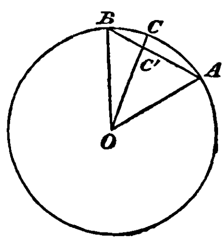
Some Theorems of Kinematics on a Sphere. *By* E. B. Elliott.

[*Read Feb.* 10*th*, 1881.]

1. The theorems in question have to do with the spherical areas passed round by points of a spherical figure as it moves upon its sphere without changing size or form through a complete cycle of positions ending with its original one, and are the analogues of the theorem of plane kinematics, known as Holditch's, and of others given by Messrs. Leudesdorf and Kempe, in the "Messenger of Mathematics" for 1877 and 1878.

Let the radius of our sphere be R, and consider AB an arc of great circle upon it of given length, and consequently subtending a given angle, $\alpha+\beta$ say, at the sphere's centre. Let this arc be divided at C into two constant parts α and β. The chord joining AB is, of course, also of constant length $2R\sin\frac{1}{2}(\alpha+\beta)$, and, if the radius to C meet it in C', then $AC' : C'B = \sin\alpha : \sin\beta$, a constant ratio. Thus the consideration of the kinematics of a constant arc moving on a sphere, and a point always dividing it into two constant parts α, β, resolves itself into that of a rod of constant length, moving with its two ends on a sphere, and the point C' which always divides it in a constant ratio $\sin\alpha : \sin\beta$.



Again, since $\dfrac{OC'}{OA} = \dfrac{\sin C'AO}{\sin AC'O} = \dfrac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$, the concentric sphere on which C' moves is of radius $R\dfrac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$.

2. Now the rod AB, in any position, has three distinct motions open to it,—(1) a rotation about the centre of the sphere in the great circle plane which contains it; (2) a rotation about its own middle point in the plane, which contains it, and is at right angles to this one; and (3) a translation at right angles to itself in this second plane. Let it be given an infinitesimal displacement $d\theta$ of the first kind; then the two ends A, B are displaced on the sphere in virtue of it, through distances each $Rd\theta$, and the point C' through a distance $R\dfrac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}d\theta$. Again, let the rod be given simultaneously or successively displacements $d\phi$, ds of the second and third kinds respectively; then, writing $2c$ instead of $2R\sin\frac{1}{2}(\alpha+\beta)$, the length, the consequent displacements of A and B on the sphere will be $ds+cd\phi$ and $ds-cd\phi$ respectively, in directions at right angles to the first displacements of those points, and that of C' upon its own sphere will be $ds+\dfrac{\sin\beta-\sin\alpha}{\sin\beta+\sin\alpha}cd\phi$ at right

angles to its first displacement. Thus, $d(A)$ and $d(B)$ being rectangular elements of area on the given sphere contained by these rectangular displacements of A and B respectively,

$$d(A) = Rd\theta\,(ds + cd\phi)$$

and

$$d(B) = Rd\theta\,(ds - cd\phi),$$

and there is a corresponding rectangular element on the sphere which is the locus of C', viz.,

$$d(C') = R\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}d\theta\left\{ds - \frac{\sin\alpha - \sin\beta}{\sin\alpha+\sin\beta}cd\phi\right\}.$$

From these, eliminating $d\theta ds$ and $d\theta d\phi$, we get

$$(\sin\alpha+\sin\beta)\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}\,d(C') = \sin\beta\, d(A) + \sin\alpha\, d(B),$$

i.e.,

$$d(C') = \frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\cdot\frac{\sin\beta d(A) + \sin\alpha d(B)}{\sin\alpha+\sin\beta}.$$

Now, suppose that the one end A of our rod pass just completely round the perimeter of an area (A) on the sphere, and that meanwhile the other B pass also just entirely round the perimeter of another area (B), so that C' also passes round the perimeter of an area (C') on its own sphere. Then, if the two areas (A), (B) are such that they can just completely be covered by pairs of corresponding points, each pair of which are possible simultaneous positions of the two ends of our rod, the areas can be split up entirely into elements $d(A)$, $d(B)$, $d(C')$, to which the above result applies. Summing, then, we have a like relation between the whole areas

$$(C') = \frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\cdot\frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin\alpha+\sin\beta} \quad\ldots\ldots\ldots\ldots\ldots (1).$$

