Philosophical Magazine Series 5

ISSN: 1941-5982 (Print) 1941-5990 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm16

LVII. On the beats of consonances of the form h:1

R.H.M. Bosanquet

To cite this article: R.H.M. Bosanquet (1881) LVII. On the beats of consonances of the form h:1, Philosophical Magazine Series 5, 11:70, 420-436, DOI: 10.1080/14786448108627039

To link to this article: http://dx.doi.org/10.1080/14786448108627039

LVII. *On the Beats of Consonances of the Form* h : 1. *By* R. H. M. Bosanquet[*].

[Plates IV.–VII.]

Ohm's Law, and the Hypothesis of Resultant Displacements.

1. THE doctrine known as Ohm's law states that the simplest form of motion by which definite musical pitch is defined to the ear is the pendulum-vibration. It may be extended as follows:—In all cases in which Ohm's law holds, the ear resolves any complex of two or more simultaneous pendulum-vibrations into the original pendulum-vibrations of which they consist, and hears them as distinct and independent sounds.

I rest my belief in Ohm's law mainly on ordinary phenomena, not on refinements or difficult observations; and I shall endeavour to make this my course throughout.

2. So long as, in our mechanical arrangements, we approximate more and more nearly to the condition of things by which we know that simple harmonic vibrations must be produced, we also approximate in the character of the resulting sound to a pure and simple note of definite pitch, free from harmonics and other accompaniments.

So far as simple sounds are concerned, therefore, we receive Ohm's law as being at all events approximately true generally, and in all probability absolutely true when sounds of small intensity alone are considered.

3. When two different sounds are heard together, we have phenomena of which the following is a slight sketch.

If the two sounds are very nearly of the same pitch, they are not heard according to Ohm's law, *i. e.* separately and independently, but in the form of resultant displacements. The most important case is that in which the two sounds are of nearly equal intensity. In this case one sound is heard, intermediate in pitch between the two primaries, and oscillating in intensity between a certain maximum and nothing. These oscillations are what are called the beats of imperfect unisons. Now, as the two notes separate from one another in pitch, the character of the phenomenon changes; and at a certain point the two notes begin to be heard separately and independently, beside the beats which accompany them. It is this phenomenon that is accounted for by Helmholtz's theory of the existence of vibrating bodies, in the ear, having sympathy of a certain definite degree with the various notes.

4. Helmholtz ascertained the approximate degree of this

---

[*] Communicated by the Physical Society.

sympathy by considerations of a somewhat indirect character. I wish to point out the important bearing, on the theory, of the direct determination of the interval which separates the region in which two notes are heard only as resultant displacements, from that in which they begin to be heard separately, in accordance with Ohm's law.

5. The experiments I have made on this point have been mostly conducted by means of my enharmonic organ, in which I have a collection of notes separated for the most part by single commas.

The results, so far as I have gone, are:—1. The critical interval, at which two notes begin to be heard beside their beats, or resultant displacements, is about two commas, throughout that medium portion of the scale which is used in practical music. 2. This critical interval appears not to be exactly the same for all ears. In my own case two notes two commas apart are not heard distinctly beside the beats. In the case of Mr. Parratt, who has kindly examined the point with me, two notes two commas apart are distinctly heard beside the beats. In both cases the beats alone are heard with an interval of one comma, and the two notes are quite clear beside the beats with an interval of three commas.

I propose to undertake further experiments, with the view of determining this initial interval more accurately. So far as the above results go, they are quite consistent with Helmholtz's assumption as to the degree of sympathy of the ear.

6. Independently of any theory, the fact that at a certain point the ear begins to separate out two independent pendulum-vibrations from the resultant displacements, is one that must be recognized. It is easy to show that it is inconsistent with that representation of facts which assumes that beats arise only from the resultant forms exhibited by the superposition of the two vibrations on one receptive mechanism. This is shown as follows.

7. If we combine two vibrations of equal amplitude, which we may take $=1$, $\cos pt$ and $\cos(qt-\epsilon)$, on the same receptive mechanism, the effect is to produce a resultant displacement represented by

$$2\cos\frac{(p+q)t-\epsilon}{2}\cdot\cos\frac{(p-q)t+\epsilon}{2}.$$

This would be heard, by a hypothetical ear receiving the whole disturbance on the same sensorium, as a note whose frequency is the arithmetic mean between the frequencies of the two primaries, and having oscillations of intensity whose frequency is defined by a pendulum-vibration of frequency equal to half the difference of the frequencies of the primaries. This is what is actually heard in the case of two notes less than two commas apart.

8. When the interval is greater than two commas, this ceases to represent the whole phenomenon perceived by the ear as it exists. The separate notes step in beside the resultant form represented by the above expression, with its beats and note having the frequency of the arithmetic mean. As the interval increases, the separate notes become more and more prominent, and the beats diminish in loudness and distinctness, till, by the time that a certain interval is reached, which is about a minor third in the middle of the scale, the beats practically disappear and the two notes alone survive.

9. It has been supposed by some that the beats disappear only in consequence of their rapidity*; and it is clear that under this supposition, as ordinarily made, lies the assumption that the mass of tone continues to be received in the same manner all the time—*i. e.* that the phenomena of the beats of imperfect consonances and combination-tones are to be explained by reasoning analogous to that of the above formula, which supposes the whole displacement reduced to its resultant on one receptive mechanism. This, for instance, is assumed whenever Smith's or Young's theories of beats are admitted as sufficient explanations of the phenomena.

10. In such cases, (*a*) it is forgotten that the fundamental assumption carries another consequence with it than those it was desired to explain; (*b*) the explanation itself also fails in an important point.

(*a*) The other consequence is, that if it were true that the receptive mechanism of the ear received a resultant displacement, so that the combination was as represented by the above formula, then the primary notes would not be heard at all, and the note that would be heard would have the arithmetic mean of the frequencies of the primaries.

