*Note on the Theory of the Pellian Equation, and of Binary Quadratic Forms of a Positive Determinant.** *By* HENRY J. STEPHEN SMITH, Savilian Professor of Geometry in the University of Oxford.

(*Read May* 11*th*, 1876.)

Art. 1. Let $\theta_0 = \mu_0 + \frac{1}{\mu_1} + \frac{1}{\mu_2} + \dots$ be any continued fraction, of which $\theta_1, \theta_2, \dots$ are the complete quotients; $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \dots$ the successive convergents; so that

$$p_0 = \mu_0,\quad q_0 = 1,\quad \dots\dots\quad p_i = \mu_i p_{i-1} + p_{i-2},\quad q_i = \mu_i q_{i-1} + q_{i-2},$$

$$\theta = \frac{\theta_i p_{i-1} + p_{i-2}}{\theta_i q_{i-1} + q_{i-2}},\quad \theta_i = -\frac{p_{i-2} - \theta q_{i-2}}{p_{i-1} - \theta q_{i-1}}.$$

Also, let $\quad p_{i-1} - \theta q_{i-1} = (-1)^i \epsilon_{i-1}, \quad$ so that $\quad \theta_i = \frac{\epsilon_{i-2}}{\epsilon_{i-1}};$

we have $\quad \theta_1 = \frac{1}{\epsilon_0}, \quad$ and hence $\quad \epsilon_{i-1} = \frac{1}{\theta_1 \theta_2 \dots \theta_i}.$

This expression, which supplies a measure of the rate of decrease of the difference ϵ_{i-1}, admits of an interesting application to the theory of the Pellian equation, and of binary quadratic forms of a positive determinant.

---

* The following summary of the contents of this Note may be of use to the reader:—

Art. 1.—The relation, in a continued fraction, between the quantities ϵ and θ.

Art. 2.—The theorem that $T + U\sqrt{D}$ is equal to the product of the complete quotients in the development of $\sqrt{D}$.

Art. 3.—The same theorem for the period of complete quotients in the development of any quadratic surd.

Art. 4.—Theorems as to the number of different periods of complete quotients; viz., equations (1)—(4).

Art. 5.—Theorems as to the number of non-equivalent classes of quadratic forms; viz., equations (5) and (6).

Art. 6.—Equations arising from a comparison of the formulæ (5) and (6) with those of Dirichlet.

Arts. 7—13.—Discussion of the nature of the periods in the more important special cases.

Art. 14.—On the symmetry of any periodic series.

Art. 15.—On the arithmetical conditions under which the various special cases present themselves.

[It would be difficult to say that anything in the Addition (Arts. 7—15) is new: the discussion there attempted has never been given completely (see Art. 7); but this may have been because no one has thought it worth giving.]

2. Let D be any positive integer, not a perfect square. In the development of $\sqrt{D}$ in a continued fraction, let

$$\frac{\sqrt{D}+Q_1}{P_1}, \quad \frac{\sqrt{D}+Q_2}{P_2}, \quad \ldots\ldots \quad \frac{\sqrt{D}+Q_i}{P_i}$$

be the period of complete quotients; so that, if a is the integral number next inferior to $\sqrt{D}$,

$$\frac{\sqrt{D}+Q_1}{P_1} = \frac{\sqrt{D}+a}{D-a^2}, \quad \frac{\sqrt{D}+Q_i}{P_i} = \sqrt{D}+a.$$

Let T and U be the least integral numbers satisfying the equation

$$T^2-DU^2 = (-1)^i,$$

and let $$\sqrt{D} = a+\frac{1}{\mu_1+}\,\frac{1}{\mu_2+}\cdots\frac{1}{\mu_{i-1}+}\,\frac{1}{2a+}\cdots;$$

we have $T = p_{i-1}, \quad U = q_{i-1}, \quad (-1)^i \epsilon_{i-1} = T-U\sqrt{D};$

whence, by the preceding theorem,

$$T+U\sqrt{D} = \prod_{s=1}^{s=i} \frac{\sqrt{D}+Q_s}{P_s}.$$

Example.—The continued fraction equivalent to $\sqrt{13}$ is

$$\sqrt{13} = 3+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\ldots\ldots,$$

and the period of complete quotients is

$$\frac{\sqrt{13}+3}{4}, \quad \frac{\sqrt{13}+1}{3}, \quad \frac{\sqrt{13}+2}{3}, \quad \frac{\sqrt{13}+1}{4}, \quad \sqrt{13}+3,$$

giving $$(\sqrt{13}+3)^2(\sqrt{13}+1)^2(\sqrt{13}+2)\times\frac{1}{144} = 18+5\sqrt{13},$$
and $$18^2-25\times 13 = -1.$$