But now each point C' is, as has been seen, the central projection of the point C which divides the arc AB into two parts α, β; and each element $d(C)$ surrounded by C is simply the corresponding $d(C')$ multiplied by the square of the ratio of the radii of the two spheres on which they lie. Consequently, we see that, if the two ends of a moving arc of great circle of given angular length $\alpha+\beta$ pass simultaneously on the sphere just all round closed areas (A), (B), which can be covered by corresponding points as above described, the point C which divides it into two constant arcs α, β, will in the same motion pass just all round an area (C) given by

$$(C) = \frac{\cos^2\frac{1}{2}(\alpha-\beta)}{\cos^2\frac{1}{2}(\alpha+\beta)}\cdot\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\cdot\frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin\alpha+\sin\beta}$$

$$= \frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin(\alpha+\beta)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In verification, proceeding to the limit when the sphere becomes a plane, this agrees with the generalised form of Holditch's theorem,

$$(C) = \frac{m(B)+n(A)}{m+n} - \frac{mn}{(m+n)^2}S,$$

for it is a case where the relative area S vanishes.

4. Now it is not always that the two spherical areas (A), (B), passed round by the two ends of an arc that moves through a complete cycle of positions to its old one again, can be entirely covered by pairs of points, each of which is distant from its conjugate by an arc equal to the moving one. It always is possible when each closed curve lies entirely without the other, in which case the moving arc returns to its first position without having made a complete rotation; but when, as is the case if one area (A) lies within the other (B), the arc has to make a complete rotation before returning to its old place (or rather, considering the arc rigidly fixed to a closely fitting spherical surface which slips on the fixed sphere as it moves, when in the motion this spherical surface returns to its first position, having in the mean while completely turned through 2π about some one of its diameters), it cannot be done. The relation (2) then has not been proved for such areas.

As preliminary to discovering the relation which takes its place, let us consider first a special motion. Let one end A of the moving arc remain fixed, so that the other B describes a circle of angular radius $\alpha+\beta$ with it as centre, and so C one of radius α. Call the spherical areas of these circles (b) and (c); then

$$(b) = 4\pi R^2 \sin^2 \tfrac{1}{2}(\alpha+\beta), \quad (c) = 4\pi R^2 \sin^2 \tfrac{1}{2}\alpha \;\ldots\ldots\ldots\ldots \;(3, 4).$$

Now, generally, taking our given surrounded areas (A), (B), call (A) the inner; and within it let there be described an infinite succession of continually smaller curves, each differing infinitesimally from the preceding, and the last one being a point. Then there will be a corresponding infinite succession of curves, the first differing infinitesimally from (B), and each infinitesimally from the preceding, such that arcs of length $\alpha+\beta$ can move all round with one end on any one of them, and the other upon the corresponding one of the other system; and the last of these must be a circle (b) about the point which is the last of the first system as centre. Each of the second system of curves will lie mostly, as a rule altogether, within the preceding, and consequently the last of them, the circle (b), will generally lie entirely within the first one (B). But the method does not fail in case part of the circle lie without it, provided the natural convention be made, as has already been done implicitly, that areas passed round by a point in one sense of rotation being considered positive, those passed round in the other sense are negative.

So, also, corresponding to each curve of the A-system, and the conjugate one of the B-system, there will be a curve of a C-system; and the last of these, corresponding to the point (a) and the circle (b), will be a circle (c) of angular radius a.

Now the areas (A), $(B)-(b)$, $(C)-(c)$ are made up of elements such as those which composed the (A), (B), (C) of equation (2). We have, therefore, $(C)-(c) = \dfrac{\sin\alpha\,\{(B)-(b)\}+\sin\beta\,(A)}{\sin(\alpha+\beta)}.$

Hence, substituting for (b) and (c) from (3, 4), we have

$$(C)-\frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin(\alpha+\beta)} = 4\pi R^2\left\{\sin^2\frac{\alpha}{2}-\frac{\sin^2\frac{1}{2}(\alpha+\beta)\sin\alpha}{\sin(\alpha+\beta)}\right\}$$

$$= -4\pi R^2\frac{\sin\frac{1}{2}\alpha\,\sin\frac{1}{2}\beta}{\cos\frac{1}{2}(\alpha+\beta)} \quad\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

