

1882.] *Systems of Formulæ for the sn, cn, and dn of $u+v+w$.* 97

if $m < 2p + 1$, and both m and p positive integers;" (2.) "On the surface

$$\frac{a^2x^2}{r^2(a^2-r^2)} + \frac{b^2y^2}{r^2(b^2-r^2)} + \frac{c^2z^2}{r^2(c^2-r^2)} = 0,$$

its Form, Conical Points, and singular Tangent Planes." Mr. C. E. Bickmore.

"Note of Proofs of the Addition-Theorem for the Second Elliptic Integral, and of Fagnani's Theorem by Confocal Conics:" Mr. J. J. Walker.

The following presents were received:—

"Educational Times," March, 1882.

"Reprint of Mathematics from the Educational Times," Vol. xxxvi.: from the Publishers.

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück 2, No. 2.

"Bulletin de la Société Mathématique de Paris," Tome ix., No. 5.

"Monatsbericht," December, 1881.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Fasc. 6°; Roma.

"The Mathematical Magazine, a Journal of Elementary Mathematics," edited and published by Artemas Martin, M.A., Erie, Pa., Pt. 1; 1882.

"Sitzungsberichte der Physikalisch-medizinischen Societät zu Erlangen," 13 Heft; November 1880, bis August 1881.

"Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. Fr. Gomes Teixeira, Vol. iii., Nos. 4, 5, 6; Coimbra.

"Boletín de la Institucion libre de Enseñanza," Año vi., Dec. 31, 1881.

Systems of Formulæ for the sn, cn, and dn of $u+v+w$. By WILLIAM WOOLSEY JOHNSON, Professor of Mathematics in the Naval Academy, Annapolis, Maryland, U.S.

[Read March 9th, 1882.]

1. In the *Messenger of Mathematics*, for July, 1881, Mr. Glaisher has given formulæ for the sn, cn, and dn of $u+v+w$, in which the common denominator is a rational function of $\text{sn}^2 u$, $\text{sn}^2 v$, and $\text{sn}^2 w$; and he has also recorded formulæ first given by Professor Cayley, in which the common denominator is not rational, but which reduce when $w=0$ to the usual forms of the sn, cn, and dn of $u+v$, in which the common denominator is rational in $\text{sn}^2 u$ and $\text{sn}^2 v$. This communication contains the two systems of formulæ for $u+v+w$ of which those just mentioned are members.

2. If for sn, cn and dn we substitute $\frac{s}{n}$, $\frac{c}{n}$ and $\frac{d}{n}$, and use the suffixes 1 and 2 to refer to the arguments u and v , and the large letters
VOL XIII.—NO. 186. H

S, C, D, N to refer to the argument $u+v$, the complete system of formulæ for the functions of $u+v$ consists of four sets of values of the ratios $S : C : D : N$. Distinguishing these values by the suffixes s, c, n , and d , we may write

$$\begin{aligned} S_n &= s_1 n_1 c_2 d_3 + s_2 n_2 c_1 d_1, & S_s &= n_1^2 s_2^2 - s_1^2 n_2^2, \\ C_n &= c_1 c_2 n_1 n_2 - s_1 s_2 d_1 d_2, & C_s &= s_1 c_1 d_1 n_2 - s_2 c_2 d_1 n_1, \\ D_n &= d_1 d_2 n_1 n_2 - k^2 s_1 s_2 c_1 c_2, & D_s &= s_1 d_1 c_2 n_2 - s_2 d_2 c_1 n_1, \\ N_n &= n_1^2 n_2^2 - k^2 s_1^2 s_2^2, & N_s &= s_1 n_1 c_2 d_2 - s_2 n_2 c_1 d_1; \\ S_c &= s_1 c_1 d_2 n_2 + s_2 c_2 d_1 n_1, & S_d &= s_1 d_1 c_2 n_2 + s_2 d_2 c_1 n_1, \\ C_c &= n_1^2 n_2^2 - s_1^2 n_2^2 - n_1^2 s_2^2 + k^2 s_1^2 s_2^2, & C_d &= c_1 c_2 d_1 d_2 - k^2 s_1 s_2 n_1 n_2, \\ D_c &= c_1 c_2 d_1 d_2 + k^2 s_1 s_2 n_1 n_2, & D_d &= n_1^2 n_2^2 - k^2 s_1^2 n_2^2 - k^2 n_1^2 s_2^2 + k^2 s_1^2 s_2^2, \\ N_c &= c_1 c_2 n_1 n_2 + s_1 s_2 d_1 d_2, & N_d &= d_1 d_2 n_1 n_2 + k^2 s_1 s_2 c_1 c_2. \end{aligned}$$

Putting herein each n equal to unity, we return to the usual notation in which s, c , and d stand for sn, cn , and dn respectively.