3. Again, let $$\Omega = a+2bu+cu^2 = 0$$

represent any properly primitive equation of determinant D (*i. e.*, any quadratic equation whatever, in which a, b, c are integral numbers satisfying the equation $b^2-ac = D$, and a, $2b$, c have no common divisor). If

$$\frac{\sqrt{D}+Q_1}{P_1}, \quad \frac{\sqrt{D}+Q_2}{P_2}, \quad \ldots\ldots \quad \frac{\sqrt{D}+Q_i}{P_i}$$

is the period of complete quotients obtained by the development of either root of Ω, we shall have, as before,

$$T+U\sqrt{D} = \prod_{s=1}^{s=i} \frac{\sqrt{D}+Q_s}{P_s}.$$

For the equations of the period are of the type,

$$
(\Omega)\ldots\ldots\left\{\begin{array}{l}
a_0+2\beta_0 u_0-a_1 u_0^2=0, \\
-a_1-2\beta_1 u_1+a_2 u_1^2=0, \\
a_2+2\beta_2 u_2-a_3 u_2^2=0, \\
\ldots\ \ldots\ \ldots\ \ldots\ \ldots \\
(-1)^{i-1}a_{i-1}+(-1)^{i-1}\,2\beta_{i-1}\,u_{i-1}+(-1)^i\,a_0 u_{i-1}^2=0, \\
(-1)^i[a_0+2\beta_0 u_0-a_1 u_0^2]=0, \\
\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots
\end{array}\right.
$$

so that, if $\mu_0, \mu_1, \mu_2, \ldots\ldots \mu_{i-1}$, is the period of integral quotients, we have

$$u_0=\mu_0+\frac{1}{\mu_1+}\ldots\ldots\frac{1}{\mu_{i-1}+}\,\frac{1}{u_0},$$

where

$$u_0=\frac{\beta_0+\sqrt{D}}{a_1}.$$

Hence

$$
\begin{aligned}
\frac{1}{e_{i-1}} &= \frac{(-1)^i}{p_{i-1}-q_{i-1}u_0}, \\
&= u_0 u_1 u_2 \ldots u_{i-1}, \\
&= \prod_{s=1}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.
\end{aligned}
$$

But, by a known theorem (see the Report on the Theory of Numbers, in the Report of the British Association for 1861, Art. 96 (i.), p. 315), we have

$$
(Q.)\ldots\ldots\ldots\ldots\ldots\ldots\left\{\begin{array}{l}
T=\frac{1}{2}(p'_{i-1}+q_{i-2}), \\
U=\frac{p^{i-2}}{a_0}=\frac{p_{i-1}-q_{i-2}}{2\beta_0}=\frac{q_{i-1}}{a_1};
\end{array}\right.
$$

whence

$$p_{i-1}-q_{i-1}u_0=p_{i-1}-q_{i-1}\frac{\beta_0+\sqrt{D}}{a_1}=T-U\sqrt{D},$$

or

$$T+U\sqrt{D}=\frac{(-1)^i}{T-U\sqrt{D}}=\prod_{s=i}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.$$

If the given equation (Ω) is improperly primitive (*i.e.*, if the numbers a, $2b$, c have 2 for their greatest common divisor), we have to replace the numbers T and U in the equations (Q), by $\frac{1}{2}T_1$ and $\frac{1}{2}U_1$, where T_1 and U_1 are the least numbers which satisfy the equation $T_1^2-DU_1^2=(-1)^i 4$; and we find

$$\tfrac{1}{2}(T_1+U_1\sqrt{D})=\prod_{s=1}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.$$

4. Every primitive quadratic equation of determinant D, of which one root is positive and greater than unity, and the other negative and less in absolute magnitude than unity, occurs in one, and only in one, of the periods of equations of determinant D (see the Report cited, p. 309, Art. 93). Hence, every expression of the form $\frac{\sqrt{D}+Q}{P}$, in which, P and Q being positive, Q is less than $\sqrt{D}$, P is a divisor of $D-Q^2$ intermediate between $\sqrt{D}-Q$ and $\sqrt{D}+Q$, and the three numbers

P, Q, $\frac{D-Q^2}{P}$ are relatively prime, is the root of an equation contained in a period of equations of determinant D; and, for any given determinant, the number of periods of complete quotients is equal to the number of periods of quadratic equations, if we regard two quadratic equations such as

$$a+2b\theta+c\theta^2=0,$$
$$-a-2b\theta-c\theta^2=0,$$

which differ only in sign, as identical with one another.

Let $P'=\frac{D-Q^2}{P}$, and let k be the number of periods of properly primitive complete quotients of determinant D; we have evidently

$$(1)\ldots\ldots\ldots\ldots(T+U\sqrt{D})^k=\Pi\,\frac{\sqrt{D}+Q}{P},$$

or

$$k=\frac{1}{\log(T+U\sqrt{D})}\,\Sigma\log\frac{\sqrt{D}+Q}{P},$$

the sign of multiplication Π, and the sign of summation Σ, extending to all positive numbers Q which do not surpass $\sqrt{D}$, and to all divisors P of $D-Q^2$, which are intermediate between $\sqrt{D}+Q$ and $\sqrt{D}-Q$, and are such that the three numbers P, 2Q, P′ admit of no common divisor other than unity. Let $\psi(Q)$ be the number of such divisors of $D-Q^2$; observing that, if $\frac{\sqrt{D}+Q}{P}$ is a complete quotient in a period, $\frac{\sqrt{D}+Q}{P'}=\frac{P}{\sqrt{D}-Q}$ is also a complete quotient in the same or in a different period, we may write

$$(2)\ldots\ldots\ldots\ldots(T+U\sqrt{D})^k=\Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\frac{1}{2}\psi(Q)};$$

where it will be noticed that, if $P=P'$, $D=P^2+Q^2$, the two identical complete quotients $\frac{\sqrt{D}+Q}{P}$ and $\frac{P}{\sqrt{D}-Q}$ are each of them equal to $\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\frac{1}{2}}$.

