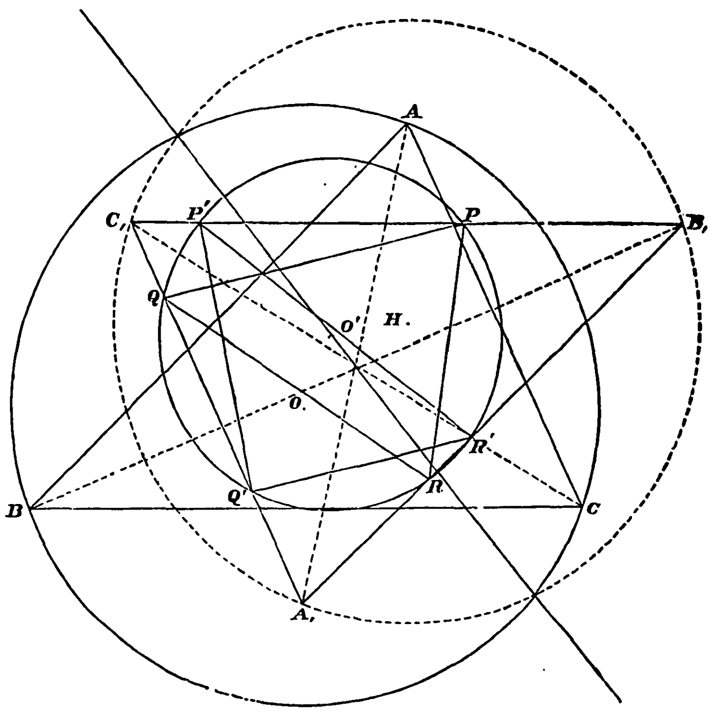

Fig. ii.

Hence, completing the triangle $A_1B_1C_1$, we see that the loci of P, (Q, R), are the lines determined by the sides of this triangle, which we propose to call the *Director* triangle.

P, Q, R are, of course, the mid-points of the arcs MN, NL, LM, and are, further, the orthocentres of the triangles $AM'N'$, $BN'L'$, $CL'M'$.

10. The triangle PQR is similar to ABC, and therefore $= L'M'N'$.

11. By § 9, the circle ANM has P for its centre. If we put

$$C \equiv a\beta\gamma + \ldots + \ldots, \quad L \equiv a\alpha + \ldots + \ldots,$$

the circles *AMN*, *BNL*, *CLM* are given by

$$a \,.\, C = L\,(c\nu'\beta + b\mu\gamma) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(\text{i.}),$$

$$b \,.\, C = L\,(a\lambda'\gamma + c\nu a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.}),$$

$$c \,.\, C = L\,(b\mu' a + a\lambda\beta) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (\text{iii.}),$$

of which the radical axes are, (i., ii.), (ii., iii.),

$$\nu ca a - \nu' bc\beta + (a^2\lambda' - b^2\mu)\,\gamma = 0,$$

$$(b^2\mu' - c^2\nu)\,a + \lambda ab\beta - \lambda' ca\gamma = 0;$$

these intersect in $\quad a \cos A = \gamma \cos C,$

hence the radical centre is *H*.

The straight lines *PL*, *QM*, *RN* consequently pass through *H*, and the sides *PQ*, *QR*, *RP* of *PQR* are perpendicular to *HN*, *HL*, *HM*; hence *H* is the orthocentre of *PQR*, and the centre of perspective of *PQR* and *LMN*.

12. If *O* be the circumcentre, we have

$$\tan OL'B = 2R \cos A \cos C / (a \cos C - \mu b),$$

and $\quad \tan HLC = 2R \cos B \cos C / (\lambda' a - b \cos C).$

Now (§ 3) $\quad a \cos B \cos C - \mu q = \lambda' p - b \cos C \cos A,$

hence $\quad \angle\, OL'B = \angle\, HLC,$

and *L'O* meets the "H." circle in the point *P'*, where it is cut by B_1C_1 and so for the points *Q'*, *R'*. Hence the triangles *P'Q'R'* *L'M'N'* have *O* for their centre of perspective.

13. Further, by § 9, $\angle\, PLM = \angle\, PLN$, therefore *H* is the in-centre of *LMN*, and it is also readily seen to be the circumcentre of $A_1B_1C_1$.

14. Since

$$\angle\, PL'C = (90^\circ - A) + (180^\circ - 2C) = 90^\circ + B - C = \angle\, P'LB,$$

$$\therefore \quad R \cos (B-C) = PL' \sin (90^\circ + B - C) = PL' \cos (B-C),$$

i.e., $PL' = R = P'L$, and so for the other points *Q*, *Q'*; *R*, *R'*.

Hence, if we take any point *P* on a side of the director triangle, the "H." circle, through *P*, can be easily constructed.*

15. We have $\quad R = PL' = 2\rho \sin PP'L' = 2\rho \sin OL'B,$

therefore $\quad OL' = 2\rho \cos A.$

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* A variation in the construction is suggested by § 32, *infra*.

Now, since AO makes, with BC, an angle $= 90^\circ + \widehat{B-C} = \angle PL'C$,

therefore $PL' =$ and is parallel to AO,

therefore $AP =$ and is parallel to $OL' = 2\rho \cos A$.

Similarly, $BQ = 2\rho \cos B, \quad CR = 2\rho \cos C$.

Hence $AP \sin APP' = R \cos A = 2\rho \cos A \sin APP'$,

therefore $\sin APP' = R/2\rho = \sin BQQ' = \sin CRR'$.

16. The equations to the circles $AM'N'$, $BN'L'$, $CL'M'$ are

$$2 \cos B \cos C \, . \, C = L\,(\lambda \cos C\beta + \lambda' \cos B\gamma) \ldots\ldots\ldots\ldots \text{(i.)},$$
$$2 \cos C \cos A \, . \, C = L\,(\mu \cos A\gamma + \mu' \cos C\alpha) \ldots\ldots\ldots\ldots \text{(ii.)},$$
$$2 \cos A \cos B \, . \, C = L\,(\nu \cos B\alpha + \nu' \cos A\beta) \ldots\ldots\ldots\ldots \text{(iii.)}.$$

The radical axes of (i., ii.), (ii., iii.), (iii., i.) are

$$\mu' \sec A\alpha - \lambda \sec B\beta + (\mu - \lambda') \sec C\gamma = 0,$$
$$(\nu - \mu') \sec A\alpha + \nu' \sec B\beta - \mu \sec C\gamma = 0,$$
$$-\nu \sec A\alpha + (\lambda - \nu') \sec B\beta + \lambda' \sec C\gamma = 0.$$

The radical centre of these is the circumcentre O.