In each set one of the expressions is rational, and the suffix of the set refers to the letter which takes a rational form in that set.

3. The identities which result from a comparison of the various values of $sn(u+v)$, &c., are of the form

$$S_n N_s = S_s N_n, \quad S_c N_d = S_d N_c, \quad \&c.;$$

viz., we may in any product interchange the suffixes.

4. It is noticeable that the sixteen expressions give also the values of the functions of $u-v$; for example,

$$sn(u-v) = \frac{N_s}{N_n} = \frac{S_s}{S_n} = \frac{C_s}{C_n} = \frac{D_s}{D_n},$$

the suffixes being determined by the required function, and the large letter being arbitrary.

The product of a function of $u+v$ by a function of $u-v$ is the ratio of two of the sixteen expressions; for example,

$$sn(u+v) cn(u-v) = \frac{S_d}{N_d} \frac{C_c}{C_n} = \frac{S_c C_d}{N_n C_d} = \frac{S_c}{N_n}.$$

In like manner, using cs for $\frac{cn}{sn}$, nd for $\frac{1}{dn}$, &c.,

$$cs(u+v) nd(u-v) = \frac{C'_n}{S'_d};$$

the large letters depending on the function of $u+v$, and the suffixes upon the function of $u-v$.

5. Using the suffix 3 to refer to the argument w , and the letters S, C, D, N to refer to the argument $u+v+w$, the process by which Mr. Glaisher's formulæ were derived may be exhibited as follows.

Employing the forms S_n , C_n , D_n and N_n in adding the arguments w and $u+v$, we have S , C , D , and N proportional to

$$\begin{aligned} s_1 n_2 CD + c_2 d_2 SN, \\ c_2 n_2 CN - s_2 d_2 SD, \\ d_2 n_2 DN - k^2 s_2 c_2 SC, \\ \text{and} \quad n_2 N^2 - k^2 s_2^2 S^2; \end{aligned}$$

then, if we multiply each of these expressions by

$$n_2^2 N^2 - k^2 s_2^2 S^2,$$

where the accented letters refer to the argument $u-v$, it is found that the products are all symmetrical functions of u , v , and w .

Since the ratios only of these products are considered, it follows from 4, that we may in all of them for NN' , SN' , &c., substitute N_n , S_n , &c. Adopting the expressions so found as the values S , C , D , and N , we have, for example,

$$\begin{aligned} N &= (n_2^2 N^2 - k^2 s_2^2 S^2) (n_2^2 N^2 - k^2 s_2^2 S^2) \\ &= n_2^4 N_n^2 - k^2 n_2^2 s_2^2 (S_n^2 + N_n^2) + k^4 s_2^4 S_n^2, \end{aligned}$$

which, putting each $n = 1$, is

$$(1 - k^2 s_1^2 s_2^2)^2 - 2k^2 s_2^2 (s_1^2 c_1^2 d_1^2 + s_2^2 c_1^2 d_1^2) + k^4 s_2^4 (s_1^4 - s_2^4)^2,$$

and this reduces to Mr. Glaisher's expression denoted by N_n in the table below.

6. In like manner, if we employ the forms S_n , C_n , D_n , and N_n in forming expressions proportional to S , C , D , and N , the expressions are rendered symmetrical by multiplying by

$$n_2^2 S^2 - s_2^2 N^2,$$

and the products may be denoted by S_n , C_n , D_n , and N_n .

Again, a third set of values denoted by S_n , C_n , D_n , and N_n , and a fourth set S_n , C_n , D_n , N_n may be derived from the corresponding forms for $u+v$. The complete system is as follows, the first set being Mr. Glaisher's formulæ.