When $D\equiv1$, mod. 4, let k_1 be the number of periods of improperly primitive complete quotients of determinant D, we find as before

$$(3)\ldots\ldots\ldots\ldots(\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^{k_1}=\Pi\,\frac{\sqrt{D}+Q}{P},$$

or

$$k_1=\frac{1}{\log\frac{1}{2}(T_1+U_1\sqrt{D})}\,\Sigma\log\frac{\sqrt{D}+Q}{P},$$

the symbols Π and Σ extending to all positive uneven numbers Q which are less than $\sqrt{D}$, and to all divisors P of $D-Q^2$ which are intermediate between $\sqrt{D}+Q$ and $\sqrt{D}-Q$, and are such that the three numbers P, 2Q, P′ have 2 for their greatest common divisor. If

$\psi_1(Q)$ be the number of such divisors of $D-Q^2$, we may also write

$$(4)\dots\dots\dots\dots(\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^{k_1} = \Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\psi_1(Q)}.$$

It will be observed that, if Q is even, we have always $\psi_1(Q) = 0$.

5. Let h and h_1 respectively denote the numbers of properly and improperly primitive classes of quadratic forms of determinant D, and let $[\tau, v]$, $[\tau_1, v_1]$ be the least numbers which satisfy the equations $\tau^2-Dv^2 = +1$, $\tau_1^2-Dv_1^2 = +4$; so that, when the equations

$$(T)\dots\dots\dots\dots x^2-Dy^2 = -1,\quad x^2-Dy^2 = -4,$$

are resoluble [they are either both resoluble or both irresoluble],

$$\tau+v\sqrt{D} = (T+U\sqrt{D})^2,\quad \tfrac{1}{2}\tau_1+\tfrac{1}{2}v_1\sqrt{D} = (\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^2;$$

and when the equations (T) are irresoluble,

$$\tau+v\sqrt{D} = T+U\sqrt{D},\quad \tau_1+v_1\sqrt{D} = T_1+U_1\sqrt{D}.$$

We can now establish the equations

$$(5)\dots\dots\dots\dots(\tau+v\sqrt{D})^h = \Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\psi(Q)},$$

$$(6)\dots\dots\dots\dots(\tfrac{1}{2}\tau_1+\tfrac{1}{2}v_1\sqrt{D})^{h_1} = \Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\psi_1(Q)}$$

For this purpose it is only necessary to show that, when the equations (T) are not resoluble, we have,

$$(a)\dots\dots\dots\dots h = 2k,\quad h_1 = 2k_1;$$

but when these equations are resoluble, we have, instead,

$$(b)\dots\dots\dots\dots h = k,\quad h_1 = k_1.$$

To the single period of complete quotients

$$(7)\dots\dots\dots\dots \frac{\sqrt{D}+\beta_0}{a_1},\ \frac{\sqrt{D}+\beta_1}{a_2},\ \frac{\sqrt{D}+\beta_2}{a_3},\ \dots\dots$$

there correspond two periods of reduced quadratic forms, viz.,

$$(8)\dots\dots\dots\dots(a_0, \beta_0, -a_1),\ (-a_1, \beta_1, a_2),\ (a_2, \beta_2, -a_3),\ \dots\ \dots\ \dots\ \dots\ \dots$$

and

$$(9)\dots\dots\dots\dots(-a_0, \beta_0, a_1),\ (a_1, \beta_1, -a_2),\ (-a_2, \beta_2, a_3),\ \dots\ \dots\ \dots\ \dots\ \dots$$

For the complete quotients (7) are the positive roots of the equations (Ω); they are also the positive roots of the same equations with their signs changed; and the period (9) is related to the period ($-\Omega$) exactly as the period (8) is related to the period (Ω); viz., the coefficients of the forms are the same as the coefficients of the equations, except that in the periods of equations the middle coefficients are alternately positive and negative, whereas in the periods of forms these coefficients are all positive. The two periods of forms are, in general, but not always, distinct; and we shall now prove (what is indeed well known) that these two periods are, or are not, identical, according as the equations (T) are, or are not, resoluble. We may observe that the form $(a, -b, c)$ is termed the *opposite* of the form (a, b, c), and $(-a, -b, -c)$ the *negative* of (a, b, c); thus $(-a, b, -c)$ is the negative of the opposite of (a, b, c).