Now $N'L' = 2\rho \sin BNL' = 2\rho \sin B$,

therefore radius of circle $BN'L'$ (and of the others) is ρ; *i.e.*, these circles are equal to the "H." circle.*

17. Assume the equation to circle LMN to be

$$C = L\,(\lambda_1\alpha + \mu_1\beta + \nu_1\gamma).$$

We get

$$\left.\begin{aligned} a\lambda\lambda' &= \mu_1\lambda'c + \nu_1\lambda b, \\ b\mu\mu' &= \lambda_1\mu c + \nu_1\mu' a, \\ c\nu\nu' &= \lambda_1\nu' b + \mu_1\nu a; \end{aligned}\right\}$$

whence

$$\lambda_1 \begin{vmatrix} 0, & \lambda'c, & \lambda b \\ \mu c, & 0, & \mu' a \\ \nu' b, & \nu a, & 0 \end{vmatrix} = \begin{vmatrix} \lambda\lambda' a, & \lambda' c, & \lambda b \\ \mu\mu' b, & 0, & \mu' a \\ \nu\nu' c, & \nu a, & 0 \end{vmatrix};$$

i.e.,

$$\lambda_1\, abc\,(\lambda\mu\nu + \lambda'\mu'\nu') = a\mu'\nu\,(-\lambda\lambda' a^2 + \lambda\mu b^2 + \lambda'\nu' c^2),$$
$$\mu_1 \quad ,, \qquad ,, \qquad = b\nu'\lambda\,(-\mu\mu' b^2 + \mu\nu c^2 + \lambda'\mu' a^2),$$
$$\nu_1 \quad ,, \qquad ,, \qquad = c\lambda'\mu\,(-\nu\nu' c^2 + \nu\lambda a^2 + \mu'\nu' b^2);$$

* If O_2 be the centre of the circle, then $O'L'O_2N'$ is a square.

whence $$\frac{\lambda_1 \cos A}{\mu'\nu} = \frac{\mu_1 \cos B}{\nu'\lambda} = \frac{\nu_1 \cos C}{\lambda'\mu}$$

$$= \frac{\cos^2 B - 2\lambda \cos B \sin C \sin A + \lambda^2 \sin^2 A}{\sec A \cos B \cos C \,(\lambda\mu\nu + \lambda'\mu'\nu')}.$$

The radical axis of ABC and LMN is

$$L' \equiv \mu'\nu \sec A \,.\, \alpha + \nu'\lambda \sec B \,.\, \beta + \lambda'\mu \sec C \,.\, \gamma = 0,$$

and the equation to LMN is

$$2C = LL'.$$

18. The projection of OL' on BC

$$= (a \cos C - \mu b) / 2 \cos C = (c \cos A \cos B - \lambda p) / 2 \cos B \cos C,$$

therefore

$$OL'^2 = R^2 \cos^2 A \sec^2 B \sec^2 C \,(\cos^2 B - 2\lambda \cos B \sin C \sin A + \lambda^2 \sin^2 A);$$

hence (by § 15) $$4\rho^2 \cos^2 B \cos^2 C = R^2 (\ldots\ldots),$$

or $$12\rho^2 \cos^2 A \cos^2 B \cos^2 C$$

$$= R^2 \left\{ \begin{array}{l} \cos^2 A \cos^2 B - 2\lambda \sin A \cos^2 A \cos B \sin C + \lambda^2 \sin^2 A \cos^2 A \\ + \cos^2 B \cos^2 C - 2\mu \sin B \cos^2 B \cos C \sin A + \mu^2 \sin^2 B \cos^2 B \\ + \cos^2 C \cos^2 A - 2\nu \sin C \cos^2 C \cos A \sin B + \nu^2 \sin^2 C \cos^2 C \end{array} \right\}.$$

19. If T_a, T_b, T_c be the tangents to LMN from A, B, C, then

$$T_a^2 = \mu' b \,.\, \nu c / 2 \cos A, \text{ \&c.},$$

and $$T_a^2 \,.\, T_b^2 \,.\, T_c^2 = \lambda\lambda' a^2 \,.\, \mu\mu' b^2 \,.\, \nu\nu' c^2 / 8 \cos A \cos B \cos C$$

$$= (LM' \,.\, MN' \,.\, NL')\,(LN' \,.\, ML' \,.\, NM') / 8 \cos A \cos B \cos C,$$

which can be otherwise expressed, since

$$8 \cos A \cos B \cos C = (LM \,.\, MN \,.\, NL) / (L'M' \,.\, M'N' \,.\, N'L').$$

20. If O' be the centre of LMN, then the projection (x say) of BO' on $BC = \frac{1}{2}\left[(\lambda+1)\, a - \mu b / 2 \cos C\right]$.

Assume α, β, γ to be trilinear coordinates of O', then

$$\gamma + \alpha \cos B = x \sin B = \sin B \left[q - \lambda\,(p - 2a \cos B \cos C)\right] / 4 \cos B \cos C,$$

$$\alpha \sin B = \left[q \cos B + \lambda a \sin B \sin (B - C)\right] / \ldots,$$

whence $$M \equiv \gamma \cos B + \alpha \cos 2B = \lambda a \sin 3B / 4 \cos B \ldots\ldots\ldots\ldots \text{(i.)},$$

and $$N \equiv \alpha \cos C + \beta \cos 2C = \mu b \sin 3C / 4 \cos C \ldots\ldots\ldots\ldots \text{(ii.)},$$

$$L \equiv \beta \cos A + \gamma \cos 2A = \nu c \sin 3A / 4 \cos A \ldots\ldots\ldots \text{(iii.)}.$$

From (ii.), (iii.), with the aid of §2, we get

$$4\cos A\cos B\cos C\left[N\sin(B-C)/\sin 3C+L\sin(C-A)/\sin 3A\right]$$
$$=-\lambda p\cos C\sin(A-B);$$

hence, making use of (i.), we have

$$L\sin(C-A)\sin 3B\sin 3C+M\sin(A-B)\sin 3C\sin 3A+\ldots=0,$$

wherein the coefficient of $\alpha=\sin 3A\left[\Sigma(\cos 2A)-\Sigma(\cos 4A)\right]$,

therefore locus of O' is

$$\alpha\sin 3A+\beta\sin 3B+\gamma\sin 3C=0.*$$

This straight line is perpendicular to OH, being the radical axis of ABC, $A_1B_1C_1$, and passes through the centre of the "N.P." circle, as it should do, since that circle is a circle of this system.

21. Since $\angle P'LB=90^\circ+B-C$,

and $\angle Q'LB=BLN'-Q'QN'=B-90^\circ+C$,

therefore $\angle P'R'Q'=\angle P'L'Q'=180^\circ-2C$,

therefore $\Delta P'Q'R'$ is similar to the pedal triangle and $=\Delta LMN$;

also, $\angle P'OQ'=\angle L'OM'=180^\circ-C$,

therefore O is the incentre of $P'Q'R'$.

Now $\angle M'L'P'=\angle M'PP'=90^\circ-B$,

therefore O is the orthocentre of $L'M'N'$.

From §13,

$$\rho^2(1-8\cos A\cos B\cos C)=O'O^2=O'H^2=\rho^2-2\rho r_1,$$

if r_1 is the inradius of LMN (or $P'Q'R'$);

hence $r_1=4\rho\cos A\cos B\cos C$.

Also, $Q'L'=PM$ (or PN),

$R'M'=QN$ (or QL),

and $P'N'=RL$ (or RM).