In case the slipping spherical surface, to which the moving arc is supposed rigidly attached, has to make n revolutions instead of one in the course of the complete motion, the circular areas will have to be reckoned n times instead of once. Thus the general result, of which (2) and (5) are special cases, is

$$(C) = \frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin(\alpha+\beta)} - 4n\pi R^2\,\frac{\sin\frac{1}{2}\alpha\,\sin\frac{1}{2}\beta}{\cos\frac{1}{2}(\alpha+\beta)} \quad\ldots\ldots\ldots\ldots (6).$$

The area of (C'), enclosed upon its own sphere by the curve which C', dividing the chord AB in the constant ratio $\sin\alpha : \sin\beta$, describes, is of course found at once from that of (C) by multiplying by the square of the ratio of the radii. Thus

$$(C') = \frac{\cos^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}\left\{\frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin(\alpha+\beta)} - 4n\pi R^2\,\frac{\sin\frac{1}{2}\alpha\,\sin\frac{1}{2}\beta}{\cos\frac{1}{2}(\alpha+\beta)}\right\} \quad\ldots\ldots\ldots (7).$$

The limiting form which (5) or (6) takes when R is made infinite, is Holditch's theorem of plane kinematics, as extended by Woolhouse.

5. In (6) let $(B) = (A)$, *i.e.*, let the two ends A, B of our moving arc go round either the same spherical curve or two curves of equal area; then we get, reducing,

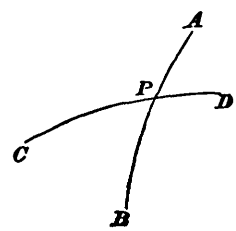
$$(C)\cos\tfrac{1}{2}(\alpha+\beta) = (A)\cos\tfrac{1}{2}(\alpha-\beta) - 4n\pi R^2\sin\frac{\alpha}{2}\sin\frac{\beta}{2},$$

which may be written

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{(C)-(A)}{(C)+(A)-4n\pi R^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

in which n is zero and 1 respectively in the cases of the most simple and usual complete motions contemplated in (2) and (5), and is always a positive or negative integer.

6. We have already considered our moving arc AB as a part of a spherical figure, or indeed of a whole spherical surface, fitting the fixed sphere closely, and moving on it through a closed cycle of positions. Let us discuss now the areas surrounded by different points of this moving surface that are not all on the same great circle; and firstly find the locus of points which in the complete motion pass round areas equal to a given one.



Let A, B, C, D be four points of the moving surface which pass round equal areas (A), and let the arcs AB, CD meet in P. Then all the arcs in the figure are constant, and the above results apply. Using then (8), we obtain a value for $\tan\frac{AP}{2}\tan\frac{PB}{2}$ in terms of (A), (P), and constants; and, again, precisely the same value for $\tan\frac{CP}{2}\tan\frac{PD}{2}$. Thus

$$\tan\frac{AP}{2}\tan\frac{PB}{2}=\tan\frac{CP}{2}\tan\frac{PD}{2}.$$

But this is the necessary and sufficient condition that A, B, C, D lie upon the same circle, small or great, of the sphere. The locus required, then, for any given value of (A) is a circle.

Let S be the centre of this circle, ρ its radius, $\rho'=SP$; then we have

$$\tan\tfrac{1}{2}(\rho+\rho')\tan\tfrac{1}{2}(\rho-\rho')=\tan\frac{AP}{2}\tan\frac{PB}{2}=\text{const.},$$

by (8), if (P) is constant. Thus $(P)=\text{const.}$ necessitates that ρ' as well as ρ be constant. The circle, then, which is the locus of points passing round the area (P), is concentric with that giving (A). The various locus circles are therefore concentric.