(i.) If the equations (T) are resoluble, any form (a, b, c) of determinant D is properly equivalent to the negative of its opposite; viz., (a, b, c) is transformed into $(-a, b, -c)$ by

$$\begin{vmatrix} bU-T, & -cU \\ -aU, & bU+T \end{vmatrix}$$

Hence the reduced forms $(\alpha_0, \beta_0, -\alpha_1)$, $(-\alpha_0, \beta_0, \alpha_1)$ are properly equivalent; either of them is therefore contained in the period of the other; *i.e.*, the two periods are identical.

(ii.) If the two periods (8) and (9) are identical, the form $(-\alpha_0, \beta_0, \alpha_1)$ must occur in the period of $(\alpha_0, \beta_0, -\alpha_1)$; and because its first coefficient is negative, it must occupy an even place in that period. Hence the period of complete quotients (7) consists of an uneven number of terms; and we infer from the formulæ (Q) of Art. 3 that the equations (T) are resoluble.

6. If $D\equiv 1$, mod. 8, we have $h=h_1$; if $D\equiv 5$, we have $h=3h_1$ when τ_1 and v_1 are even, but $h=h_1$ when τ_1 and v_1 are uneven, in which case

$$(\tfrac{1}{2}\tau_1+v_1\sqrt{D})^3 = \tau+v\sqrt{D}.$$

We thus find, in both cases alike,

$$(12) \quad \ldots\ldots\ldots\ldots\ldots\ldots \Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\psi(Q)} = \Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\sigma\psi_1(Q)},$$

where $\sigma=1$, or $=3$, according as $D\equiv 1$, or $\equiv 5$, mod. 8.

Again, since, by the formulæ of Lejeune Dirichlet, we have (see the Report cited, Art. 101)

$$h\log[\tau+v\sqrt{D}] = 2\sqrt{D}\,\Sigma\left(\frac{D}{n}\right)\frac{1}{n},$$

where the sign of summation extends to all numbers prime to 2D, and $\left(\frac{D}{n}\right)$ is the generalized symbol of quadratic reciprocity, we obtain

$$2\sqrt{D}\,\Sigma\left(\frac{D}{n}\right)\frac{1}{n} = \Sigma\psi(Q)\log\frac{\sqrt{D}+Q}{\sqrt{D}-Q},$$

or

$$(13) \quad \ldots\ldots\ldots\ldots\ldots \Sigma\left(\frac{D}{n}\right)\frac{1}{n} = \Sigma\psi(Q)\left[\frac{Q}{D}+\frac{Q^3}{3D^2}+\frac{Q^5}{5D^3}+\ldots\ldots\right].$$

Similarly, from the formulæ (see *ibid.*)

$$h_1\log[\tfrac{1}{2}\tau_1+\tfrac{1}{2}v_1\sqrt{D}] = \frac{2}{\sigma}\sqrt{D}\,\Sigma\left(\frac{D}{n}\right)\frac{1}{n},$$

where σ has the same meaning as before, we infer

$$(14)\ldots\ldots\ldots\Sigma\left(\frac{D}{n}\right)\frac{1}{n} = \sigma\Sigma\psi_1(Q)\left[\frac{Q}{D}+\frac{Q^3}{3D^2}+\frac{Q^5}{5D^3}+\ldots\ldots\right].$$

It is probable that a direct demonstration of the equations (12), (13), (14), of which any two involve the third, would offer considerable difficulties.

Addition to the preceding Note.

7. As the preceding determination (equations 5 and 6) of the number of non-equivalent classes for a positive determinant depends on the equations (a) and (b), which assign the relation between the number of periods of complete quotients and the number of periods of reduced forms, it is worth while, for the sake of distinctness, to describe fully the characteristic appearances presented by these periods in certain special cases which are of some importance.

Every form, or class of forms, is, of course, properly equivalent to itself, and improperly equivalent to its opposite. But a form, or class of forms, may be—

(i.) Properly equivalent to its opposite, and improperly equivalent to itself (in this case the class is ambiguous);

(ii.) Properly equivalent to its negative, and improperly equivalent to the negative of its opposite;

(iii.) Properly equivalent to the negative of its opposite, and improperly equivalent to its negative.

Since, if any two of these specialities coexist, they necessarily involve the third, there are four cases to be considered, viz., the cases (i.), (ii.), (iii.), in which the specialities (i.), (ii.), (iii.) exist singly, and the case (iv.) in which they all exist simultaneously. We shall briefly refer to each of these cases in succession. We may observe, however, that the case (i.), which is that of an ambiguous class, has been fully considered by Gauss (*Disq. Arith.*, Art. 187, Obs. 6, 7, 8); of the rest, the case (ii.) has, perhaps, attracted less attention than might have been expected.