22. Referring to §17, we can write the equation to the radical axis $L'=0$ in the form (multiplying all through by $abc\cos A\cos B\cos C$),

$$a\alpha(p+q-r-\lambda p)(r-q+\lambda p)+b\beta\lambda p(q-\lambda p)$$
$$+c\gamma(1-\lambda)p(r-p+\lambda p)=0,$$

* *Cf. Proc.*, xv., p. 129.

or
$$\left.\begin{array}{l}(p+q-r)(r-q)\,a\alpha+p\,(r-p)\,c\gamma \\ +\lambda p\left[\{p+2\,(q-r)\}\,a\alpha+qb\beta+(2p-r)\,c\gamma\right] \\ -\lambda^2p^2\,(a\alpha+b\beta+c\gamma)=0,\end{array}\right\}\begin{array}{l}\equiv\mathfrak{A} \\ \equiv-2\mathfrak{B}\lambda p \\ \equiv\mathfrak{C}\lambda^2p^2.\end{array}$$

Hence, for the envelope, we have $\mathfrak{A}\mathfrak{C}=\mathfrak{B}^2$.

Substituting and reducing, we obtain

$$p^2a^2\alpha^2+q^2b^2\beta^2+r^2c^2\gamma^2+2a\alpha b\beta\left[2r\,(p+q-r)-pq\right]$$
$$+2b\beta c\gamma\left[2p\,(q+r-p)-qr\right]+2c\gamma a\alpha\left[2q\,(r+p-q)-rp\right]=0;$$

i.e., $p^2a^2\alpha^2+\ldots+\ldots+2pq\alpha\beta\,(4c^2\cos C-ab)+\ldots+\ldots=0.$

To obtain the envelope of the ellipses, we have merely to subtract from the sinister side of the above result

$$4\,(a\alpha+b\beta+c\gamma)\,\frac{2\,(a\beta\gamma+\ldots)}{a\alpha+\ldots}\,pqr;$$

i.e., $$8\,(a\beta\gamma+\ldots+\ldots)\,pqr.$$

Hence the envelope is

$$p^2a^2\alpha^2+\ldots+\ldots-2pq\,ab\,\alpha\beta-\ldots-\ldots=0.$$

This is the conic $\sqrt{pa\alpha}+\sqrt{qb\beta}+\sqrt{rc\gamma}=0.$

The centre of this conic is the "N.P." centre, and it touches the sides of ABC in (say) t_1, t_2, t_3, so that At_1, Bt_2, Ct_3 intersect in T. Its foci are O and H, its (semi-minor axis)$^2=2R^2\cos A\cos B\cos C$, and its major axis is R; hence the "N.P." circle is its auxiliary circle.

23. Let λ, λ_1 be values corresponding to centres equidistant from the "N.P." centre; then, §20,

$$4\cos B\cos C\sin B.R\cos(B-C)=2q\cos B+(\lambda+\lambda_1)\,a\sin B\sin(B-C),$$

i.e., $2b\cos B\left[\cos C\cos(B-C)-\cos B\right]=(\lambda+\lambda_1)\,a\sin B\sin(B-C),$

therefore $(\lambda+\lambda_1)\,a=2c\cos B.$

By § 22, the radical axes are given by

$$\mathfrak{A}-2\mathfrak{B}\lambda p+\mathfrak{C}\lambda^2p^2=0,$$
$$\mathfrak{A}-2\mathfrak{B}\lambda_1p+\mathfrak{C}\lambda_1^2p^2=0,$$

whence $a\mathfrak{B}=p\mathfrak{C}\,c\cos B,$

i.e., $(q-r)\,a\alpha+(r-p)\,b\beta+(p-q)\,c\gamma=0,$

or the radical axis of any two "conjugate" circles of the system is

OH. Hence locus of radical centre of two "conjugate" circles and ABC is the same straight line.

24. From § 16 we see that $N'L'$ envelopes a parabola touching the sides BC, BA, with O for its focus.

If we take, for the moment, a', b', c' to be the mid-points of the sides BC, CA, AB, then Ob' is the direction of the axis of the curve, and $c'a'$ is the tangent at the vertex; hence the latus rectum $= 4R\cos C\cos A$, and the directrix is the side C_1A_1 of the *director* triangle. Similar results hold for the envelopes of $L'M'$, $M'N'$.

25. The equation to the above parabola in trilinear coordinates is

$$\left[(a\alpha-b\beta)\sin(A-B)+c\gamma\sin C\right]^2 = 4ab\,\alpha\beta\sin 2A\sin 2B,$$

so that the chord of contact is

$$(a\alpha-b\beta)\sin(A-B)+c\gamma\sin C = 0.$$

Referred to BC, BA as axes of x and y, the envelope of $N'L'$, *i.e.*, of

$$x/\nu'c+y/\lambda a = 1/2\cos B,$$

is

$$\sqrt{2x\cos C}+\sqrt{2y\cos A} = \sqrt{b}.$$

Hence, if K, K' are the points of contact, we have

$$BK = b/2\cos C,\quad BK' = b/2\cos A,$$

and therefore equation to KK' is

$$2x\cos C+2y\cos A = b.$$

The equation to CA is $\dfrac{x}{a}+\dfrac{y}{c} = 1$;

hence CA meets KK' in $a/2$, $c/2$; *i.e.*, CA is bisected.

To find direction of axis, we must join B to the mid-point of KK' $(b/4\cos C,\ b/4\cos A)$; the equation to this line is

$$y\cos A = x\cos C;$$

i.e., it passes through H. If it cuts KK' in K'', then mid-point of BK'' is on the curve.

Since the sides of the *director* triangle are directrices of the parabolas, we obtain, from a tangent property, another proof of § 9.

26. From § 11, we see that the envelope of NL is a parabola touching BC, BA, with H as focus. In this case, if AD, BE, CF are the perpendiculars on BC, CA, AB, DF is the tangent at the vertex, and the latus rectum $= 8R\cos A\cos B\cos C$.

In trilinear coordinates, the equation is

$$\left[a\alpha\,(q-r)+b\beta q-c\gamma\,(p-q)\,\right]^2=4c\gamma\,a\alpha\,rp\,;$$

and, referred to BC, BA, it is

$$\sqrt{x\cos A}+\sqrt{y\cos C}=\sqrt{q}.$$

If K_1, K_1', are the points of contact, then

$$BK_1=q\sec A,\quad BK_1'=q\sec C\,;$$

hence K_1K_1' makes the same angle with BC that KK' does with BA.

The equation to K_1K_1' is

$$x\cos A+y\cos C=q.$$

The equation to the bisector (from B) of K_1K_1' is

$$y\cos C=x\cos A,$$

and this is a line passing through O; hence BO is the direction of the axis.

