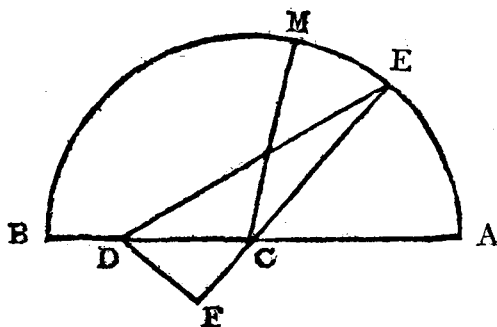

IX. *A NEW and UNIVERSAL SOLUTION of KEPLER'S PROBLEM.* By *JAMES IVORY, Esq.* Communicated by *JOHN PLAYFAIR, Professor of Mathematics and F. R. S. Edin.*

[Read July 7. 1800.]

1. **K**EPLER, having discovered the laws that regulate the motion of a planet in its orbit, proposed the following problem, for determining the true place of a planet at any given time: "To draw a straight line DE, from an eccentric point D in the diameter of a semicircle AEB, so that the whole semicircle may be to the sector ADE, in a given ratio."



IN resolving this problem, we are to take the quadrature of the circle for granted: and therefore, if C be the centre of the circle, and if the sector ACM be taken, such, that the whole semicircle is to the sector ACM in the required given ratio, the problem may be otherwise enunciated: "To draw a straight line DE from an eccentric point D, to cut off a sector ADE, that shall be equal to the given sector ACM."

THE given arch AM, or the given angle ACM, is usually called the mean anomaly; and the arch AE, or the angle ACE, the anomaly of the eccentric: the problem, therefore, is reduced

ced to this: "The mean anomaly being given, to find the anomaly of the eccentric."

2. DRAW the straight line DF at right angles to the diameter passing through the point E: Then, since the sector ADE is equal to the sector ACM, and the space ACE is common to both, the triangle EDC will be equal to the circular sector ECM: therefore, it is manifest, that the straight line DF is equal to the circular arch EM.

SUPPOSE that the radius of the circle is unity; and let $m = \text{arch AM}$, $\mu = \text{arch AE}$, and $\varepsilon = \text{eccentricity DC}$: Then, since $DF = \varepsilon \sin \mu$, we shall have this equation, expressing the relation between the arch of mean anomaly and the arch of eccentric anomaly:

$$m - \mu = \varepsilon \sin \mu.$$

3. IN the equation just found, let us put $m = 2n$ and $\mu = 2\nu$: and, remarking that $\sin \mu = \sin 2\nu = 2 \sin \nu \times \cos \nu$, we shall readily obtain,

$$n - \nu = \varepsilon \sin \nu \times \cos \nu:$$

And, if we further suppose $e = \varepsilon \times \frac{\sin (n - \nu)}{n - \nu}$, and, by means of this formula, exterminate ε , we shall find

$$\sin (n - \nu) = e \sin \nu \times \cos \nu.$$

It may be remarked here, for the greater precision, that, from the nature of the problem, the arches, m and μ , never exceed 180° ; and, of consequence, the arches n and ν , never exceed 90° .

4. IF we consider e as a known or given quantity, it is evident, that the equation last found will no longer have the form of a transcendental equation; and that the arch ν will be determined, when the arch n is given, by a finite equation, resolvable by known methods.

IN strictness, indeed, we cannot consider e as a given quantity; for the exact value of e depends upon the arch ν , and cannot be known unless the length of that arch were known;
in

in other words, unless the problem, of which we are treating, were already resolved. But it is easy to demonstrate, that e is always very nearly equal to the eccentricity ϵ ; and that, therefore, we may assume $e = \epsilon$, at least for a first approximation to the value of ν .

For it is clear, from the most elementary principles, that the maximum value of $\sin \nu \times \cos \nu$ is equal to $\frac{1}{2}$: therefore the arch $n - \nu$, determined by the equation $\sin (n - \nu) = \epsilon \sin \nu \times \cos \nu$, when greatest of all, can never exceed $\frac{\epsilon}{2}$. It is also evident, from the nature of KEPLER'S problem, that ϵ can never be greater than unity; because the point D is supposed to be always taken in the diameter, and never without the circle. Therefore, even in the extreme case, when $\epsilon = 1$, the arch $n - \nu$ can never be greater than $\frac{1}{2}$.

BUT small arches of a circle are very nearly equal to their sines; a proposition that we may extend, without great error, to all arches not exceeding 30° . Now, we have shewn, that the length of the arch $n - \nu$ can never exceed $\frac{1}{2}$, and therefore, that arch will always be less than the arch of 30° , the sine of which is $\frac{1}{2}$. Therefore, the fraction $\frac{\sin (n - \nu)}{n - \nu}$ will always be very nearly equal to unity; and, consequently, the value of e determined by the formula $e = \epsilon \times \frac{\sin (n - \nu)}{n - \nu}$, will, in all cases whatsoever, differ but little from ϵ .