E. g., in the case of a fifth (4 : 6) the note heard would be the major third (5), which would beat very rapidly; just as, when I myself hear the resultant of notes two commas apart, it is one note midway between them beating rapidly. But, as a matter of fact, the note 5 is not heard at all in the above case.

(*b*) Again, supposing that in some unexplained way the beats whose speed is $\frac{p-q}{2}$ in the above notation gave rise to a note, as supposed by König. Then the speed of that note

---

* This is absolutely disproved by the argument in Helmholtz's *Tonempf.* p. 286, ed. 4.

does not agree with that required for König's first beat-note, which has the same speed as Helmholtz's difference-tone, or $(p-q)$ in the above notation.

11. The relationship of these resultant displacements to the phenomena in the general case, is most conveniently studied by means of the curves drawn by Donkin's harmonograph. The instrument in my possession has a rather restricted number of change-wheels; and one of my first tasks in the St. John's College laboratory has been to cut additional change-wheels for this instrument, for the purpose of illustrating this subject graphically*. (See Plates IV. to VII.)

12. The examination of the curves leads us to the following conclusions.

In every case, whether of beats of unisons, or of beats of imperfect consonances, the examination of the curves shows a portion of a harmonic curve lying through the vertices of the single resultant vibrations, which portion corresponds in duration to the beats as given either by Smith's rule or the ordinary rule for beats.

The durations of these harmonic curves are different in different cases. Three principal types may be distinguished:—

Let E, F be the amplitudes; $p : q$ the ratio in lowest terms of the exact consonance whose small variation is considered $(q>p)$.

(1) If $E \mid p$ is considerably less than $F \mid q$, there are q complete harmonic curves both at top and bottom, and the duration of each is q times that of the Smith's beat.

(2) If $E \mid p = F \mid q$, there are $p+q$ complete harmonic curves which may be called external, passing both top and bottom, and the duration of each external curve is $p+q$ times that of the Smith's beat; also there are $q-p$ internal curves, which lie nearer the middle; the duration of each internal curve is $q-p$ times that of the Smith's beat.

(3) If $E \mid p$ is considerably greater than $F \mid q$, there are p harmonic curves both at top and bottom. They are not complete, but appear to form portions of curves of long period.

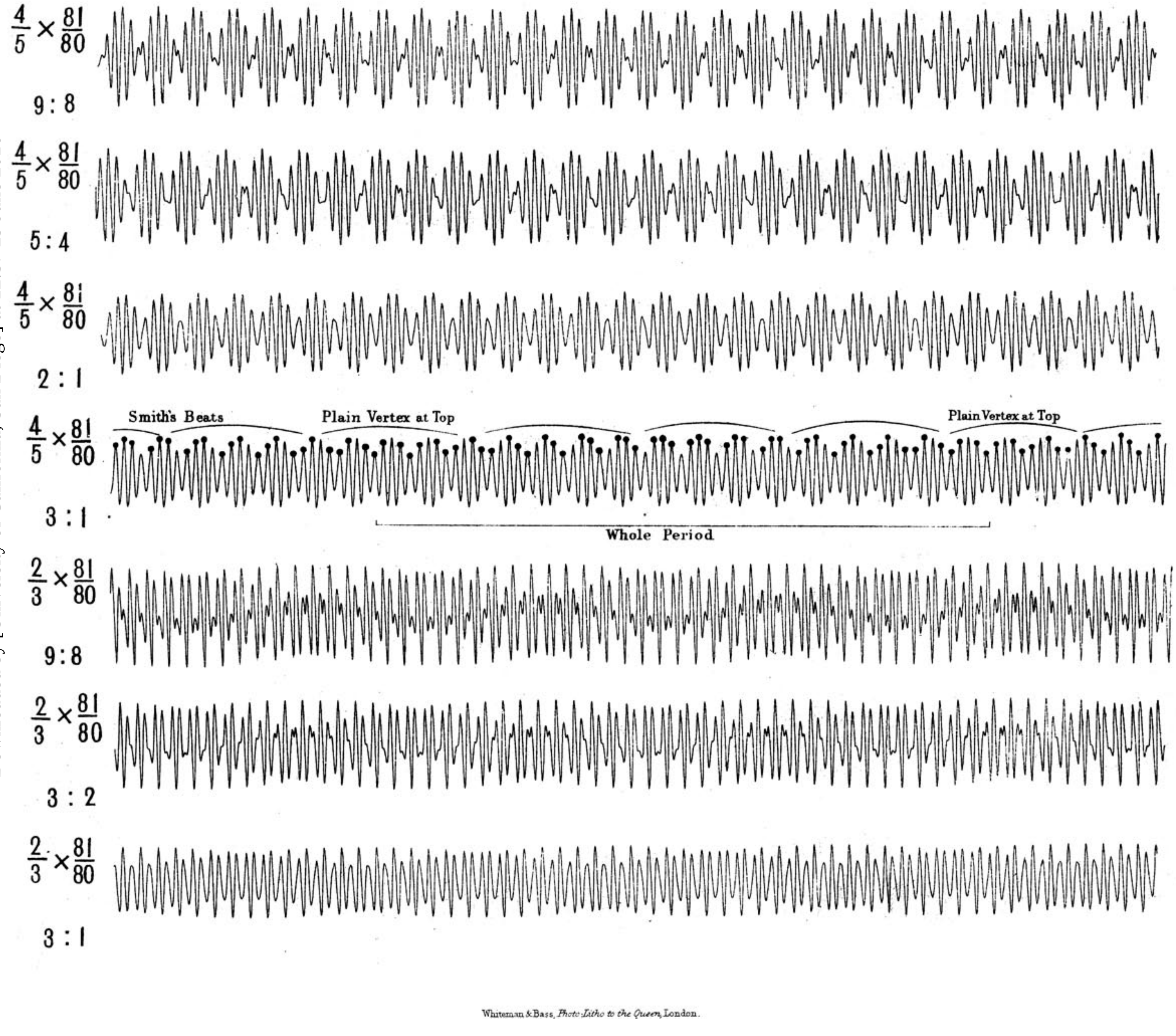
13. In all cases the curves which correspond to the beats, as ascertained by Smith's method or the ordinary formula, lie like series of bows, one series at the top and the other at the bottom.