8. If the period of reduced forms equivalent to (a, b, c) is

$$(8)\ \ldots\ldots\ (\alpha_0, \beta_0, -\alpha_1),\ (-\alpha_1, \beta_1, \alpha_2)\ \ldots\ldots\ (-\alpha_{2k-1}, \beta_{2k-1}, \alpha_0),$$

the periods of reduced forms equivalent to $(a, -b, c)$, $(-a, -b, -c)$, $(-a, b, -c)$ are respectively

$$(10)\ldots\ldots(\alpha_0, \beta_{2k-1}, -\alpha_{2k-1}),\ (-\alpha_{2k-1}, \beta_{2k-2}, \alpha_{2k-2})\ \ldots\ldots\ (-\alpha_1, \beta_0, \alpha_0),$$

$$(11)\ldots\ldots(-\alpha_0, \beta_{2k-1}, \alpha_{2k-1}),\ (\alpha_{2k-1}, \beta_{2k-2}, -\alpha_{2k-2})\ \ldots\ldots\ (\alpha_1, \beta_0, -\alpha_0),$$

$$(9)\ldots\ldots(-\alpha_0, \beta_0, \alpha_1),\quad (\alpha_1, \beta_1, -\alpha_2)\ \ldots\ldots\ldots\ldots\ (\alpha_{2k-1}, \beta_{2k-1}, -\alpha_0).$$

As in Art. 3, we designate the period (7) of complete quotients, or, which is the same thing, the period formed by the positive roots of the equations (Ω), by $\quad u_0, u_1, \ldots\ldots u_{2k-1}$.

The negative roots of the same equations we represent by

$$-\frac{1}{v_0},\ -\frac{1}{v_1},\ \ldots\ldots\ -\frac{1}{v_{2k-1}};$$

so that, if $\mu_0, \mu_1, \ldots\ldots \mu_{2k-1}$ is the period of integral quotients, we have, by a well-known theorem,

$$\mu_s = \mathrm{I}u_s = \mathrm{I}v_{s+1},$$

the symbol $\mathrm{I}x$ denoting the greatest integral number not surpassing x. The four periods of forms (8), (10), (11), (9), we represent for brevity by the symbols

$$\begin{array}{lllll} \Phi \ldots\ldots & \phi_0, & \phi_1, & \phi_2, & \ldots\ldots\ \phi_{2k-1}, \\ \Psi \ldots\ldots & \psi_{2k-1}, & \psi_{2k-2}, & \psi_{2k-3}, & \ldots\ldots\ \psi_0, \\ \bar{\Psi} \ldots\ldots & \bar{\psi}_{2k-1}, & \bar{\psi}_{2k-2}, & \bar{\psi}_{2k-3}, & \ldots\ldots\ \bar{\psi}_0, \\ \bar{\Phi} \ldots\ldots & \bar{\phi}_0, & \bar{\phi}_1, & \bar{\phi}_2, & \ldots\ldots\ \bar{\phi}_{2k-1}. \end{array}$$

Two such forms as (a, b, c), (c, b, a) are said to be *associated;* thus, ϕ_s and ψ_s, or again $\bar{\phi}_s$ and $\bar{\psi}_s$, are associated forms. The periods Φ and Ψ, and again the periods $\bar{\Phi}$ and $\bar{\Psi}$, are themselves termed associated periods.

The period of complete quotients corresponding to the periods Φ and $\bar{\Phi}$ is (Art. 5) $\quad u_0, u_1, \ldots\ldots u_{2k-1};$

and similarly the period of complete quotients corresponding to the periods Ψ and $\bar{\Psi}$ is $\quad v_{2k-1}, v_{2k-2}, \ldots\ldots v_1, v_0.$

9. Case (i.)—If (a, b, c) is properly equivalent to $(a, -b, c)$, the periods Φ and Ψ must coincide; *i.e.*, we must have, for some value of σ, $\phi_0 = \psi_{2\sigma+1}$, the suffix $2\sigma+1$ being uneven, because the extreme coefficients of $\psi_{2\sigma+1}$ must have the same signs as the extreme coefficients of ϕ_0. Hence $\phi_1 = \psi_{2\sigma}$, $\phi_2 = \psi_{2\sigma-1}, \ldots\ldots$; and finally $\phi_\sigma = \psi_{\sigma+1}$, $\phi_{\sigma+1} = \psi_\sigma$, or there occur in the period two consecutive forms of which each is the associate of the other. As we may begin the period with any form we please, we may suppose that ϕ_0 and ϕ_1 are these two consecutive forms, so that $\beta_1 = \beta_0$, $a_2 = a_0$, $2\beta_0 \equiv 0$, mod. a_1. It will be seen that, if $\phi_0 = \psi_{2\sigma+1}$, we have not only $\phi_\sigma = \psi_{\sigma+1}$, $\phi_{\sigma+1} = \psi_\sigma$, but also

$$\phi_{\sigma+k} = \psi_{\sigma+1-k} = \psi_{\sigma+k+1},\quad \phi_{\sigma+k+1} = \psi_{\sigma+k}.$$

Thus a sequence of two associated forms occurs twice in the period; and, assuming (as we have done) that $\sigma=0$, the period of forms is of the

type $\quad \psi_1, \phi_1;\ \phi_2, \phi_3 \ldots \phi_{k-1};\ \phi_k, \psi_k;\ \psi_{k-1}, \psi_{k-2}, \ldots \psi_2,$

where $\phi_{s+1} = \psi_{2k-s}$ for every value of s. The period of integral quotients is of the type

$$\lambda_0, \mu_1, \mu_2, \ldots \mu_{k-1};\ \lambda_k, \mu_{k-1}, \mu_{k-2}, \ldots \mu_1,$$

where $$\lambda_0 = \frac{2\beta_0}{a_1},\quad \lambda_k = \frac{2\beta_k}{a_{k+1}}.$$