$$\begin{aligned} S_n &= s_1 s_2 s_3 \{ -(1+k^2) + 2k^2 \Sigma s_1^2 - k^2 (1+k^2) \Sigma s_1^2 s_2^2 + 2k^4 s_1^2 s_2^2 s_3^2 \} \\ &\quad + \Sigma s_1 c_1 c_2 d_2 d_3 (1+k^2 s_2^2 s_3^2 - k^2 s_2^2 s_1^2 - k^2 s_1^2 s_3^2), \end{aligned}$$

$$\begin{aligned} C_n &= c_1 c_2 c_3 (1 - k^2 \Sigma s_1^2 s_2^2 + 2k^4 s_1^2 s_2^2 s_3^2) \\ &\quad + \Sigma s_2 s_3 c_1 d_2 d_3 (-1 + k^2 s_2^2 s_3^2 - k^2 s_2^2 s_1^2 - k^2 s_1^2 s_3^2 + 2k^4 s_1^2), \end{aligned}$$

$$\begin{aligned} D_n &= d_1 d_2 d_3 (1 - k^2 \Sigma s_1^2 s_2^2 + 2k^4 s_1^2 s_2^2 s_3^2) \\ &\quad + k^2 \Sigma s_2 s_3 c_2 c_3 d_1 (-1 + k^2 s_2^2 s_3^2 - k^2 s_2^2 s_1^2 - k^2 s_1^2 s_3^2 + 2s_2^2), \end{aligned}$$

$$N_n = 1 - 2k^2 \Sigma s_1^2 s_2^2 + 4k^2 (1+k^2) s_1^2 s_2^2 s_3^2 - 2k^4 \Sigma s_1^4 s_2^2 s_3^2 + k^4 \Sigma s_1^4 s_2^4,$$

$$S_1 = \Sigma s_1^4 - 2\Sigma s_1^2 s_2^2 + 4(1+k^2)s_1^2 s_2^2 s_3^2 - 2k^2 \Sigma s_1^4 s_2^2 s_3^2 + k^4 s_1^4 s_2^4 s_3^4,$$

$$O_1 = s_1 s_2 s_3 c_1 c_2 c_3 (2 - k^2 \Sigma s_1^2 + k^4 s_1^2 s_2^2 s_3^2) \\ + \Sigma s_1 c_1 d_1 d_2 (s_1^2 - s_2^2 - s_3^2 + 2s_2^2 s_3^2 - k^2 s_1^2 s_2^2 s_3^2),$$

$$D_1 = s_1 s_2 s_3 d_1 d_2 d_3 (2 - \Sigma s_1^2 + k^2 s_1^2 s_2^2 s_3^2) \\ + \Sigma s_1 d_1 c_1 c_2 (s_1^2 - s_2^2 - s_3^2 + 2k^2 s_2^2 s_3^2 - k^2 s_1^2 s_2^2 s_3^2),$$

$$N_1 = s_1 s_2 s_3 \{2 - (1+k^2) \Sigma s_1^2 + 2k^2 \Sigma s_1^2 s_2^2 - k^2 (1+k^2) s_1^2 s_2^2 s_3^2\} \\ + \Sigma s_1 c_2 c_3 d_2 d_3 (s_1^2 - s_2^2 - s_3^2 + k^2 s_1^2 s_2^2 s_3^2).$$

$$S_2 = s_1 s_2 s_3 c_1 c_2 c_3 (2 - k^2 - k^2 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) \\ + \Sigma s_1 c_1 d_1 d_2 \{1 - s_1^2 - s_2^2 - s_3^2 + (2 - k^2) s_2^2 s_3^2 + k^2 s_2^2 s_3^2 \\ + k^2 s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2\},$$

$$O_2 = 1 - 2\Sigma s_1^2 + 2(1+k^2) \Sigma s_1^2 s_2^2 + \Sigma s_1^4 - 2k^2 \Sigma s_1^4 s_2^2 - 4(1+k^4) s_1^4 s_2^2 s_3^2 \\ + 2k^2 (1+k^2) \Sigma s_1^2 s_2^2 s_3^2 + k^4 \Sigma s_1^4 s_2^4 - 2k^4 \Sigma s_1^4 s_2^4 s_3^2 + k^4 s_1^4 s_2^4 s_3^4,$$

$$D_2 = c_1 c_2 c_3 d_1 d_2 d_3 (1 - \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2) \\ + (1 - k^2) \Sigma s_2 s_3 c_1 d_1 (1 + s_1^2 - s_2^2 - s_3^2 + k^2 s_2^2 s_3^2 - k^2 s_1^2 s_2^2 s_3^2 \\ - k^2 s_1^2 s_2^2 + k^2 s_1^2 s_2^2 s_3^2),$$