The complete period of the pendulum-vibration, of which each of these bows forms a part, is always longer than the single bow or Smith's beat, according to the above rules.

14. Now, according to a well-known principle of mechanics, no pendulum-vibration can give rise to one of another period,

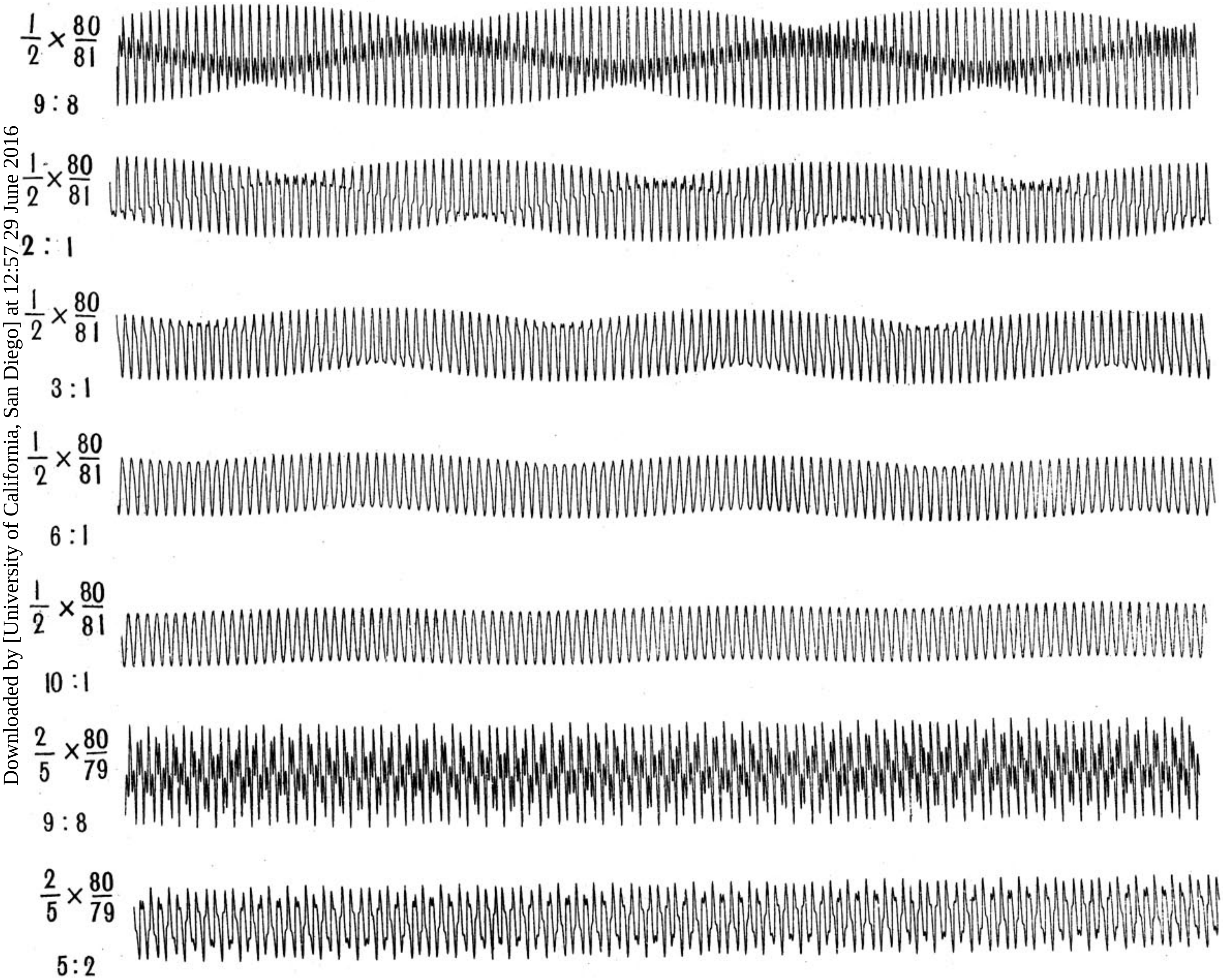
---

* These curves are of such interest that I devote some space to their discussion, § 77 &c.

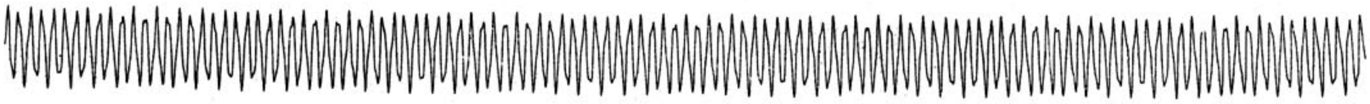


Whiteman & Bass, *Photo-Litho to the Queen*, London.

Phil. Mag. S. 5. Vol 11. Pl.V.



Whiteman & Bass, *Photo-Litho to the Queen*, London.