27. The $L'N'$ series of envelopes can be written in the form

$$\sqrt{\gamma\cos C}+\sqrt{\alpha\cos A}=\sqrt{R}\sin B\,\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

$$\sqrt{\alpha\cos A}+\sqrt{\beta\cos B}=\sqrt{R}\sin C\,\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)},$$

$$\sqrt{\beta\cos B}+\sqrt{\gamma\cos C}=\sqrt{R}\sin A\,\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)};$$

these respectively pass through

$(X_1)\quad 4R\alpha/c^2\cos A,\quad 2\beta/c\sin A,\quad 4R\gamma/a^2\cos C,$

$(X_2)\quad 4R\alpha/b^2\cos A,\quad 4R\beta/a^2\cos B,\quad 2\gamma/a\sin B,$

$(X_3)\quad 2\alpha/b\sin C,\quad 4R\beta/c^2\cos B,\quad 4R\gamma/c^2\cos A,$

and AX_1, BX_2, CX_3 cointersect in

$$a\alpha\tan A=b\beta\tan B=c\gamma\tan C=2\Delta\tan\omega.$$

28. The LN series of envelopes touch the sides in (K_c, K_b), (L_a, L_c), (M_b, M_a), say.

The equations to the chords are

$$a\alpha\sin A\cos A+b\beta\cos B\sin(C-A)-c\gamma\cos C\sin(A-B)=0\ldots\text{(i.)},$$

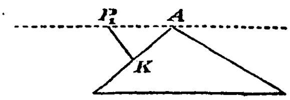
$$-a\alpha\cos A\sin(B-C)+b\beta\sin B\cos B+c\gamma\cos C\sin(A-B)=0\ldots\text{(ii.)},$$

$$a\alpha\cos A\sin(B-C)-b\beta\cos B\sin(C-A)+c\gamma\sin C\cos C=0\ldots\text{(iii.)};$$

(ii.), (iii.) intersect in

$$\frac{a\alpha}{\sin A/\sin(B-C)} = \frac{b\beta}{1} = \frac{c\gamma}{-1} = \frac{2\Delta}{\sin A/\sin(B-C)}$$

i.e., these chords intersect in points (P_1, Q_1, R_1) on lines through A, B, C parallel to the sides:



$$P_1K \text{ (in figure)}$$
$$= b\sin(B-C),$$

i.e., $AP_1 = 2R\sin(B-C)$.

The equation to the circle $P_1Q_1R_1$ is

$$C + L\left[4\{a\cos A\sin 2A\sin^2(B-C) + \ldots + \ldots\}/\sin 2A\sin 2B\sin 2C\right] = 0.$$

29. In Fig. iii., S, V, W are the mid-points of the arcs $M'N'$, NL',

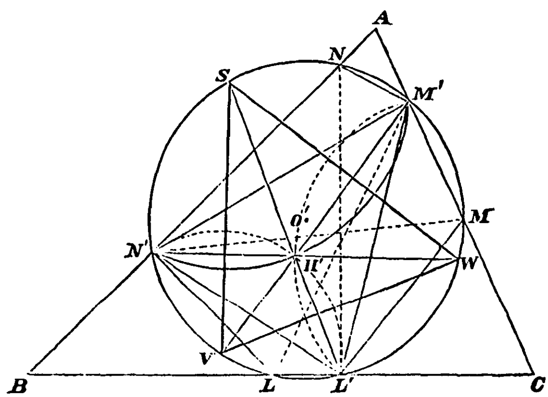


Fig. iii.

$L'M'$; *i.e.*, NS, MS bisect the angles at N' and M'; then the angles S, V, W are, respectively,

$$(B+C)/2, \quad (C+A)/2, \quad (A+B)/2,$$

i.e., the triangle is similar to the intriangle of ABC whose circum-circle is the incircle of ABC.

Since $\angle SWN' = \dfrac{A}{2}$,

therefore $N'W$ is perpendicular to SV,

$L'S$ " VW,

and $M'V$ " WS.

Let H' be the orthocentre; then circle whose centre is S, and which passes through $M'N'$, also passes through H', and so for the allied circles; whence we see that the sides of SVW bisect $H'L'$, $H'M'$, $H'N'$; hence these distances equal

$$4\rho \sin\frac{B}{2}\sin\frac{C}{2}, \text{ \&c.}$$

Further, H' is clearly the incentre of $L'M'N'$, and also the centre of perspective of this triangle and SVW.

We have, if r_1' be the inradius of $L'M'N'$,

$$\rho^2\left(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = (O'H')^2 = \rho^2-2\rho r_1';$$

therefore $$r_1' = 4\rho\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

30. If the circle $N'H'M'$ cuts BA in X, then

because $$\angle H' = 90°+\frac{A}{2},$$

therefore $$\angle NXM' = 90°-\frac{A}{2},$$

and $$NX = NM' = AM'.$$

31. Assume $$\angle N'L'B = \phi,$$

then $$\frac{\sin(B+\phi)}{\sin\phi} = \frac{BL'}{BN'} = \frac{\nu' c}{\lambda a} = \frac{q-\lambda p}{\lambda a\cos C},$$

therefore $$\cot\phi = \frac{\cos B-\lambda\sin A\sin C}{\lambda\sin A\cos C};$$

whence, § 18, $\cos\phi = \cos A\,(\cos B-\lambda\sin A\sin C)/D$,

$$\sin\phi = \lambda\sin A\cos C\cos A/D,$$

where D is a symmetrical expression.

Now $$a\text{ (of }H') = H'L'\sin\left(\frac{A}{2}+\phi\right)$$

$$= (\S\ 29)\ \frac{2R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos B\cos C}(\lambda'\cos B+\lambda\cos C);$$

hence coordinates are given by

$$\frac{a\alpha}{p(\lambda'\cos B+\lambda\cos C)}=\frac{b\beta}{q(\mu'\cos C+\mu\cos A)}=\frac{c\gamma}{r(\nu'\cos A+\nu\cos C)};$$

or, using § 3,

$$\frac{a\alpha}{p\cos B-\lambda p(\cos B-\cos C)}$$

$$=\frac{b\beta}{\cos A\cos B(2q+b-2c\cos A)-\lambda p(\cos C-\cos A)}$$

$$=\frac{c\gamma}{\cos B(b\cos A+r-q)-\lambda p(\cos A-\cos B)}$$

$$=\frac{a\alpha+b\beta+c\gamma}{2(a+b+c)\cos A\cos B\cos C}=\frac{p\alpha+q\beta+r\gamma}{(a+b+c)\cos A\cos B\cos C};$$

therefore locus of H' is

$$(a-2p)\,\alpha+(b-2q)\,\beta+(c-2r)\,\gamma=0,$$

a straight line which passes through $(b+c)/a$, $(c+a)/b$, $(a+b)/c$,* and is parallel to

$$\alpha\sin 2A+\beta\sin 2B+\gamma\sin 2C=0.$$

32. Since $\angle NN'Q=90°-B=\angle P'PM'$,

therefore arc NQ = arc $P'M'$,

and therefore $P'Q=M'N$,

and $\angle QP'C_1=\angle PRQ=\angle C$;

hence $P'QR'PQ'R$ is an equal hexagram.

The negative isoscelians are

$$P'Q\,(=M'N),\quad Q'R\,(=N'L),\quad R'P\,(=L'M);$$

and the positive ones are

$$QR'\,(=M'L),\quad PQ'\,(=N'M),\quad RP'\,(=L'N).$$

33. With centre N', the circle $C_1P'Q'$ passes through O, and the radius $=2\rho\cos C$; and N' is the orthocentre of ΔPQC.