$$N_2 = c_1 c_2 c_3 \{1 - \Sigma s_1^2 - k^2 \Sigma s_1^2 s_2^2 + k^2 (1 - 2k^2) s_1^2 s_2^2 s_3^2\} \\ + \Sigma s_2 s_3 c_1 d_1 d_2 \{1 + (1 - 2k^2) s_1^2 - s_2^2 - s_3^2 + k^2 s_2^2 s_3^2 \\ + k^2 s_2^2 s_3^2 + k^2 s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2\}.$$

$$S_3 = s_1 s_2 s_3 d_1 d_2 d_3 (2k^2 - 1 - k^2 \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) \\ + \Sigma s_1 d_1 c_1 c_2 \{1 - k^2 s_1^2 - k^2 s_2^2 - k^2 s_3^2 - k^2 (1 - 2k^2) s_2^2 s_3^2 \\ + k^2 s_2^2 s_3^2 + k^2 s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2\},$$

$$O_3 = c_1 c_2 c_3 d_1 d_2 d_3 (1 - k^2 \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) \\ - (1 - k^2) \Sigma s_2 s_3 c_1 d_1 (1 + k^2 s_1^2 - k^2 s_2^2 - k^2 s_3^2 + k^2 s_2^2 s_3^2 - k^2 s_1^2 s_2^2 s_3^2 \\ - k^2 s_1^2 s_2^2 + k^4 s_1^2 s_2^2 s_3^2),$$

$$D_3 = 1 - 2k^2 \Sigma s_1^2 + 2k^2 (1 + k^2) \Sigma s_1^2 s_2^2 + k^4 \Sigma s_1^4 - 2k^4 \Sigma s_1^4 s_2^2 \\ - 4k^2 (1 + k^4) s_1^2 s_2^2 s_3^2 + 2k^4 (1 + k^2) \Sigma s_1^2 s_2^2 s_3^2 + k^4 \Sigma s_1^4 s_2^4 \\ - 2k^4 \Sigma s_1^4 s_2^4 s_3^2 + k^4 s_1^4 s_2^4 s_3^4,$$

$$N_3 = d_1 d_2 d_3 \{1 - k^2 \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 + k^2 (1 - 2k^2) s_1^2 s_2^2 s_3^2 \\ + k^2 \Sigma s_2 s_3 c_2 c_3 d_1 \{1 - (2 - k^2) s_1^2 - k^2 s_2^2 - k^2 s_3^2 + k^2 s_2^2 s_3^2 \\ + k^2 s_2^2 s_3^2 + k^2 s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2\}.$$

7. Some of the above expressions were calculated in the manner explained in 5, but in most cases it was more convenient to derive them

by transformation from the first set. The second set may be derived by adding iK' to the argument u , which converts

$$\begin{array}{cccc} s_1, & c_1, & d_1, & n_1, \\ \text{respectively into} & \frac{n_1}{k}, & -i\frac{d_1}{k}, & -ic_1, \quad s_1, \end{array}$$

and produces a similar change in S , O , D , and N : or they may be found by adding iK' to each of the arguments, thereby adding $3iK'$ to $u+v+w$, and converting S , O , D , N into $\frac{N}{k}$, $\frac{iD}{k}$, iO , and S respectively.

In like manner, the third set may be derived by adding $K+iK'$ to one or all of the arguments, and the fourth set by adding K to one or all of the arguments.

The third set may also be obtained by transforming the first set to modulus k' , and the fourth set by transforming the third set to the modulus $\frac{1}{k}$.

8. If $w=0$, the expressions reduce to the corresponding expressions for $u+v$. Putting $u=v=w$, they were found to reduce to the known expressions for $3u$, multiplied in the case of the second set by s^2 , in the case of the third set by c^2 , and in the case of the fourth set by d^2 .

9. Professor Cayley's formulæ consist of the expressions denoted by S_n , O_n , D_n , N_n , in the system given below. In the case of these formulæ, the addition of iK' , K , or $K+iK'$ to one of the arguments does not produce a symmetrical result, while the addition of the same quantities to each of the arguments merely reproduces the formulæ. But, by transforming to the modulus k' , we have the expressions S_n , O_n , D_n , and N_n ; and by transforming the latter to the modulus $\frac{1}{k}$, we have the expressions S_n , O_n , D_n , and N_n . The system, which appears to be complete without containing expressions of the form S_n , &c., is as follows:—