$\frac{2}{5} \times \frac{80}{79}$

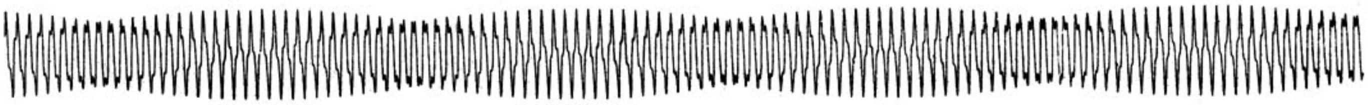
9 : 2

$\frac{27}{80}$

9 : 8

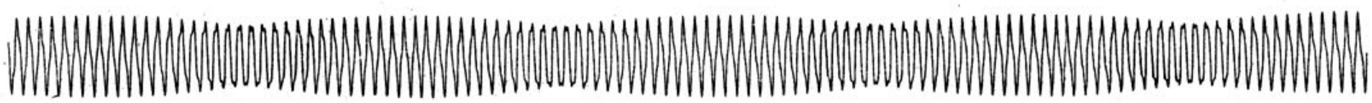
$\frac{27}{80}$

2 : 1



$\frac{27}{80}$

3 : 1



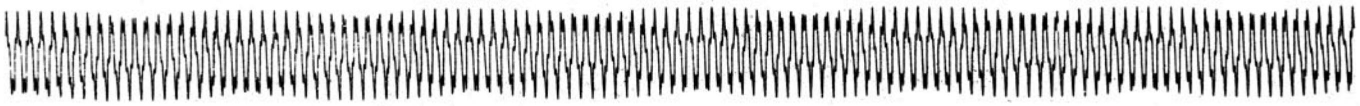
$\frac{27}{80}$

9 : 2

$\frac{1}{4} \times \frac{78}{79}$

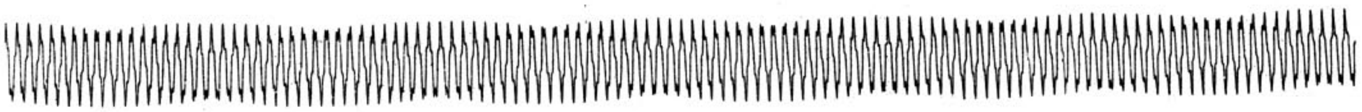
9 : 8

Whiteman & Bass, *Photo-Lith^s to the Queen*, London.



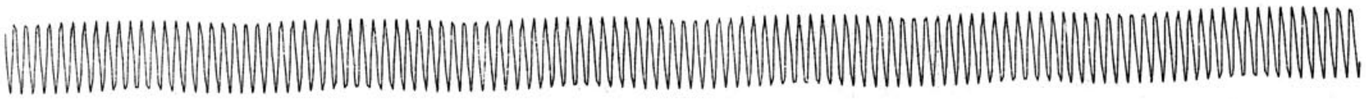
$\frac{1}{4}\times\frac{78}{79}$

3 : 1



$\frac{1}{4}\times\frac{78}{79}$

4 : 1

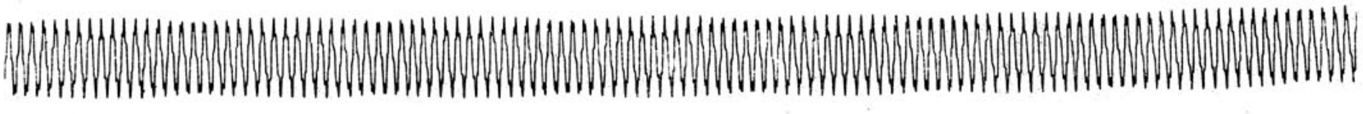


$\frac{1}{4}\times\frac{78}{79}$

10 : 1

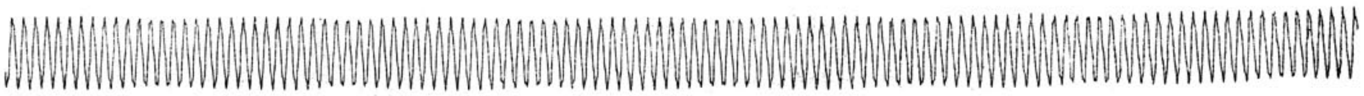
$\frac{1}{5}\times\frac{80}{79}$

3 : 1



$\frac{1}{5}\times\frac{80}{79}$

5 : 1



$\frac{1}{5}\times\frac{80}{79}$

10 : 1

Whiteman & Bass, *Photo-Litho to the Queen*, London.

in a system in which the forces are proportional to the displacements.

15. In the present case, if we find a term present whose whole period is that of the Smith's beat, it must therefore arise by transformation, *i. e.* through the presence of terms of higher orders than the first. We shall use generally the expression "transformation" to signify the effect on a system of terms higher than the first in the expression for the forces. This is substantially Helmholtz's explanation of the difference-tone, which is identical with the lowest beat-note of König.

16. We shall show that all König's beat-notes can be accounted for in a similar manner, by the assumption that terms of higher orders become important in the mechanism of the ear when the displacements are considerable.

17. We can illustrate further the difference produced in the curves by the admission of the difference-tone or beat-note as part of the mass of sound. The characteristic difference is, that the medial line is itself bent into a curve, whose whole period is that of the Smith's beat. I have not been able to draw any long curve to show this; but the appearance of the curve at the top of illustration A (p. 425) is very like it in a general way. This illustration represents the square term of the force developed by a fifth (2 : 3) in a transforming system. B is the figure of the total disturbed force in a similar case; but the throwing-up of the medial line is not so prominent as it would be in a longer curve. I have, however, no machine that will draw the combination; and the construction of a long curve of this kind is not worth the labour it would entail.

18. We sum up this part in the following conclusions:—

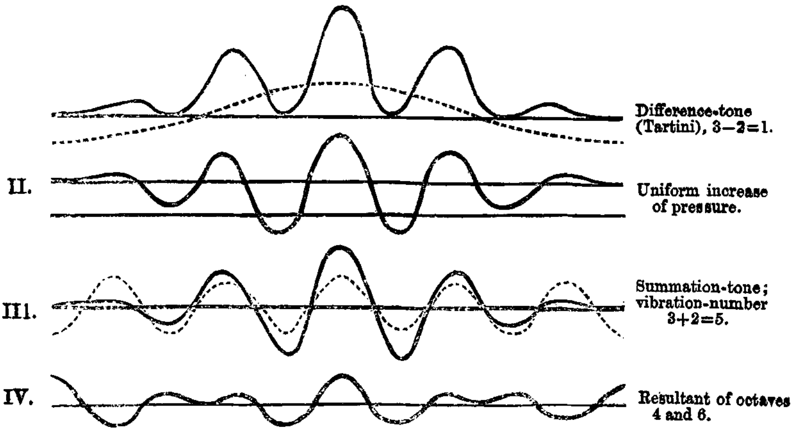
As two notes of equal amplitudes separate from unison, they are at first received by the ear in the manner of resultant displacements, consisting of the beats of a note whose frequency is midway between that of the primaries.