ASSUMING, therefore, $e = \epsilon$, if we denote by π the value of ν corresponding to this value of e , in the equation $\sin (n - \nu) = e \sin \nu \times \cos \nu$; we may consider π as a first approximation to half the arch of eccentric anomaly.

5. HAVING thus found one approximate value of ν , it is easy to find as many others as we please, by means of the formulas already investigated.

FOR, in the formula $e = \varepsilon \times \frac{\sin(n - \nu)}{n - \nu}$, let π , the first approximate value already found, be substituted for ν , and let ε' denote the value of e that will result from the substitution; or, which is the same thing, let $\varepsilon' = \varepsilon \times \frac{\sin(n - \pi)}{n - \pi}$: And further, let ε' be substituted for e in the equation $\sin(n - \nu) = e \sin \nu \times \cos \nu$, and let π' denote the corresponding value of ν : then will π' be a second approximation to the arch ν .

IN like manner, π' being substituted for ν in the formula $e = \varepsilon \times \frac{\sin(n - \nu)}{n - \nu}$, will give a new value of e , denoted by ε'' : and, by means of the equation $\sin(n - \nu) = e \times \sin \nu \times \cos \nu$, this new value of e will give a third approximation to the arch ν , denoted by π'' . And it is manifest, that the series of arches, $\pi, \pi', \pi'', \&c.$ approximating to the value of ν , may be continued indefinitely.

6. I now say, that the arches $\pi, \pi', \pi'', \&c.$ which constitute the series of approximations to the value of ν , are alternately too small and too great: that is, the first, third, fifth, $\&c.$ terms in the series are all less; but the second, fourth, sixth, $\&c.$ terms in the series are all greater, than the exact value of ν .

FOR, if, in the equation $\sin(n - \nu) = e \times \sin \nu \times \cos \nu$, we write $\sin n \cos \nu - \cos n \sin \nu$, for $\sin(n - \nu)$, and divide both sides by $\sin \nu \times \cos \nu$, we shall get

$$e = \frac{\sin n}{\sin \nu} - \frac{\cos \nu}{\cos \nu}:$$

in this formula e vanishes when $n = \nu$: and, supposing the arch ν to decrease, it is manifest that the positive part, $\frac{\sin n}{\sin \nu}$, will

increase, and that the negative part, $\frac{\cos n}{\cos \nu}$, will decrease: therefore e will increase when ν decreases; and the less the arch ν is, the greater will be the value of e . This, it is evident, must also be true, when taken inversely; that is, the greater e is, the less will be the arch ν .

LET

LET us now consider π , the first term in the series of approximate arches; this arch is the value of ν in the equation $\sin(n - \nu) = e \times \sin \nu \times \cos \nu$, when ε is substituted for e ; but, since we have $e = \varepsilon \times \frac{\sin(n - \nu)}{n - \nu}$, it is evident, that e is less than ε , and consequently ν will be greater than π .

AGAIN, take π' , the second term in the series of approximate arches. This arch is the value of ν in the equation $\sin(n - \nu) = e \times \sin \nu \times \cos \nu$, when ε' is substituted for e : Now, $\varepsilon' = \varepsilon \times \frac{\sin(n - \pi)}{n - \pi}$, and $e = \varepsilon \times \frac{\sin(n - \nu)}{n - \nu}$; and since ν has been shewn to be greater than π , it is evident that $n - \pi$ will be greater than $n - \nu$: but the greater arch has to its sine the greater ratio; therefore the fraction $\frac{\sin(n - \pi)}{n - \pi}$, will be less than the fraction $\frac{\sin(n - \nu)}{n - \nu}$: consequently ε' will be less than ε ; and therefore π' will be greater than ν .

AND, in general, if, in the formula $e = \varepsilon \times \frac{\sin(n - \nu)}{n - \nu}$, we substitute a greater arch for ν , we shall have a value of e greater than its true value; but, if we substitute a less arch for ν , we shall have a value of e less than its true value: but we have demonstrated, that, in the equation $\sin(n - \nu) = e \sin \nu \times \cos \nu$, the greater e is, the less will the arch ν be: from which considerations the truth of what we have asserted above is evident, viz. that the arches $\pi, \pi', \pi'', \&c.$ continued indefinitely, are alternately too small and too great.

7. LET us now compare together the alternate terms in the series of approximate arches; and it will not be difficult to perceive, that the first, third, fifth, &c. terms, which have been shewn to be all less than half the arch of eccentric anomaly, continually increase; but that the second, fourth, sixth, &c. terms, which have been shewn to be all greater than half the arch of eccentric anomaly, continually decrease.