* This point is the mid-point of TT' (see "A Group of Isostereans," *Proc.*, Vol. XIX., p. 219).

Since $$\angle PHQ = 180^\circ - C,$$

therefore circle O_1PQ passes through H.

34. Bisect the arcs PQ, QR, RP in W', S', V', respectively;

then $$\Delta S'V'W' = \Delta SVW,$$

because $$W'P' \text{ bisects } \angle QP'C_1, \text{ \&c.}$$

Now $$\angle W'V'Q = \angle W'RQ = C/2,$$

therefore $V'Q$ is perpendicular to $S'W'$; hence if H'' be the orthocentre of $S'V'W'$, the circle, centre S', through QR, will pass through H''; this point is also the incentre of ΔPQR.

[The envelopes of PQ, QR, RP, and of $P'Q'$, $Q'R'$, $R'P'$, can be easily obtained by reference to the foregoing results.]

35. We collect here some other results of interest. Suppose

BM, CN intersect in f, $\quad (a\alpha\mu'\nu = b\beta\nu'\mu' = c\gamma\mu\nu)$,

CN, AL „ in g, $\quad (a\alpha\lambda\nu = b\beta\lambda\nu' = c\gamma\lambda'\nu')$,

AL, BM „ in h, $\quad (a\alpha\lambda'\mu' = b\beta\lambda\mu = c\gamma\lambda'\mu)$.

Similarly, let

BM', CN' intersect in f', $\quad \left(\dfrac{\alpha}{\lambda\lambda'\cos A} = \dfrac{\beta}{\lambda'\mu'\cos B} = \dfrac{\gamma}{\lambda\nu\cos C}\right)$,

CN', AL' „ in g', $\quad \left(\dfrac{\alpha}{\lambda\mu\cos A} = \dfrac{\beta}{\mu\mu'\cos B} = \dfrac{\gamma}{\mu'\nu'\cos C}\right)$,

AL', BM' „ in h', $\quad \left(\dfrac{\alpha}{\nu'\lambda'\cos A} = \dfrac{\beta}{\mu\nu\cos B} = \dfrac{\gamma}{\nu\nu'\cos C}\right)$;

then the equations to ff', gg' are

$$a\alpha\left[\nu r\,.\,\lambda\mu\nu - \mu' q\,.\,\lambda'\mu'\nu'\right] + b\beta\left[\lambda p\,.\,\lambda'\mu'\nu' - \nu' r\,.\,\lambda\mu\nu\right]$$
$$+ c\gamma\left[\mu q\,.\,\lambda'\mu'\nu' - \lambda' p\,.\,\lambda\mu\nu\right] = 0,$$

$$a\alpha\left[\nu r\,.\,\lambda'\mu'\nu' - \mu' q\,.\,\lambda\mu\nu\right] + b\beta\left[\lambda p\,.\,\lambda\mu\nu - \nu' r\,.\,\lambda'\mu'\nu'\right]$$
$$+ c\gamma\left[\mu q\,.\,\lambda'\mu'\nu' - \lambda' p\,.\,\lambda\mu\nu\right] = 0;$$

whence we find that ff', gg', hh' cointersect in T, which is therefore the centre of perspective of the triangles fgh, $f'g'h'$.

36. If $(MN, M'N')$ meet in f'',

$(NL, N'L')$ „ in g'',

and $(LM, L'M')$ „ in h'',

then equation to Af'' is

$$\beta \cos C\,(\mu'\nu' q - \nu\lambda p) = \gamma \cos B\,(\lambda'\mu' p - \mu\nu r),$$

and Af'', Bg'', Ch'' cointersect in T'';

i.e.,

$$\frac{a}{\cos A\,(\nu'\lambda' r - \lambda\mu q)} = \beta/(\ldots) = \gamma/(\ldots).$$

37. For the intersections of $(MN', M'N;\, f''')$, $(NL', N'L;\, g''')$, $(LM', L'M;\, h''')$, we have equation to Ch''',

$$\frac{a}{\cos A\,(\lambda\mu q - \lambda'\nu' r)} = \frac{\beta}{\cos B\,(\lambda'\mu' p - \mu\nu r)};$$

hence CA, CT', CB, Ch''' form a harmonic pencil.

38. The equation to the circle $A_1B_1C_1$ is

$$4 \sin A \sin B \sin C \,.\, C = L\,(a \sin 3A + \beta \sin 3B + \gamma \sin 3C),$$

and the equation to B_1C_1 is

$$-aa \cos A + (b\beta + c\gamma) \cos (B-C) = 0.$$

39. The conic through intersections of ABC, $A_1B_1C_1$ is

$$\frac{\lambda}{ap \cos (C-A) \cos (A-B)} = \ldots = \ldots$$

$$= \frac{2\lambda'}{bc \cos (B-C) \left[\cos A \cos 2A - \cos (B-C)\right]} = \ldots = \ldots$$

If we call the points of intersection (of the triangles) on BC (1, 2), on CA (3, 4), on AB (5, 6), then (14), (25), (36) cointersect in the "N.P." centre; and if

(35, 46) intersect in λ_1

(51, 62) „ in μ_1

and (13, 24) „ in ν_1

then $A\lambda_1$, $B\mu_1$, $C\nu_1$ cointersect in

$$\frac{a}{\sin 2A \cos (B-C)} = \ldots = \ldots,$$

which is the point (β) of Question 9950 of the *Educational Times.*

40. The centroid of $L'M'N'$ is given by

$$\alpha/(\lambda' c\cos B+\lambda b\cos C)\,p = \beta/(\mu' a\cos C+\mu c\cos A)\,q$$
$$= \gamma/(\nu' b\cos A+\nu a\cos B)\,r.$$

Assume $\quad A'\equiv b\cos C-c\cos B,\quad B'\equiv c\cos A-a\cos C,$

$$C'\equiv a\cos B-b\cos A,$$

then the above may be written

$$\frac{\alpha}{pc\cos B+\lambda pA'}=\frac{\beta}{c\cos B\cos A\,(b-2B')+\lambda pB'}$$
$$=\frac{\gamma}{\cos B\,(ar-bC')+\lambda pC'},$$

or
$$\frac{A'\beta-B'\alpha}{A'c\cos A\,(b-2B')-B'cp}=\frac{A'\gamma-C'\alpha}{A'\,(ar-bC')-C'pc}$$
$$=\frac{C'\beta-B'\gamma}{C'c\cos A\,(b-2B')-B'(ar-bC')};$$

whence, after reduction, and writing

$$K\equiv a^2+b^2+c^2,$$

we get locus required to be

$$a\alpha\,(3a^2-K)+b\beta\,(3b^2-K)+c\gamma\,(3c^2-K)=0.$$

41. The centroid of LMN is given by

$$\frac{a\alpha}{\mu+\nu'}=\frac{b\beta}{\nu+\lambda'}=\frac{c\gamma}{\lambda+\mu'}=\frac{2\Delta}{3};$$

i.e.,
$$\frac{a\alpha qr}{qa\cos(B-C)-2cr\cos A\cos B-\lambda p\,(q-r)}$$
$$=\frac{b\beta r}{2r-q-\lambda\,(r-p)}=\frac{c\gamma q}{2c\cos A\cos B-\lambda\,(p-q)}$$
$$=\frac{b\beta r+c\gamma q}{r+p+\lambda\,(q-r)}=\frac{r\,(a\alpha+c\gamma)}{q+r+\lambda\,(r-p)}.$$