When the interval reaches about two commas, the ear begins to resolve the resultant displacements, and the primary notes step in beside the beats.

When the interval reaches a minor third in the ordinary parts of the scale, neither the beats nor the intermediate pitch of the resultant note are any longer audible, at least as matter of ordinary perception; but the resultant displacement which reaches the ear is decomposed, and produces the sensation of the two primary notes, perfectly distinct from each other: that is to say, Ohm's law has set in, and is true, for ordinary perceptions and in the ordinary regions of the scale, for the minor third and all greater intervals.

A.

Statical Analysis of Disturbance proportional to Square of Displacement, showing origin of the combination-tone. Fifth 2 : 3.



I. From disturbance proportional to square of displacement subtract simple tone. Amplitude ½ of disturbance. Period = that of the beat. (Vibration-number = difference of vibration-numbers, 3−2=1.)

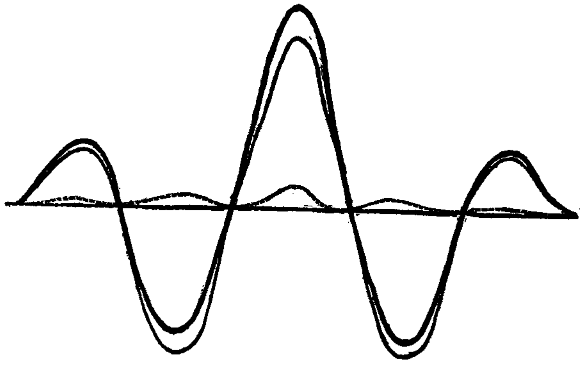
II. Subtract ¼ of amplitude of disturbance.

III. Subtract a simple tone. Amplitude ¼ of that of disturbance. Period = ⅕ that of resultant vibration in original beat. (Vibration-number = sum of vibration-numbers of original tones, 3+2=5.)

IV. There remains a succession of beats like the original beats of the fifth, an octave higher, *i. e.* the combination of the octaves of the original notes.

B.

Beat of Fifth, 2 : 3. *Disturbed by term proportional to square of displacement, showing origin of the combination-tone.*



19. We may notice here incidentally that it is necessary that the resolved primaries should be uniform and steady, in order that the beats exhibited in the resultant forms may retain their regularity. Those who support the Young-and-Smith theory generally have a sort of confused idea that the primaries are modified when superposed into their resultant.

20. How, then, do the beats of mistuned consonances arise? They may be regarded as springing from interference of new notes, which arise by transformation, in the passage of the resultant forms through the transmitting mechanism of the ear, before the analysis by the sensorium.

Experiments.

21. The engine and bellows* being adjusted to run continuously and quietly, I began to follow the course of König's experiments at the point where he deals with the combinations of the note C, following his form not accurately, but with such divergences as the difference in the apparatus suggested. After going through one or two sets in the way hereinafter described, I concentrated my attention on the analysis of beats, and specially on those of mistuned consonances of the form $h:1$. It will be seen that after a time I entirely discarded resonators, having convinced myself that, so far as they were concerned, the beats of mistuned consonances, other than unisons, with the beat-notes, difference- and combination-tones of all orders, and, in fact, all that I had to observe, were of a purely subjective nature, and were extinguished by resonators properly used, so far as my arrangements enabled me to perceive.

22. The mode in which I then pursued the observations on the beats of the mistuned consonances in question was, to adjust the notes and leave them sounding uniformly and continuously by the hour together. I then walked about the room listening to the combination in all the various forms in which it presented itself, went outside and came in again, always keeping in view the question, what are the sounds that these beats consist of?

23. It is hard to believe that a question to which the answer is tolerably simple could be so difficult. Yet it is *very* difficult; it is one of the most difficult things I ever tried to do, to analyze these apparently complicated sounds into their elements by the ear alone. And when I state my results, I must not be taken to mean that the elements I mention are all that are present. In fact, one of the great difficulties is that there appear to be such a number of different sounds.

---

* See Phil. Mag. Oct. 1880.

Some of these are probably due to the imagination; others probably exist in small intensity. And I am satisfied that there still exists a large field of work in the further prosecution of this subject. But of the main result I have no doubt whatever; and that is:—

24. The beats of mistuned consonances of the form $h:1$, where h is nearly some whole number, consist mainly of variations of intensity of the lower note when the beats of the harmonics are eliminated.

25. I was prepared for this result in the case of the octave by my preliminary experiments (Phil. Mag. viii. p. 293), but did not proceed further till I had verified it and got my ear to perceive it readily under the new conditions, which required two or three days. I then got Mr. Parratt to come and listen. He was much disturbed by the trifling noises from the engine, belts, &c.; and I blew the bellows myself for a time. Eventually he came to the same conclusion, but with an amount of hesitation and difficulty which showed me what an important element practice is in these observations.

26. I then started these observations with the mistuned twelfth, proceeding in the same way. I seemed to have the same difficulty as before in seizing the phenomenon; and when I eventually decided that these beats were also on the lower note, it was not in pursuance of any preconceived conclusion; for I had no idea at that time of the explanation I now give, and certainly none of the presence of the second difference-tone, or its identity with one of König's beat-notes.

27. Having got so far, I found the remaining verification, of the beats of the mistuned double octave, somewhat easier. These are also on the lower note when they are heard. I have never heard the beats of a mistuned consonance with any wider interval, with the notes I employ, as clear and unmistakable phenomena. Such beats may be discernible by more acute ears, or with notes of a more powerful quality, as they were discerned by König. But in such cases it will be incumbent on the observer to purge the beats from the suspicion of containing the beats of harmonics, as I have done.

28. Mr. Parratt subsequently convinced himself, as before, that the beats of the twelfth and double octave were all heard on the lower notes. I endeavoured, as far as possible, to make his observations independent by avoiding communicating my conclusions to him beforehand.