The period of complete quotients is of the type

$$v_1, u_1;\ u_2, u_3, \ldots u_{k-1};\ u_k, v_k;\ v_{k-1}, v_{k-2}, \ldots v_2,$$

where $u_{s+1} = v_{2k-s}$ for every value of s. The periods of the coefficients a and β are respectively of the types

$$a_1;\ a_2 \ldots a_k;\ a_{k+1};\ a_k \ldots a_2;$$
$$\beta_1, \beta_1;\ \beta_2, \ldots \beta_{k-1};\ \beta_k, \beta_k;\ \beta_{k-1}, \ldots \beta_2.$$

The two *ambiguous* forms are ϕ_1 and ψ_k.

10. Case (ii.)—If (a, b, c) is properly equivalent to $(-a, -b, -c)$, the periods Φ and $\overline{\Psi}$ must coincide. Hence we must have (for some even value of the suffix 2σ) $\phi_0 = \overline{\psi}_{2\sigma}$; whence $\phi_\sigma = \overline{\psi}_\sigma$, and also $\phi_{\sigma+k} = \overline{\psi}_{\sigma-k} = \overline{\psi}_{\sigma+k}$. The equation $\phi_\sigma = \overline{\psi}_\sigma$ is equivalent to the equation $a_\sigma = a_{\sigma+1}$; we thus see that the period contains two forms, in each of which the extreme coefficients are equal in absolute magnitude; so that (supposing, as before, that $\sigma = 0$), we have

$$D = \beta_0^2 + a_0^2,\quad D = \beta_k^2 + a_k^2.$$

The period of reduced forms is of the type

$$\overline{\psi}_0 = \phi_0;\ \phi_1, \phi_2, \ldots \phi_{k-1};\ \phi_k = \overline{\psi}_k;\ \overline{\psi}_{k-1}, \overline{\psi}_{k-2} \ldots \overline{\psi}_1,$$

where $\phi_s = \overline{\psi}_{2k-s}$; the period of complete quotients is of the type

$$u_0 = v_0;\ u_1, u_2, \ldots u_{k-1};\ u_k = v_k;\ v_{k-1}, v_{k-2}, \ldots v_1,$$

where $u_s = v_{2k-s}$.

Lastly, the periods of integral quotients and of the coefficients a and β are of the types

$$\mu_0, \mu_1, \ldots \mu_{k-1};\ \mu_{k-1}, \mu_{k-2}, \ldots \mu_1, \mu_0;$$
$$a_1, a_2 \ldots a_{k-1}, a_k, a_k, a_{k-1}, \ldots a_2, a_1;$$
$$\beta_0;\ \beta_1, \beta_2 \ldots \beta_{k-1};\ \beta_k;\ \beta_{k-1}, \beta_{k-2} \ldots \beta_1.$$

11. Case (iii.)—If (a, b, c) is properly equivalent to $(-a, b, -c)$, the periods Φ and $\overline{\Phi}$ coincide, and we must have $\phi_0 = \overline{\phi}_{2\sigma+1}$, $\overline{\phi}_0 = \phi_{2\sigma+1}$, where $2\sigma+1$ is less than $2k$. From these equations we infer $\overline{\phi}_0 = \overline{\phi}_{2(2\sigma+1)}$, or $2\sigma+1 = k$, since if $\overline{\phi}_0 = \overline{\phi}_m$, m is a multiple of $2k$. The period of forms is therefore of the type

$$\phi_0, \phi_1, \ldots \phi_{k-1}, \overline{\phi}_0, \overline{\phi}_1, \ldots \overline{\phi}_{k-1},$$

k being an uneven number; the periods of complete quotients, of integral quotients, and of the coefficients a and β, consist each of a period of k terms, twice repeated. The period of equations (Ω) in like manner consists of a period of k equations, twice repeated; but each equation appears in the second half of the period with its sign changed.

12. Case (iv.)—If (a, b, c) is properly equivalent to any two of the forms $(a, -b, c)$, $(-a, -b, -c)$, $(-a, b, c)$, and therefore to all three of them, the nature of the periods is most readily ascertained by considering the series of integral quotients. Since the conditions characteristic of the cases (i.) and (iii.) must be united, the semi-period

$$\lambda_0, \mu_1, \mu_2, \ldots\ldots \mu_{k-1}$$

must be term for term identical with the semi-period

$$\lambda_k, \mu_{k-1}, \mu_2, \ldots\ldots \mu_1,$$

k being an uneven number $2i+1$. Hence $\lambda_0 = \lambda_k$, and the period is of the type $$\lambda, \mu_1, \mu_2 \ldots \mu_i, \mu_i \ldots \mu_2, \mu_1,$$
twice repeated; which combines the characters of the periods of integral quotients in the cases (i.), (ii.), (iii.)