After reduction, we get for locus

$$a\alpha\,(p^2+q^2+r^2-3qr)+\ldots+\ldots=0.$$

42. To find the symmedian point of $L'M'N'$, we must cut $M'N'$

in a point P, so that

$$M'P : PN' = c^2 : b^2.$$

The coordinates of P will be given by

$$\alpha/(\lambda r+\lambda' q)\cos A, \quad \beta/\mu' r\cos B, \quad \gamma/\nu q\cos C,$$

and the equation to $L'P$ is

$$\alpha\sec A\,(\mu'\nu' r-\mu\nu q)-\beta\sec B\,\nu'\,(\lambda r+\lambda' q)+\gamma\sec C\,\mu\,(\lambda r+\lambda' q)=0.$$

Similarly, the symmedian line through M' is

$$\alpha\sec A\,\nu\,(\mu p+\mu' r)+\beta\sec B\,(\nu'\lambda' p-\nu\lambda r)-\gamma\sec C\lambda'\,(\mu p+\mu' r)=0.$$

Eliminating γ, and removing the factor $r(\lambda\mu\nu+\lambda'\mu'\nu')$, we get coordinates of symmedian point to be

$$\frac{\alpha\sec A}{\lambda r+\lambda' q}=\frac{\beta\sec B}{\mu p+\mu' r}=\frac{\gamma\sec C}{\nu q+\nu' p}$$

$$=P=2\Delta/\left[2\,(pq+qr+rp)-(p^2+q^2+r^2)\right]=2\Delta/(-R), \text{ say.}$$

Hence
$$a\alpha=P\left[p\,\{q-\lambda\,(q-r)\}\right],$$
$$b\beta=P\left[2pr-p^2-r^2+qr-p\lambda\,(r-p)\right],$$
$$c\gamma=P\left[q\,(p-q+r)-\lambda p\,(p-q)\right];$$

therefore $pa\alpha+qb\beta+rc\gamma=3pqrP=3pqr\,(a\alpha+b\beta+c\gamma)/[\ldots]$,

i.e., $a\alpha\,(pR+3pqr)+\ldots+\ldots=0$, the locus required.

This passes through

$$a\alpha/(q-r)=\ldots=\ldots,$$

and is, hence, parallel to $pa\alpha+qb\beta+rc\gamma=0$.

43. In like manner, to find the symmedian point of the triangle LMN, we cut MN in P (say), so that

$$MP : PN = r^2 : q^2.$$

The coordinates of P will be given by

$$\alpha/bc\,(r^2\nu'+q^2\mu), \quad \beta/car^2\nu, \quad \gamma/abq^2\mu',$$

and the equation to LP is

$$a\alpha\left[\lambda'\mu' q^2-\nu\lambda r^2\right]+b\beta\lambda\left[\nu' r^2+q^2\mu\right]-c\gamma\lambda'\left[\nu' r^2+q^2\mu\right]=0;$$

similarly, the symmedian line through M is

$$-a\alpha\mu'\left[p^2\lambda'+r^2\nu\right]+b\beta\left[\mu'\nu' r^2-\lambda\mu p^2\right]+c\gamma\mu\left[p^2\lambda'+r^2\nu\right]=0.$$

Eliminating γ, and removing the factor $r^2(\lambda\mu\nu+\lambda'\mu'\nu')$, we have

$$\frac{a\alpha}{r^2\nu'+q^2\mu}=\frac{b\beta}{p^2\lambda'+r^2\nu}=\left[\frac{c\gamma}{q^2\mu'+p^2\lambda}=\frac{2\Delta}{p^2+q^2+r^2}\right].$$

These may be written

$$\frac{a\alpha}{q(2r-p)+\lambda p(q-r)}=\frac{b\beta}{p^2+r^2-qr+\lambda p(r-p)}$$

$$=\frac{c\gamma}{q(p+q-r)+\lambda p(p-q)}=\frac{pa\alpha+qb\beta+cr\gamma}{pq(2r-p)+q(p^2+r^2-qr)+qr(p+q-r)}$$

$$=\frac{pa\alpha+qb\beta+cr\gamma}{3pqr};$$

i.e., $$3pqr(a\alpha+b\beta+c\gamma)=(p^2+q^2+r^2)(pa\alpha+qb\beta+rc\gamma);$$

whence $$a\alpha\left[p(p^2+q^2+r^2)-3pqr\right]+\ldots+\ldots=0$$

is the locus of the symmedian point of LMN. This line is also parallel to $pa\alpha+\ldots+\ldots=0$, and is therefore parallel to the line in § 42.

II. *Negative Hexagrams.**

44. The construction is made as in I., 1.

45. Write
$$\left.\begin{array}{ll} BL'=\lambda a, & CL'=\lambda' a \\ CM'=\mu b, & AM'=\mu' b \\ AN=\nu c, & BN'=\nu' c \end{array}\right\}.$$

Here λ, λ' may have the same values as in I. 3, but the μ, ν will be different.

Now $$2\nu' c\cos B=BL,$$
$$2\mu b\cos C=CL;$$

whence
$$\left.\begin{array}{l} a=2\mu b\cos C+2\nu' c\cos B \\ b=2\nu c\cos A+2\lambda' a\cos C \\ c=2\lambda a\cos B+2\mu' b\cos A \end{array}\right\};$$

* The geometrical properties of the two hexagrams are, of course, identical; but the analytical work is very different. Sometimes results come more easily by the negative hexagram equations.

i.e.,
$$2\mu' b\cos A = c-2\lambda a\cos B,$$
$$2\nu c\cos A = b-2a\cos C+2\lambda a\cos C,$$
$$2\nu' c\cos A = b-2\lambda a\cos C,$$
$$2\mu b\cos A = 2b\cos A-c+2\lambda a\cos B.$$

46. The trilinear coordinates may be written

$$\left.\begin{matrix} L', & 0, & \lambda' c, & \lambda b \\ M', & \mu c, & 0, & \mu' a \\ N', & \nu' b, & \nu a, & 0 \end{matrix}\right\}; \qquad \left.\begin{matrix} L, & 0, & \mu\cos C, & \nu'\cos B \\ M, & \lambda'\cos C, & 0, & \nu\cos A \\ N, & \lambda\cos B, & \mu'\cos A, & 0 \end{matrix}\right\}.$$

47. From § 45, we have the identical relation

$$0 = \begin{vmatrix} -1, & 2\mu\cos C, & 2\nu'\cos B \\ 2\lambda'\cos C, & -1, & 2\nu\cos A \\ 2\lambda\cos B, & 2\mu'\cos A, & -1 \end{vmatrix},$$

whence
$$8(\lambda\mu\nu+\lambda'\mu'\nu')\cos A\cos B\cos C-1$$
$$+4\left\{\mu'\nu\cos^2 A+\nu'\lambda\cos^2 B+\lambda'\mu\cos^2 C\right\}=0.$$