29. The elimination of the beats of the harmonics depends on the following considerations. The notes employed were examined, with and without resonators, as to the presence of harmonics. These, so far as they are objective, are readily

detected with resonators. The beats of the harmonics, where they existed objectively, were also examined with resonators. After a little practice the sound of these beats became familiar enough to prevent their being confused with the beats of the low notes, and the two sets of beats could be observed independently.

30. The only harmonic that exists in these notes in sensible intensity is the twelfth; and this does not appear to originate in the same manner as the principal note. It is heard separately, as it were, and as if it had an independent origin. It seems probable that it arises in connexion with the movements of the air about the mouth-piece, and not by resonance in the cavity of the bottle, like the principal note. At all events, whatever the cause may be, the effect is that the presence of this note is readily distinguished and allowed for, and there is no risk of its being mixed up with the rest of the phenomenon.

31. The notes employed are of moderate strength. It seems to me that the employment of notes of great power is open to the objection that it introduces all sorts of transformations depending on the greatness of the diplacements; and in this respect alone König's procedure is open to considerable objection. I have confined myself to notes of moderate strength, lying in those regions of the scale which are in ordinary use in music. It is phenomena thus presented that we really seek to understand; and I do not think that any thing is gained by pushing the investigation into those extreme regions where it is possible and highly probable that the ordinary laws of hearing become modified.

32. The first series of notes examined in the above manner were the set of pairs

C : *c*
C : *g*
C : *c′*

C : *e′*
C : *g′*
C : *c″*

The beats produced by mistuning, when cleared of the harmonic beats, were heard only in the first three cases.

The second set of pairs was

c : *c′*
c : *g′*

---

c : *c″*
c : *e″*
c : *g″*

The beats in question were only heard in the first two cases.

The third set was

$$\frac{c' : c''}{\begin{matrix} c' : g'' \\ c' : c''' \end{matrix}}$$

The beats in question were only heard in the first case.

33. In the few experiments hitherto made with notes of higher pitch, the beats of mistuned consonances of the form $h:1$ were not heard when the beats of the harmonics were eliminated, unless the power of the notes was very greatly increased. In this region, however, König's own observations are very full and complete.

34. We notice at once the decrease in the range within which the phenomena are heard as we rise in the scale. This is at once accounted for on the hypothesis of transformation, by the consideration that the displacements to which the higher notes give rise are much smaller than those of the lower notes. If we knew the law of the decrease, we might obtain a relation between the coefficients of the different terms in the expression for the character of the transforming mechanism. König has attempted to formulate a law of decrease; and I have done so on a previous occasion; but this part of the subject is as yet too hypothetical to admit of satisfactory treatment.

Objective and Subjective Phenomena.

Resonators.

35. On beginning work I endeavoured, in the first instance, to ascertain what evidence resonators are capabable of furnishing as to the nature of binary combinations. There are a few points connected with their use which require attention.

36. I have always found difficulty in getting results of a definite character with resonators, whether applied directly to one ear in the manner described by Helmholtz, or connected with one ear by means of a flexible tube, as practised by others. There are three difficulties which occur: (1) pressing the tube or orifice into the ear is apt to close the inner passage of that organ; (2) if the tube or orifice is applied lightly, it does not completely occupy the passage, and external sound comes past it into the ear; and (3) it is impossible so to stop the unused ear as to prevent the external impressions from arriving there and causing confusion.

37. The method I ultimately adopted was as follows:—A copper tube of $\frac{1}{4}$ inch diameter was bent into a semicircle, the diameter of which was nearly 8 inches. At the middle of the tube, and at right angles to its plane, another copper tube was

inserted, 2 inches long, which tapered down to an orifice $\frac{1}{8}$ inch in diameter; this served to communicate with the interior of a resonator by means of a small flexible tube. The extremities of the semicircle were turned inwards and upwards; and into them two brass tubes were inserted, $\frac{3}{4}$ inch long and $\frac{1}{8}$ inch in internal diameter, screwed on the outside. Over each of these was fitted a brass tube, screwed inside, carrying an ivory nipple, such as is used for ear-trumpets. I generally covered the nipple with a couple of thicknesses of thin india-rubber tube.

38. When used, the semicircle is passed under the chin with the resonator-attachment projecting in front. The nipples are at first screwed back as far as possible, brought opposite the orifices of the ears, and then screwed forward until they enter the ears. They are then gradually advanced until the passages are closed to external sounds. Something depends on the way the tube is held. With practice it is possible to hold it so that the passages are closed to external sounds without screwing the nipples in very tight. When they are screwed very tight, it is rather unpleasant, and even painful. But it is necessary constantly to be on one's guard against being deceived by an occasional entrance of external sounds if the nipples are not quite tight. This instrument was made for me some time ago by Mr. Walters of Moorgate Street; it has already been described (Proc. Mus. Assoc. 1879–80, p. 18).

39. The resonators I employ are bottles fitted with corks having apertures of various sizes. I sometimes tune them with water, in the same way as the bottle-notes; sometimes I insert tubes into the apertures to lower the pitch. A bit of small glass tubing passed through the cork is connected by an india-rubber tube with the above-described ear-piece.

40. By means of these arrangements I some time ago examined the nature of the ordinary first difference-tone, and convinced myself that it is not capable of exciting a resonator (*l. c.* p. 20). This conclusion has also been arrived at by others[*]. In short, the difference-tone of Helmholtz, or first beat-note of König, as ordinarily heard, is not objective in its character. It is therefore subjective. (See Helmholtz, *Tonempfindungen*, 4th ed. p. 259.) In making the experiment of listening for the difference-tone through a resonator, it is necessary to be careful that the ears are both closed to external sounds; otherwise the external notes will penetrate through, the difference-tone will appear, and the completeness of the cut-off effected by the resonator will be entirely lost.

---

[*] Preyer, *Akustische Untersuchungen*, p. 13.