$$\begin{aligned} S_n &= -s_1 s_2 s_3 (1 + k^2 - k^2 \Sigma s_1^2 + k^4 s_1^2 s_2^2 s_3^2) & + \Sigma s_1 c_2 c_3 d_2 d_3, \\ O_n &= c_1 c_2 c_3 (1 - k^4 s_1^2 s_2^2 s_3^2) & - d_1 d_2 d_3 \Sigma s_1 s_2 c_1 d_1, \\ D_n &= d_1 d_2 d_3 (1 - k^2 s_1^2 s_2^2 s_3^2) & - k^3 c_1 c_2 c_3 \Sigma s_1 s_2 c_1 d_1, \\ N_n &= 1 - k^2 \Sigma s_1^2 s_2^2 + k^2 (1 + k^2) s_1^2 s_2^2 s_3^2 & - k^2 s_1 s_2 s_3 \Sigma s_1 c_2 c_3 d_2 d_3. \\ S_o &= s_1 s_2 s_3 (2 - k^2 - \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) & + c_1 c_2 c_3 \Sigma s_1 c_1 d_2 d_3, \\ O_o &= c_1 c_2 c_3 (1 - \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) & - (1 - k^2) s_1 s_2 s_3 \Sigma s_1 c_1 d_2 d_3, \\ D_o &= d_1 d_2 d_3 (1 - \Sigma s_1^2 + \Sigma s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2) & + (1 - k^2) \Sigma s_1 s_2 c_3 c_3 d_1, \\ N_o &= 1 - \Sigma s_1^2 + \Sigma s_1^2 s_2^2 - k^2 (2 - k^2) s_1^2 s_2^2 s_3^2 & + d_1 d_2 d_3 \Sigma s_1 s_2 c_3 c_3 d_1. \end{aligned}$$

$$\begin{aligned}
S_s &= s_1 s_2 s_3 (2k^2 - 1 - k^4 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) + d_1 d_2 d_3 \Sigma s_1 c_1 c_2 d_1, \\
O_s &= c_1 c_2 c_3 (1 - k^2 \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2) - (1 - k^2) \Sigma s_2 s_3 c_1 d_1 d_2, \\
D_s &= d_1 d_2 d_3 (1 - k^2 \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2) + k^2 (1 - k^2) s_1 s_2 s_3 \Sigma s_1 c_1 c_2 d_1, \\
N_s &= 1 - k^2 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 + k^2 (1 - 2k^2) s_1^2 s_2^2 s_3^2 + k^2 c_1 c_2 c_3 \Sigma s_2 s_3 c_1 d_1 d_2.
\end{aligned}$$

10. If we put $w = 0$, the second set reduce to the corresponding expressions for $u + v$ multiplied by $c_1 c_2$; and the third set reduce to the same quantities multiplied by $d_1 d_2$. When $u = v = w$, the formulæ reduce directly to the known expressions for $3u$.

It is noteworthy that the expressions for the functions of $u + v + w$, recently communicated to the Society by Mr. M. M. U. Wilkinson,* reduce, when $w = 0$, to the forms S_s , O_s , D_s , and N_s ; and that, when rendered homogeneous by the introduction of the letter n , they are symmetrical with respect to the four letters s , c , d , and n , and therefore constitute of themselves a complete system.

*Note on a Geometrical Theorem. By H. M. TAYLOR and
R. C. ROWE.*

[Read March 9th, 1882.]

1. This note contains the solution of a generalized form of a problem interesting, if only from the names of the mathematicians whose attention it attracted half a century ago. The problem is,—“To find the number of independent ways in which a polygon of m sides can be divided into triangles by means of non-intersecting diagonals.”

The solution seems to have been first discovered by Euler, who gives, but without proof, the number required for polygons up to the enneagon. The proof is supplied by J. A. de Segner, in a paper in the Petersburg Transactions [Nov. Comm., t. vii., pp. 203–210].

The question was next taken up in a series of four papers in Liouville's Journal for 1838–9. Proposed by Terquem, it was answered first by Lamé in a letter to Liouville (t. iii., pp. 505–7), and further discussed by O. Rodrigues (t. iii., pp. 547–8), M. J. Binet (t. iv., pp. 79–90), and E. Catalan (t. iv., pp. 91–94).

2. The result is most conveniently expressed in terms, not of the number (m) of angles of the polygon, but of the number (n) of triangles into which it can be divided. This number will be called the ‘order’ of the polygon; and it is clear that $m = n + 2$. We write P_n for the required number of partitions in the case of a polygon of the n^{th} order.

* [See infra, pp. 106–109, and p. 70.]