The triangles $L'M'N'$, LMN are respectively similar, as in § 8, to ABC, and its pedal triangle; and we find

$$\Delta L'M'N' = \Delta(\lambda\mu\nu+\lambda'\mu'\nu') = \rho'^2\Delta/R^2,$$

where ρ' is the hexagram radius. Also,

$$\Delta LMN = 8\Delta(\lambda\mu\nu+\lambda'\mu'\nu')\cos A\cos B\cos C.$$

48. We may note here that for the positive hexagram

$$LN'.ML'.NM' = \lambda\mu\nu\, abc/8\cos A\cos B\cos C,$$
$$LM'.MN'.NL' = \lambda'\mu'\nu'\, abc/8\cos A\cos B\cos C;$$

whereas for the negative hexagram

$$LN'.ML'.NM' = \lambda'\mu'\nu'\, abc,$$
$$LM'.MN'.NL' = \lambda\mu\nu\, abc.$$

49. We give here some results, referring to the corresponding articles for the positive hexagram.

§ 17: $\dfrac{\lambda_1}{\mu'\nu\cos A} = \dfrac{\mu_1}{\nu'\lambda\cos B} = \dfrac{\nu_1}{\lambda'\mu\cos C} = \dfrac{1-4\lambda\sin^2 A+4\lambda^2\sin^2 A}{2\cos^2 A(\lambda\mu\nu+\lambda'\mu'\nu')}.$

The equation to the radical axis is

$$\mu'\nu\alpha\cos A+\nu'\lambda\beta\cos B+\lambda'\mu\gamma\cos C = 0,$$

§ 18: projection $= a\,(2\lambda-1)/2 = R\sin A\,(2\lambda-1)$.

$$OL'^2 = R^2\,(1-4\lambda\sin^2 A+4\lambda^2\sin^2 A) = 4\rho'^2\cos^2 A.$$

§ 20:
$$2\alpha\cos A = -\left\{R\cos 2B+\lambda a\sin(B-C)\right\},$$
$$2\beta\cos A = \left\{R\cos(2A-C)-\lambda a\sin(C-A)\right\},$$
$$2\gamma\cos A = \left\{R\cos B-\lambda a\sin(A-B)\right\};$$

multiplying these respectively by $\sin 3A$, $\sin 3B$, $\sin 3C$, we get

$$\alpha\sin 3A+\ldots+\ldots = 0.$$

§ 23:

$$2R\cos A\cos(B-C) = -2R\cos 2B-(\lambda+\lambda_2)\,a\sin(B-C),$$

i.e., $R\,(\cos 2B+\cos 2C-2\cos 2B) = (\lambda+\lambda_2)\,a\sin(B-C)$;

therefore $\lambda+\lambda_2 = 1$.

NOTE.

[50. Referring to § 9, through P draw a parallel to BC, cutting the circle in P', and the perpendicular (AD) in J. Then

$$\angle PM'L' = \tfrac{1}{2}AM'N+\angle MM'L',$$

and $\angle MM'L' = \angle MNL' = C-\angle MNM'$

$$= C-(AM'N-NPM'),$$

therefore $\angle PM'L' = C+NPM'-\tfrac{1}{2}AM'N$

$$= C+A-90^\circ+NPM' = \angle APP';$$

then $PL' = 2\rho\sin PM'L' = 2\rho\sin APP'$,

and $AJ = AP\sin APP' = PN\sin APP'$

$$= 2\rho\cos A\sin APP' = PL'\cos A.$$

Again, by § 14, perpendicular from P on $BC = PL'\cos(B-C)$; hence J is the mid-point of AH, and $PL' = R$.

51. Let the normal at a point P of an ellipse cut the axes in G, g, and parallels to the axes, through P, meet them in N, n; then, with the usual notation, if $Pg = r$,

because $PF\,.\,Pg = AC^2 = a^2$,

we have $r^2-a^2 = r\,.\,gF = Cg\,.\,gn$ (because C, F, P, n are concyclic);

but since $CG = e^2\,.\,Pn$,

therefore $r^2-a^2 = (Cg)^2/e^2 = d^2/e^2$, say.

Hence, if from any point on the minor axis as centre, with gP as radius, we describe a circle to touch the ellipse, we have the above relation.

Now, if we call the distance between O' and the "N.P." centre d, and put θ for the angle which the line connecting them makes with BC and $2a$, ϵ, for the major axis and eccentricity of the ellipse of § 22, we have

$$2d\cos\theta = BL+BL'-R\{2\sin A+\sin(C-B)\}$$

$$= \frac{1}{2\cos B\cos C}\{2\lambda a\cos B\cos C+b\cos B-\lambda a\cos A\}-R\{\ldots\}$$

$$= \frac{R}{\cos B\cos C}\{2\cos B\cos C-\cos A\}\{\lambda\sin A-\cos B\sin C\};$$

but $$OH\cos\theta = 2\cos B\cos C-\cos A,$$

therefore $$d\cos B\cos C = (\lambda\sin A-\cos B\sin C)\,a\epsilon,$$

i.e., $$d^2\cos^2 B\cos^2 C = a^2\epsilon^2.\{\lambda^2\sin^2 A-2\lambda\sin A\cos B\sin C+\cos^2 B-\cos^2 B\cos^2 C\}$$

$$= a^2\epsilon^2.\{4\rho^2/R^2-1\}\cos^2 B\cos^2 C, \text{ by § 18},$$

i.e., $$\frac{d^2}{\epsilon^2} = a^2\left[\frac{4\rho^2}{R^2}-1\right] = \rho^2-a^2;$$

hence, if we take $\rho = r$, we see that the "H." circles touch the in-ellipse of § 22.

From the above result we at once get

$$\rho^2\epsilon^2 = d^2+a^2\epsilon^2 = OO'^2 \text{ (or } O'H^2\text{)},*$$

therefore $$OO' = O'H = \rho\epsilon;$$

and $$2\rho\cos O'OH = 2a = R, \; cf. \text{ § 15}.$$

52. If we suppose the points L, L' to coincide† so that BC is a tangent to the H. circle, then, if ρ_a be the radius of the circle, we

have $$LN' = 2\rho_a\sin B, \quad LM = 2\rho_a\sin 2C;$$

hence $$a = 2\rho_a\{\sin 2B+\sin 2C\},$$

i.e., $$2R\sin A = 4\rho_a\sin A\cos(B-C),$$

* This result follows also from §§ 21, 22.
† The reader is requested to draw the figure.

or
$$\left.\begin{aligned} R &= 2\rho_a \cos(B-C), \\ &= 2\rho_b \cos(C-A), \\ &= 2\rho_c \cos(A-B), \end{aligned}\right\}$$

if ρ_b, ρ_c are the radii when the circle touches CA, AB respectively.

The points L, M, N (when M, M'; N, N' coalesce) are readily seen to be the points of contact of the sides with the in-ellipse, so that AL, BM, CN (in this case) cointersect in T.