41. When beginning the regular course of experiments according to the general outline of König's work, I was careful, in the first instance, to examine the various masses of sound presented, with resonators arranged as above indicated. In examining, for instance, the intervals made by the note C with the various notes of the octave above it (up to *c*), I first fixed the resonator at some one pitch, and then ran the movable note up through the octave. Then, as this did not seem a good process for analysis, I set the mistuned octave beating, or any other combination it was desired to examine, and ran the pitch of the resonator up and down with water to see if any thing could be detected. And here I came across an observation that puzzled me for some time.

42. Suppose the mistuned octave C : *c* was sounding, and I examined the lower note with the resonator: sometimes it appeared loud and steady, at other times as if beating powerfully. On removing the resonator-attachment from the ear, the lower note was always heard to beat powerfully. The explanation was simple. When the nipples of the resonator-attachment fitted tightly into the ears, nothing reached the ear but the uniform vibrations of the resonator sounding C. But if there was the slightest looseness between the nipple and the passage of either ear, the second note (*c*) of the combination got in, and gave rise to the subjective difference-tone (first beat-note of König), by interference of which with the C I explain the beats on that note. *These beats are therefore subjective.*

43. A considerable number of combinations, including examples of the principal forms of beat, rattle, or roll, were examined in this way; and when the precautions above indicated were attended to, the results were in all cases to negative the objective existence of all forms of beats, and beat-notes or difference-tones, except the beats which arise from the interference of approximate unisons, which beats arise from both notes acting on the resonator simultaneously. This of course includes the beats produced by objective harmonics.

Course of General Experiments.

44. The following is the detailed examination of the combinations of the note C, made in a continuous and connected manner. The results have a general correspondence with those of König. The numerous rattles and rolls of beats mentioned were not further analyzed for the most part: the analysis of these is very difficult; and, as has been already stated, a separate investigation is required in every such case. Some attention was devoted to beats of the mistuned fifth, both in

the case mentioned and in others; but no final result was arrived at. In two different cases of mistuned fifths (2 : 3, nearly), I had a strong impression that the note 7 formed an important part of the beat. This would be a summation-tone of the second order, thus $2\times2+3$. I am confident that it did not arise from harmonics.

These experiments were made after some experience had been gained.

C_1 : C.

Rattle up to

C : F.

Slow beats up to

C : G, smooth fifth. Roll only perceptible when the ear is held close to the two sources of sound.

——, 5 beats sharp. Perception of pitch very difficult in this part of the scale. There is a heavy beat like a knock, which appears to affect the whole mass of sound*. The low beat of C_1 : C is only distinguishable with difficulty, or hardly at all.

(Another occasion.) Mr. Parratt describes the fifth C : G, beating slowly, as consisting of E♭ and C_1 : C in addition to the primary notes; the mass of the beat is at least partly on E♭. I do not hear the E♭, but seem to hear the note E.

(Another occasion.) Mr. Parratt is clear that the beat of the mistuned fifth C : G is on C_1 : C alone; but he still hears the E♭ in the mass of tone. I seem to hear the beats both on C and C_1 : C; but I have a difficulty in separating the octaves in this deep pitch.

C : G, 8 beats sharp. Clear rattle, with suspicion of roll beside it.

—— 10 beats sharp. Beats just distinguishable. Roll.

C : B♭. Rattle emerges.

Below

C : *c*, 8 beats can be counted.

——, 4 beats very distinct. Consist entirely of variation of intensity of lower note. This effect is very clear and remarkable.

——, a very slow beat flat. Here it was easy to recognize the effect of the shift of phase in the apparent great increase of volume of the lower note at one period of the change. The upper note was not perceptibly affected.

C : *c*. A slight rich roll with smooth tone. The production of the roll depends a good deal on the phase, as is seen by leading up to *c* with a very slow beat.

---

* I take this entry to show that no progress had been made with the resolution of the phenomenon into its elements.

The twelfth of the C was plainly distinguishable, but it appeared to keep separate from the mass of tone; it was perfectly steady and unaffected by combination with *c*.

C : *c*, 2 beats sharp. Phenomena undistinguishable from 2 beats below.

——, 4 beats sharp. Perhaps a little less roll in the strong part of the beat.

——, 8 beats sharp. The mass of the beats is of pitch near C; but the exact pitch is very difficult to distinguish. It is a deep heavy rattle, quite distinct in pitch from the upper note.

C : *e*. If there is any slow beat in passing through this, it is very difficult to distinguish. I am inclined to negative it.

C : *f*♯. Roll.

Slow beats up to

C : *g*. These beats consist of alternations of intensity of C. They are more difficult to count than those of C : *c*. I counted them at 5 below.

——. Slow beats above.

C : *b*♭. Rattle, turning into beats easily counted at 4 below *c*′. These beats also consist of variations of intensity of the lower note.

C : *c*′.

The beats above *c*′ were also counted at 4 above, while the engine was going, without difficulty.

45. Above this, in the neighbourhood of the binary consonances C : *e*′ &c., I have never been able to obtain slow beats in such a way that they could be readily perceived (even without the engine) or certainly counted.

46. The mode adopted to examine cases in which the beats could not be perceived was, to introduce a third note, such as *c*′, which gave beats with the C, and tune it true. Then any note, such as *e*′ or *g*′, could be readily tuned so that the whole three notes gave 1, 2, 3, or 4 beats. When this had been done, the intermediate note *c*′ was removed. If the pair examined was capable of giving beats at all, they should then have been audible.

47. The details of the above course furnish no new results; I have not, therefore, thought it worth while to give similar courses for other sets of notes. Those results which are worthy of mention have been already stated.

Theory of the Beats of Mistuned Consonances of the form h : 1.

48. Let *n* be the frequency of the lowest note, *m* the number

of beats per second. Then the mistuned octave is $n : 2n \pm m$; the mistuned twelfth is $n : 3n \pm m$; and so on.