Again, if (S) be the area passed round by S, we obtain at once, by applying (6) or (8) to a bisected arc through S,

$$(A)=(S)\cos\rho+4n\pi R^2\sin^2\frac{\rho}{2}\ \ \ldots\ldots\ldots\ldots\ldots\ldots(9);$$

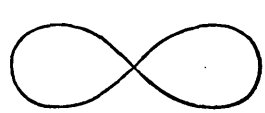
or, as it is perhaps more conveniently written,

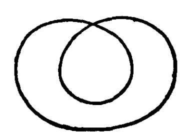
$$\cos\rho=\frac{(A)-2n\pi R^2}{(S)-2n\pi R^2}\ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

thus giving the radius of the circle which is the locus for any assigned area. We are shewn, too, that the least and greatest areas which can be passed round by any point of the sphere, are the one (S) and the other $4n\pi R^2-(S)$, which are passed round by the centre of the locus circles and the diametrically opposite point respectively. The mean area $2n\pi R^2$ is passed round by points on the polar great circle of these singular points.

The theorem of this article is the analogue of one as to plane kinematics, given by Mr. Kempe* as the interpretation of a result by Mr. Leudesdorf. The direct analogue of Mr. Leudesdorf's theorem follows in the next.

Since obtaining all the results above (and those which follow as far as Art. 8 inclusive), my attention has been called to a comprehensive paper by M. Darboux, in the "*Bulletin des Sciences Mathématiques et Astronomiques*," for August, 1878, of which a section is devoted to obtaining the locus theorem of the present article. M. Darboux's result, however, is but special, being for the case of $n = 2$, and obtaining, instead of (10), the form $\frac{4\pi R^2 - (S)}{\cos \rho}$ = constant, which it takes for that case. It is easy, however, to make his proof and result general. He follows the method of roulettes on the sphere, and bases his conclusion on the statement that the sum of an area on a unit sphere, and the integral change of direction in passing round its perimeter, is 2π. Now, use instead the general form of this special fact, namely, that the sum is $2k\pi$, where k is either zero or some integer, and the theorem follows as above, with perfect generality. As instances of cases where k has other values than unity, take spherical areas such as those in the two figures adjoining. In the first, where the area is the difference of the two loops, $k = 0$; and in the second, where it is their sum, $k = 2$.





7. We can readily connect the areas (A), (B), (C), (P) passed round on the sphere by four points, the position of one of which P is given with reference to the spherical triangle ABC of which the other three are vertices. Thus, let AP, BP, CP meet BC, CA, AB respectively in

* For the general case when n is not zero, Mr. Kempe's locus theorem, expressed by $(A) = (S) + n\pi p^2$, follows at once by proceeding to the limit with the part of the sphere near the centre of the locus circles on it. The special case of no revolution ($n = 0$) is, however, exceptional. For this case the theorem on the sphere becomes

$$(A) = (S) \cos \rho = (S) \sin p,$$

where p, the complement of ρ, is the distance from the great circle of the system of locus circles, each point of which now passes round a zero area. Proceeding then to the limit with the part of the sphere about a point of this great circle, it becomes a straight line, and the area (A), passed round by a point in what becomes the plane of motion, is seen to vary as p, the distance from that straight line. Thus Mr. Kempe's special theorem, that in the case of a non-revolutional complete motion in a plane, the loci for equal areas are straight lines instead of circles, is not, as might at first appear, inconsistent with the result above obtained on a sphere.

A', B', C'. Write down, by (6), a relation connecting (A), (A'), and (P); and another connecting (B), (C), and (A'). From these, by elimination of (A'), a result is found easily reducible to

$$(P) = \frac{\sin PA'}{\sin AA'}(A) + \frac{\sin PB'}{\sin BB'}(B) + \frac{\sin PC'}{\sin CC'}(C)$$

$$-4\pi R^2 \left\{ \frac{\sin \frac{1}{2}AP \sin \frac{1}{2}PA'}{\cos \frac{1}{2}AA'} + \frac{\sin AP}{\sin AA'} \cdot \frac{\sin \frac{1}{2}BA' \sin \frac{1}{2}A'C}{\cos \frac{1}{2}BC} \right\},$$

of which the part within the bracket must of course be capable of being brought by direct work to a symmetrical form.

To follow another method, which will at once obtain the result in a convenient shape, will, however, be interesting.