The period of forms is of the type

$$\psi_1, \phi_1; \phi_2, \phi_3, \ldots \phi_{i+1}, \overline{\psi}_i, \overline{\psi}_{i-1}, \ldots \overline{\psi}_2;$$
$$\overline{\psi}_1, \overline{\phi}_1; \overline{\phi}_2, \overline{\phi}_3, \ldots \overline{\phi}_{i+1}, \psi_i, \psi_{i-1}, \ldots \psi_2;$$

where $$\phi_s = \overline{\psi}_{2i+2-s} = \overline{\phi}_{s-2i-1} = \psi_{4i+3-s};$$

the period of complete quotients is of the type

$$v_1, u_1; u_2, u_3, \ldots u_{i+1}; v_i, v_{i+1}, \ldots v_2;$$

twice repeated, where $u_s = v_{2i+2-s}$; and the periods of the coefficients α and β are respectively

$$\alpha_1; \alpha_2, \alpha_3, \ldots \alpha_{i+1}, \alpha_{i+1}, \alpha_i, \ldots \alpha_2;$$
$$\beta_1, \beta_1; \beta_2, \ldots \beta_i, \beta_{i+1}, \beta_i, \ldots \beta_2;$$

each of them twice repeated.

13. If therefore we develope the two roots of a given primitive quadratic equation, we obtain, in the general case, two distinct associated periods of complete quotients and four distinct periods of reduced forms. In the special cases (i.) and (ii.), we have but one period of complete quotients and two periods of reduced forms; in case (i.) the two associated periods of each pair combine; in case (ii.) each period of reduced forms becomes identical with the negative of the opposite of its associated period. In case (iii.) we have two distinct periods of complete quotients; but only two distinct periods of reduced forms; the period of any form being identical with the period of the opposite of its negative, and consisting of an uneven number of forms followed by the opposites of the negatives of the same forms; the period of complete quotients contains only half as many terms as the period of reduced forms. Lastly, in case (iv.) we have but one period of complete quotients, and but one period of reduced forms, the four periods, which in the general case are distinct, being all identical with one another.

We may observe that if the equation

$$(p) \text{ .... } \nu a - 2\mu b + \lambda c = 0$$

can be satisfied by three numbers λ, μ, ν, which also satisfy the condition

$$(q) \text{ .... } \mu^2 - \lambda\nu = 1,$$

the form (a, b, c) is transformed into $(a, -b, c)$ by the substitution $\begin{vmatrix} \mu, & -\nu \\ -\lambda, & \mu \end{vmatrix}$, and consequently has a period of the type (i.) If, instead of the condition (q), the condition

$$(r) \text{ .... } \mu^2 - \lambda\nu = -1,$$

is satisfied by the three numbers (λ, μ, ν), (a, b, c) is transformed into $(-a, -b, -c)$ by $\begin{vmatrix} \mu, & \nu \\ -\lambda, & -\mu \end{vmatrix}$ and the period of (a, b, c) is of the type (ii.) Lastly, if the equation (p) can be satisfied by two different sets of numbers, of which one set satisfies the condition (q) and the other the condition (r), the period of (a, b, c) is of the type (iv.)

14. The periods which we have to consider in the cases (i.), (ii.) and (iv.) afford examples of each of the three kinds of symmetry which can exist in a periodic series. Let $\ldots c_0 c_1 c_2 \ldots c_{n-1} \ldots$ be a period of n terms repeated indefinitely in both directions; it will be found that the series thus formed may be symmetrical (*i.e.*, may be the same whether we follow it forwards or backwards) in three and only in three different ways.

(i.) Let n be even; and let the series continued from c_0 forwards coincide with the series continued from c_{2k+1} backwards so that $c_0 = c_{2k+1}$, $c_1 = c_{2k}$, ... : the period then is

$$c_0 c_1, \ldots c_k;\ c_k, \ldots c_0,\ c_{2k+2}, \ldots c_{n-1};$$

or, if $n = 2\nu$,

$$(\text{A}) \text{ .... } c_{-\nu+k+1}, c_{-\nu+k+2}, \ldots c_0, c_1, \ldots c_k, c_k, \ldots c_0, c_{-1}, \ldots, c_{-\nu+k+1};$$

where there are two centres of symmetry, one falling between the two terms c_k, the other falling between the two terms $c_{-\nu+k+1}$.

Of this type is the period of the coefficients β in case (i.); and the periods of the integral quotients and of the coefficients α in case (ii.)

(ii.) Let n still be even, but let the series continued forwards from c_0 coincide with the series continued backwards from c_{2k}; the period is

$$c_0, c_1, \ldots c_{k-1}, c_k, c_{k-1}, \ldots c_0, c_{2k+1}, \ldots c_{n-1},$$

or, if $n = 2\nu$,

$$(\text{B}) \ldots\ldots c_{-\nu+k};\ c_{-\nu+k+1} \ldots\ldots c_0, c_1 \ldots\ldots c_{k-1};\ c_k;\ c_{k-1} \ldots\ldots c_0, \ldots\ldots c_{-\nu+k+1},$$

where there are two centres of symmetry falling on the terms $c_{-\nu+k}$ and c_k respectively. The symmetry of the integral quotients, and of the coefficients α in case (i.), and of the coefficients β in case (ii.) is of this type.