49. Beats of the mistuned octave,

$$n : 2n \pm m.$$

Number of beats $= m$.

m variations of intensity of the lower note (n) are produced by interference of notes n and $n \pm m$; and $n \pm m$ is the first combination-tone (difference-tone of form $p-q$), of the primaries n and $2n \pm m$.

This rests chiefly on the observation that the beats, when the octave harmonic is eliminated, consist entirely of variations of intensity of the lower note.

The existence of the first combination-tone in question ($p-q$) is well known. It is easily demonstrated in the neighbouring case of intervals not far removed from the fifth, when the beats of the first two combination-tones are specially prominent (secondary beats of König).

50. Beats of the mistuned twelfth,

$$n : 3n \pm m,$$

Number of beats $= m$.

m variations of intensity of the lower note (n) are produced by interference of notes n and $n \pm m$. And $n \pm m$ is the second combination-tone (difference-tone of form $2p-q$) of the primaries n and $3n \pm m$.

This rests also chiefly on the observation that the beats, when the third partials are eliminated, consist entirely of variations of intensity of the lower note.

The existence of the second combination-tone in question ($2p-q$) is demonstrated in many cases by König. It is easily heard in the case of intervals near the octave high in the scale. It is also easily detected by the secondary beats which it forms with the first combination-tone in the case of intervals near the fifth—also less easily by the secondary beats which it forms with the third combination-tone in intervals near 2 : 5, at which point the second and third combination-tones coincide.

51. Beats of the mistuned fifteenth or double octave,

$$n : 4n \pm m.$$

Number of beats $= m$.

m variations of intensity of the lower note (n) are produced by interference of notes n and $n \pm m$. And $n \pm m$ is the third combination-tone (difference-tone of form $3p-q$) of the primaries n and $4n \pm m$.

This rests also chiefly on the observation that the beats, when the fourth partials are eliminated, consist entirely of variations of intensity of the lower note.

The existence of the third combination-tone in question $(3p-q)$ is demonstrated in many cases by König. It is heard not so easily as the lower combination-tones, in the case of intervals near the twelfth high in the scale. It is also less easily detected by the secondary beats which it forms with the second combination-tone in the case of intervals near 2 : 5, at which point the second and third combination-tones coincide—also much less easily by the secondary beats which it forms with the fourth combination-tone in the case of intervals near 2 : 7, at which point the third and fourth combination-tones coincide.

52. Beats of the mistuned tierce (two octaves and a major third),

$$n : 5n \pm m.$$

These beats are much less easily detected in pure notes of the ordinary strength than any of the foregoing. They are recorded by König; but I have never heard them clearly. As it is certain that König's notes were not perfectly pure, and he does not analyze the beats, we cannot tell whether the variations of the lower note were produced in his experiments. If they were, they are to be accounted for in a similar manner.

Number of beats $= m$.

m variations of intensity of lower note (n) are produced by interference of notes n and $n \pm m$. And $n \pm m$ is the fourth combination-tone (difference-tone of form $4p-q$) of the primaries n and $5n \pm m$.

The existence of the fourth combination-tone in question $(4p-q)$ is demonstrated directly by König in the case of intervals near the double octave $c''' : c^{\text{v}}$. It is also less easily detected by the secondary beats which it forms with the third combination-tone in the case of intervals near 2 : 7, at which point the third and fourth combination-tones coincide.

53. Beats of the mistuned consonance of the nineteenth are recorded by König;

$$n : 6n \pm m.$$

Number of beats $= m$.

m variations of intensity of lower note (n) might be produced by interference of n and $n \pm m$. And $n \pm m$ is the fifth combination-tone (difference-tone of form $5p-q$) of the primaries n and $6n \pm m$.

The existence of the fifth combination-tone in question $(5p-q)$ is not anywhere directly demonstrated. Secondary beats, which might be produced by its interference with the fourth combination-tone, are recorded by König in the neighbourhood of the interval $c : d''$

54. Beats of the mistuned consonance 1 : 7 are recorded by König. These might be produced by a sixth combination-tone (difference-tone of form $6p-q$) of the primaries n and $7n \pm m$.

55. Beats of the mistuned consonance 1 : 8 are recorded. These might be produced by a seventh combination-tone (difference-tone of form $7p-q$) of the primaries $8n \pm m$.

56. As far as my own experience goes, however, I have no direct and palpable evidence of beats of mistuned consonances higher than 1 : 4, or of the existence of combination-tones higher than the third ($3p-q$) in recognizable intensity. Up to this point the phenomena are quite clear; and there is no possible doubt as to their nature.

But in considering these limited results it must be remembered, (1) that I have restricted myself to notes of very moderate intensity, so that the phenomena might correspond as nearly as possible to those which are presented to our ears in practice, and (2) that, although I was unable to get rid entirely of the presence of upper partials in all cases, yet the phenomena were subjected to a careful and prolonged analysis by listening under varied conditions, until the effect of the upper partials could be separated out and eliminated with certainty. And we have at all events no security that these upper partials did not give rise to many of König's results; indeed it is almost certain that they must have entered into those results.

Note.—The present paper was written before the appearance of König's paper in Wiedemann's *Annalen* in the present year. The discussion of that paper, though necessary for a complete view of the subject, must be reserved till after the conclusion of the present paper.

[To be continued.]

LVIII. *An Analysis of Relationships.*
By A. MACFARLANE, *M.A., D.Sc., F.R.S.E.*[*]

IN this article I propose to describe some results of several papers on an Algebra of Relationship, which I have recently contributed to the Royal Society of Edinburgh[†]. The Logic of Relatives[‡] has been worked at by De Morgan, Leslie Ellis, Harley, and C. S. Peirce[§]; and the last-named

---

[*] Communicated by the Author.

[†] Proc. Roy. Soc. Edinb. May 1879, Dec. 1880, and March 1881.

[‡] Since writing this article I have had the opportunity of reading two interesting and suggestive papers on the Logic of Relatives, by Mr. J. J. Murphy.

[§] For references see Jevons's 'Principles of Science,' p. 23.