Take O the centre of a sphere, and let the tetrahedral coordinates of P with regard to the tetrahedron $ABCO$ be x, y, z, ω, so that, P being on the sphere, $x = \frac{\sin PA'}{\sin AA'}$, $y = \frac{\sin PB'}{\sin BB'}$, $z = \frac{\sin PC'}{\sin CC'}$, $\omega = 1 - x - y - z$. Take also S the centre of the locus circles found in the last article. We have the vector equality

$$\widehat{SP} = x\,\widehat{SA} + y\,\widehat{SB} + z\,\widehat{SC} + \omega\,\widehat{SO},$$

squaring each side of which, equating scalar parts and reducing, we get the relation in squares of straight lines

$$SP^2 = x\,SA^2 + y\,SB^2 + z\,SC^2 + \omega\,SO^2 - x\omega\,OA^2 - y\omega\,OB^2 - z\omega\,OC^2$$

$$-yz\,BC^2 - zx\,CA^2 - xy\,AB^2.$$

Now, let $\rho_1, \rho_2, \rho_3, \rho$ be the lengths of the arcs SA, SB, SC, SP; then, dividing this by R^2, we get

$$4\sin^2\frac{\rho}{2} = 4\left(x\sin^2\frac{\rho_1}{2} + y\sin^2\frac{\rho_2}{2} + z\sin^2\frac{\rho_3}{2}\right) + \omega\,(1 - x - y - z)$$

$$-4\left(yz\sin^2\frac{a}{2} + zx\sin^2\frac{b}{2} + xy\sin^2\frac{c}{2}\right).$$

Now, by (9), $\qquad \sin^2\frac{\rho_1}{2} = \frac{(A)-(S)}{4n\pi R^2 - 2\,(S)},$

and similarly for $\sin^2\frac{\rho_2}{2}$, $\sin^2\frac{\rho_3}{2}$, $\sin^2\frac{\rho}{2}$. Therefore, inserting,

$$4\{(P) - x\,(A) - y\,(B) - z\,(C)\}$$

$$= 2\,(S)\left\{2\omega - \omega^2 + 4\left(yz\sin^2\frac{a}{2} + zx\sin^2\frac{b}{2} + xy\sin^2\frac{c}{2}\right)\right\}$$

$$+ \left\{\omega^2 - 4\left(yz\sin^2\frac{a}{2} + zx\sin^2\frac{b}{2} + xy\sin^2\frac{c}{2}\right)\right\} 4n\pi R^2.$$

But, by the equation of the sphere, the coefficient of (S) in this equation vanishes, and that of $4n\pi R^2$ reduces to 2ω. We have, therefore, $$(P) = x(A)+y(B)+z(C)+\omega \,.\, 2n\pi R^2 \quad \ldots\ldots\ldots\ldots (11),$$

the relation required. It may be written in various forms symmetrical as to A, B, and C. Thus, for instance, at once

$$(P)-2n\pi R^2 = \{(A)-2n\pi R^2\}\frac{\sin PA'}{\sin AA'}+\{(B)-2n\pi R^2\}\frac{\sin PB'}{\sin BB'}$$
$$+\{(C)-2n\pi R^2\}\frac{\sin PC'}{\sin CC'} \quad \ldots\ldots\ldots\ldots (12).$$

Or, again, let r be the angular radius of the small circle circumscribing ABC, and r' the distance of P from its centre; then

ω = ratio (with proper sign) of distances of P and O from the plane ABC,

$$= -\frac{\cos r'-\cos r}{\cos r} = -\frac{2\sin\frac{1}{2}(r+r')\sin\frac{1}{2}(r-r')}{\cos r}$$
$$= -\frac{2\cos\frac{1}{2}(r+r')\cos\frac{1}{2}(r-r')}{\cos r}\tan\tfrac{1}{2}(r+r')\tan\tfrac{1}{2}(r-r')$$
$$= +\frac{\cos r+\cos r'}{\cos r}\,.\tan^2\frac{\tau}{2},$$

τ being the real or imaginary tangent arc from P to the circumscribing circle. Thus, inserting,

$$(P) = (A)\frac{\sin PA'}{\sin AA'}+(B)\frac{\sin PB'}{\sin BB'}+(C)\frac{\sin PC'}{\sin CC'}$$
$$+2n\pi R^2\frac{\cos r+\cos r'}{\cos r}\tan^2\frac{\tau}{2} \ldots\ldots (13);$$

which, by proceeding to the limit when R is infinite, includes Mr. Leudesdorf's theorem.