53. Since $$BN \,.\, BN' = BL^2 = 4\rho_a^2 \sin^2 2B,$$

and $$BN' = LN' = 2\rho_a \sin B,$$

therefore $$BN = 2\rho_a \sin B \,.\, 4\cos^2 B;$$

hence $$NN' = 2\rho_a \sin 3B \text{ and } MM' = 2\rho_a \sin 3C.$$

Now the angle
$$O'N'A = (\pi-2C)+(\pi-2A)-\left(\frac{\pi}{2}-B\right) = 3B-\frac{\pi}{2},$$

and $$\angle\, O'MA = \frac{3\pi}{2}-3C;$$

hence coordinates of O' are
$$\rho_a, \quad -\rho_a \cos 3C, \quad -\rho_a \cos 3B.$$

54. The equation to the circle, which passes through
$$N'\left\{b\sin B,\ a\sin(2C-B),\ 0\right\},\quad L\,(0,\ c^2\cos C,\ b^2\cos B),$$
$$M'\left\{c\sin C,\ 0,\ a\sin(2B-C)\right\},$$

is
$$a\beta\gamma+\ldots+\ldots = \frac{(a\alpha+\ldots+\ldots)}{4c\sin A\sin B\cos^2(B-C)}$$
$$\times\left[2a\alpha\cos A\sin(2C-B)\sin(2B-C)+\beta b\sin^2 2B+\gamma c\sin^2 2C\right].$$

55. If we project $O'O$ on BC, and on a perpendicular thereto, we get
$$\begin{aligned}(O'O)^2 &= \rho_a^2\left[(\sin 2B \sim \sin 2C)^2+(1+\cos 2B+\cos 2C)^2\right] \\ &= \rho_a^2\left[3+2(\cos 2A+\cos 2B+\cos 2C)\right] \\ &= \rho_a^2\left[1-8\cos A\cos B\cos C\right];\end{aligned}$$

therefore $$O'O = \epsilon\rho_a, \quad cf.\ \S\, 51.$$

Again,

$$OL^2 = \left[2\rho_a \cos A \sin (B \sim C)\right]^2 + \left[2\rho_a \cos A \cos (B \sim C)\right]^2$$
$$= 4\rho_a^2 \cos^2 A,$$

therefore $$OL : OO' : LO' = 2\cos A : \epsilon : 1;$$

and $\angle OLO' = B \sim C$, therefore $\angle OLH = 2(B \sim C)$.

From the above, we have

$$\epsilon \cos OO'L = 1 + \cos 2B + \cos 2C;$$

and $$\sin HLC = \sin OLB = R\cos A / 2\rho_a \cos A = \cos (B \sim C),$$

therefore $$\angle HLC = \frac{\pi}{2} - (C \sim B) = \phi.$$

56. If ρ_ϵ is the radius of curvature of the ellipse at L, we have

$$(\alpha\beta\rho_\epsilon)^{\frac{2}{3}} \sin^2 \phi = \beta^2;$$

therefore

$$\rho_\epsilon = \beta^2 \operatorname{cosec}^3 \phi / \alpha = 4R \cos A \cos B \cos C / \cos^3 (B - C)\big].$$

57. Let Δ' be the area of the triangle $t_1 t_2 t_3$ (§ 22), and put

$$\mathfrak{D} \equiv \cos (A - B) \cos (B - C) \cos (C - A);$$

then $$\Delta - \Delta' = R^2 \left[\sin^2 2A \sin A \cos (B - C) + \ldots + \ldots\right] / 2\mathfrak{D}$$
$$= R^2 \left[\sin 2A \sin 2B (\sin 2A + \sin 2B) + \ldots + \ldots\right] / 4\mathfrak{D}$$
$$= R^2 \sin A \sin B \sin C$$
$$\times \left[\{1 + \cos 2A + \cos 2B + \cos 2(A - B)\} + \ldots + \ldots\right] / 2\mathfrak{D}$$
$$= R^2 \sin A \sin B \sin C \left[1 + \cos 2A + \cos 2B + \cos 2C\right.$$
$$\left. + 2\cos (A - B) \cos (B - C) \cos (C - A)\right] / \mathfrak{D}$$
$$= \Delta - R^2 \sin A \sin B \sin C \,(4 \cos A \cos B \cos C) / \mathfrak{D},$$

i.e., $$\Delta' = 2\Delta \cos A \cos B \cos C / \mathfrak{D}.$$

58. From the Geometry of the Ellipse, the joins of A (B, C) and of the "N.P." centre bisect $t_2 t_3$ $(t_3 t_1, t_1 t_2)$.

59. If t_1', t_2', t_3' are the mid-points of $t_2 t_3$, $t_3 t_1$, $t_1 t_2$, then equations of $t_1 t_1'$, $t_2 t_2'$ are

$$a\alpha \sin 2A p_1 + b\beta \sin 2B p_2 - c\gamma \sin 2C p_3 = 0,$$
$$-a\alpha \sin 2A q_2 + b\beta \sin 2B q_1 + c\gamma \sin 2C q_3 = 0,$$

where $p_1 \equiv c \sin 2C \cos (A-B) - b \sin 2B \cos (C-A)$,

$p_2 \equiv$ " + "

and similar expressions for q, r; hence we get the coordinates of the centroid of $t_1 t_2 t_3$ to be

$$\alpha / p_2 \cos (B-C) = \beta / q_2 \cos (C-A) = \gamma / r_2 \cos (A-B).$$

We may put p_2 into the form

$$R \left[E + \sin 2B \sin 2C \right],$$

where $E \equiv \sin 2A \sin 2B + \ldots + \ldots$.

60. The circle $t_1 t_2 t_3$ has for its equation

$$C = L (\lambda\alpha + \mu\beta + \nu\gamma),$$

where
$$\lambda = \frac{\begin{vmatrix} 0 & c \sin 2C & b \sin 2B \\ c \sin 2C & 0 & a \sin 2A \\ b \sin 2B & a \sin 2A & 0 \end{vmatrix}}{R \sin^2 2A \begin{vmatrix} \dfrac{\sin 2B \sin 2C}{\cos (B-C)} & c \sin 2C & b \sin 2B \\ \dfrac{\sin 2C}{\cos (C-A)} & 0 & a \\ \dfrac{\sin 2B}{\cos (A-B)} & a & 0 \end{vmatrix}};$$

hence equation is

$$2abc\mathfrak{D} \,.\, C = L \begin{bmatrix} a^3\alpha \cos A \; (\sin B \sin C + 2 \cos A \cos 2B \cos 2C) \\ + \ldots \qquad\qquad + \ldots \end{bmatrix},$$

61. If we suppose our "H." circle to touch the sides CA, AB in M, N, then the hexagram is $NLMNL'MN$, *i.e.*, MN is a doubled line. From a consideration of the figure, we see that

$$A = 180^\circ - 2A,$$

i.e., AMN must be an equilateral triangle, and we can further readily prove that M, N are the points t_2, t_3.

The radius of the "H." circle in this case is

$$(\sqrt{3} + 2 \sin 2B)/c = 1/\rho = (\sqrt{3} + 2 \sin 2C)/b.$$

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