(iii.) Let $n = 2\nu+1$ be uneven; and let the series continued forward from c_0 coincide with the series continued backward from c_{2k+1}. The period is of the type

$$c_0, c_1, \dots c_k, \; c_k \dots c_0, \; c_{2k+2}, \dots c_{n-1},$$

or $$c_{-\nu+k}; c_{-\nu+k+1}, c_{-\nu+k+2}, \dots c_0, c_1 \dots c_k, c_k \dots c_0, c_{-1} \dots c_{-\nu+k+1},$$

where again there are two centres of symmetry, one falling on the term $c_{-\nu+k}$, the other between the two terms c_k. If we had supposed $c_0 = c_{2k}$ we should have obtained a period of the same form. It is evident that, if this period be doubled, it combines the symmetries of the periods (A) and (B). Of this type are the periods of integral quotients and of the coefficients α and β in case (iv.) In case (iii.) there is no symmetry; but an unsymmetrical uneven period is twice repeated.

15. Every determinant has ambiguous classes; and every ambiguous class has a period of the type (i.) But developments of the types (ii.), (iii.), (iv.) can only present themselves in the case of determinants of the form P or 2P, where P is a product of uneven prime numbers of the form $4n+1$. For in case (ii.) D must, as we have seen, be the sum of two square numbers prime to one another, and in case (iii.) the equation $T^2-DU^2 = -1$ must be resoluble, whence again D is the sum of two squares prime to one another.

If μ is the number of different primes dividing P, the number of ways in which P can be decomposed into the sum of an even and uneven square prime to one another is $2^{\mu-1}$. Let $D = P$, and let $D = A^2+B^2$ be one of these decompositions, A being uneven and B even; the forms $(-A, B, A)$, $(A, B, -A)$, $(-B, A, B)$, $(B, +A, -B)$ are all reduced, and the first two are properly, the last two improperly, primitive. We thus have $2^{\mu-1}$ properly primitive periods of reduced forms of the type (ii.); and as many improperly primitive periods of the same type; *i.e.*, since there are $2^{\mu-1}$ properly and as many improperly primitive ambiguous classes, there are as many classes having periods of the type (ii.) as there are classes having periods of the type (i.)

If $D = 2P$, we have a similar result; viz., there are $2^{\mu-1}$ equations of the form $D = 2P = A^2+B^2$, in which A and B are both uneven. We thus obtain 2^{μ} properly primitive periods of reduced forms of the type (ii.); *i.e.*, as many as there are of type (i.) There are, of course, no improperly primitive classes of a determinant of the form 2P.

When the equations (T) are not resoluble, but the determinant is of either of the forms P or 2P, the developments of the type (i.) and those of the type (ii.) are entirely distinct from one another. On the other hand, when the equations (T) are resoluble, the developments of the types (i.) and (ii.) coincide, giving rise to developments of the type (iv.), and all the remaining developments are of the type (iii.)

It is known that, when D is an uneven power of an uneven prime of the form $4n+1$, the equations (T) are always resoluble. But when D has any other value of either of the forms P or 2P, there is no known criterion for deciding whether these equations are or are not resoluble.

On the Value of a Certain Arithmetical Determinant. By HENRY J. STEPHEN SMITH, Savilian Professor of Geometry in the University of Oxford.

[*Read May* 11*th*, 1876.]

Let (m, n) denote the greatest common divisor of the integral numbers m and n; and let $\psi(m)$ be the number of numbers not surpassing m and prime to m; the symmetrical determinant

$$\Delta_m = \Sigma \pm (1, 1)(2, 2) \ldots\ldots (m, m)$$

is equal to $$\psi(1) \times \psi(2) \times \ldots\ldots \times \psi(m).$$

This theorem may be established as follows. Let $p_1, p_2, \ldots$ be all the different primes dividing m, and consider the columns (P) of which the indices are

$$m, \ \frac{m}{p_1}, \ \frac{m}{p_2}, \ldots, \ \frac{m}{p_1 p_2}, \ldots, \ \frac{m}{p_1 p_2 p_3}, \ \ldots\ldots$$

Take these columns with the signs of the corresponding terms in the product $$\psi(m) = m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\ldots;$$

and, attending to these signs, replace the terms of the last column of Δ_m by the sum of the corresponding terms in the columns (P). The value of Δ_m is not changed: the term (m, m) is evidently replaced by $\psi(m)$; and we shall now show that every other term (m, k) in the last column is replaced by zero; *i.e.*, that $\Delta_m = \psi(m) \times \Delta_{m-1}$, which is the theorem to be proved.

First, let k be prime to m; then

$$(m, k) = 1, \quad \left(\frac{m}{p}, \ k\right) = 1, \quad \&c.;$$