8. It is, of course, easy now to state all the above results as to areas passed round by points on the sphere as relations between the solid angles of the cones passed round by the different lines through a fixed point of a solid, which moves through a closed cycle of positions about that point; or, again, taking these lines as of finite lengths, either the same or different, between the sectorial volumes cut by these cones from the spheres which are the loci of the lines' extremities.

9. It occurs now, from facts as to these cones, to determine correlative ones as to the reciprocal cones; in other words, to pass from the points whose motion upon the sphere we have been considering, to the great circles of which they are the poles, and so determine properties of the curves which great circles of the moving spherical surface envelope, correlative to those already found for the curves which are the loci of its points.

We know that an intersection of two arcs of great circle is pole to the connector of their poles, and so that the relation between two spherical curves, of which the one is obtained as the envelope of great circles to which the points of the other are poles, is entirely reciprocal. Thus, while a point A moves, as in either of the cases above, round a spherical area (A), its polar great circle moves round and envelopes a curve whose point of contact with it in any position is pole of the tangent great circle at the corresponding position of A. We know also that the angle between two great circle arcs is equal to the angular distance between their poles. It follows that the angle between two consecutive tangents to the perimeter of (A) is equal to the angular length of the corresponding element of arc of the reciprocal envelope. Hence, summing for a complete motion, we obtain that the entire change of direction in passing all round the closed perimeter of (A) is equal to the entire angular length of the perimeter of the, of course closed, reciprocal curve which is the envelope of the polar great circle of A. Thus the actual length of the perimeter,

$$S_a = R \times \text{change of direction in passing round } (A)$$

$$= 2k\pi R - \frac{1}{R}(A) \quad \text{.......................................} \quad (14),$$

where k equals zero, or an integer, and is the algebraic number of loops of which (A) consists, *i.e.*, the excess of the number passed round in the positive sense over that in the negative.

Hence, at once, the closed motion being such that k is constant for the poles of all great circles of the sphere that are considered, S_a is constant whenever (A) is. The result of Art. 6, then, at once reciprocates into,—"If a spherical surface closely fitting a fixed one move on it through a closed cycle of positions, the envelope on the moving surface of great circles whose envelopes on the fixed one are of constant perimeter S_a is a circle For different values of S_a the different envelope circles are concentric. And, if S_o be the perimeter of the envelope of the great circle of which their common centre is pole, the radius ρ of any one of them is given by

$$\sin \rho = \frac{2(k-n)\pi R - S_a}{2(k-n)\pi R - S_o} \quad \text{......................} \quad (15),$$

where $k-n$ is the excess of the algebraic number of loops in the curve which is the locus of the pole of the circle which envelopes S_a, over the number of times which in the complete motion the moving sphere has made complete revolutions round the axis for which that number is greatest." Great circles, in particular, which pass through the common centre of the envelope circles, envelope a constant perimeter $2(k-n)\pi R$, depending solely on k, n and the dimensions of the sphere.

In the most usual motions $k=n$, so that (15) takes the simple form $S_a = S_o \sin\rho$. Of other possible motions the most important seem to be those where $k=n+1$, to which the simple one of Art. 3 belongs ($n=0$, $k=1$). The equivalent of the theorem of the present article has been obtained for the case $n=2$, $k=3$, by M. Darboux in the paper already referred to.

10. Still considering k to be constant for the poles of all the great circles dealt with, we can at once write down the correlative of result (6) above, by writing in it from (14) for (A), and similarly for (B) and (C). The conclusion is, that if two great circles making an angle $\alpha+\beta$ with each other envelope curves of perimeters S_a, S_b respectively in a complete cyclic motion, then the great circle which divides the angle between them into two parts α, β, shall envelope a perimeter S_c, given by an equation that is readily reduced to

$$S_c = \frac{\sin\alpha \,.\, S_b + \sin\beta \,.\, S_a}{\sin(\alpha+\beta)} - 4\,(k-n)\,\frac{\sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta}{\cos\tfrac{1}{2}(\alpha+\beta)} \quad \text{......(15).}$$

It is seen then that the common case for which $k=n$, which includes that contemplated in result (5) above, is, as far as lengths of envelopes go, one of exceptional simplicity, the motion giving the simplest result as to loci of points, ($k=1$, $n=0$) that of (2), being as to envelopes more complicated.

11. Again, we may now write down the correlative of the result of Art. 7, (11), (12), or (13). Letting ABC be the polar triangle of the one considered in that article, and letting the polar great circle of P meet its sides respectively in A', B', C', we have, that the x of that

Article $$= \frac{\sin AA'B'}{\sin AA'B} = \frac{\sin\varpi_1}{\sin p_1};$$

and, similarly, $y = \dfrac{\sin\varpi_2}{\sin p_2}$, $z = \dfrac{\sin\varpi_3}{\sin p_3}$, p_1, p_2, p_3 being the perpendicular arcs from the vertices A, B, C on the opposite sides of the triangle, and ϖ_1, ϖ_2, ϖ_3 the perpendicular arcs from those vertices upon $A'B'C'$. Thus we may state,—"If in a closed motion of a spherical figure upon its sphere the three sides of a spherical triangle upon it envelope curves of perimeters S_a, S_b, S_c respectively, the great circle on which the perpendiculars from the vertices are ϖ_1, ϖ_2, ϖ_3, shall envelope one of perimeter S_p given by

$$S_p - 2\,(k-n)\,\pi R = \{S_a - 2\,(k-n)\,\pi R\}\,\frac{\sin\varpi_1}{\sin p_1} + \{S_b - 2\,(k-n)\,\pi R\}\,\frac{\sin\varpi_2}{\sin p_2}$$
$$+ \{S_c - 2\,(k-n)\,\pi R\}\,\frac{\sin\varpi_3}{\sin p_3} \quad \text{....................(18),}$$

the correlative of (12), or again by

$$S_p = S_a \frac{\sin \varpi_1}{\sin p_1} + S_b \frac{\sin \varpi_2}{\sin p_2} + S_c \frac{\sin \varpi_3}{\sin p_3}$$
$$+2\,(k-n)\,\pi R \frac{\sin r + \sin r'}{\sin r} \tan^2 \frac{t}{2} \quad \text{.........(19)},$$

where r is the angular radius of the circle inscribed in ABC, r' the angular distance of the great circle considered from the centre of that circle, and t the angle at which that great circle cuts it." The last form of result is derived from the first, just as (13) from (12).

12. The theorem of plane kinematics obtained by taking that of the last article in the limiting case when the sphere becomes a plane, is new to me. The class of cases $k=n$ alone seems to have meaning in the limit, for in a plane the points whose motion determine k are infinitely distant ones. Now, for this class of cases, the remainder on the right of (19) vanishes, so that we may enunciate,—" In any closed motion of a plane figure in its plane, if S_a, S_b, S_c be the perimeters of the curves enveloped by the sides of a plane triangle ABC in it, that enveloped by a straight line of it, on which the perpendiculars from the vertices are p, q, r respectively, shall be of perimeter S_p, given by

$$2\Delta S_p = pa\,S_a + qb\,S_b + rc\,S_c \quad \text{.....................(20)},$$

where a, b, c are the sides, and Δ the area of the triangle ABC."

The plane theorems which are the limits of Arts. 9 and 10 have been previously obtained by M. Darboux by direct process in a plane, as also has been the remarkable relation

$$0 = a\,S_a + b\,S_b + c\,S_c,$$

obtained by giving p, q, r, in (20), infinite equal values.

It may perhaps be allowed me to make another remark on M. Darboux's paper in the "*Bulletin*," which has been so suggestive to me in the latter part of this one; viz., that his extension of Holditch's theorem to sectorial areas is one included in my earlier extension of it ("Messenger of Mathematics," Feb. 1878), which removed the restriction that the moving line be necessarily of constant length; and in like manner, that his final result on volumes, given as an extended analogue of one of mine in the same paper in the "Messenger," is, on the other hand, included in my theorem.