FOR, it is obvious, that the greatest value of e , in the formula, $e = \epsilon \times \frac{\sin(n - \nu)}{n - \nu}$, is when $n - \nu = 0$, in which case $e = \epsilon$: therefore, the arch π will be the least of all the approximate arches, $\pi, \pi', \pi'', \&c.$: therefore π is less than π'' , the third term in the series. Again, we have $\epsilon' = \epsilon \times \frac{\sin(n - \pi)}{n - \pi}$, and $\epsilon'' = \epsilon \times \frac{\sin(n - \pi'')}{n - \pi''}$: and since π is less than π'' , it is manifest that $n - \pi$ will be greater than $n - \pi''$: consequently ϵ' will be less than ϵ'' ; whence it follows, that π' , the value of ν corresponding to ϵ' , will be greater than π'' , the value of ν corresponding to ϵ'' . Thus, then, we have shewn, that the first term in the series $\pi, \pi', \pi'', \pi''', \&c.$ is less than the third term; but that the second term is greater than the fourth: and, it is clear that, by a similar mode of reasoning, we may prove that the third term is less than the fifth; but the fourth term greater than the sixth; and so on.

8. It has now been demonstrated, that the arches in the series $\pi, \pi', \pi'', \&c.$ are alternately less and greater than half the arch of eccentric anomaly; and further, that the terms in the series, less than that arch, continually increase, while the terms in the series, greater than that arch, continually decrease; whence, it is manifest, that by computing more and more terms of the series, we shall have the value of the arch of eccentric anomaly within narrower and narrower limits. The arches $\pi, \pi', \pi'', \&c.$ form a series of approximations, converging to the true length of half the arch of eccentric anomaly, and erring alternately in defect and in excess.

AND here a question occurs. It may be asked, Do the terms of the series $\pi, \pi', \pi'', \&c.$ converge slowly to the arch sought? or, Do they converge rapidly to it? According to the answer that we shall be able to give to this question, our method of solution is to be reckoned more or less perfect and valuable.

WE

WE have hitherto considered the series of approximations to the value of ν to begin with the arch π ; but, in effect, the series may be considered to begin with the arch n . For, if in the formula $e = \epsilon \times \frac{\sin(n - \nu)}{n - \nu}$, we substitute n for ν , the resulting value of e will be ϵ , to which the arch π corresponds in the equation $\sin(n - \nu) = e \times \sin \nu \times \cos \nu$. It is clear, therefore, that the arch π is derived from the arch n , precisely in the same way that π' is derived from π ; or that any other term in the series is derived from the term that immediately precedes it.

THE error of the arch n , considered as an approximation to ν , is $n - \nu$: taking the extreme case, when $\epsilon = 1$, (in which case the convergency of the series is evidently the slowest), the length of the arch $n - \nu$, when a maximum, is (Art. 3.) equal to $\frac{1}{2}$, corresponding to $28^\circ 39'$ nearly. Therefore the arch n , considered as an approximation to ν , is very wide of the truth: And, if we can prove that the error of π , the second term in the series, is nevertheless inconsiderable, we shall be entitled to conclude favourably with regard to the convergency of the series.

THE error of the arch π is $\nu - \pi$, and we are now to inquire, to what degree of magnitude this arch may attain. For this purpose let us consider the two equations,

$$(n - \nu) = \epsilon \times \sin \nu \times \cos \nu,$$

$$\sin(n - \pi) = \epsilon \times \sin \pi \times \cos \pi,$$

by means of which the arches ν and π are determined when the arch n is given. It is clear, that the quantities $\epsilon \times \sin \nu \times \cos \nu$ and $\epsilon \times \sin \pi \times \cos \pi$ are evanescent, when $\nu = 0$ and $\pi = 0$, and also when $\nu = 90^\circ$ and $\pi = 90^\circ$: therefore we shall have $n = \nu = \pi$, not only when $n = 0$, but also when $n = 90^\circ$. It has also been shewn, that the arch ν is greater than the arch π : therefore the quantity $\nu - \pi$ vanishes, when $n = 0$ and when $n = 90^\circ$, and between these two limits it is always positive; consequently there is an intermediate value of n , where the arch $\nu - \pi$ will be a maximum.

SINCE

SINCE $2 \sin \nu \times \cos \nu = \sin 2\nu$, and $2 \sin \pi \times \cos \pi = \sin 2\pi$, the two equations become,

$$n - \nu = \frac{\varepsilon}{2} \sin 2\nu,$$

$$\sin (n - \pi) = \frac{\varepsilon}{2} \sin 2\pi;$$

take the fluxions of these equations, making n , ν , and π variable; then, having brought \dot{n} to stand by itself on one side, we shall find,

$$\begin{aligned} \dot{n} &= \dot{\nu} (1 + \varepsilon \cos 2\nu), \\ \dot{n} &= \dot{\pi} \left(1 + \frac{\varepsilon \cos 2\pi}{\cos (n - \pi)} \right). \end{aligned}$$

whence, by equating these two values of \dot{n} ,

$$\dot{\nu} (1 + \varepsilon \cos 2\nu) = \dot{\pi} \left(1 + \frac{\varepsilon \cos 2\pi}{\cos (n - \pi)} \right).$$

Now $\nu - \pi$ is a maximum when $\dot{\nu} - \dot{\pi} = 0$, that is, when $\dot{\nu} = \dot{\pi}$: therefore, if we divide both sides of the preceding equation by the equal quantities $\dot{\nu}$ and $\dot{\pi}$; and further, reject what is common to both sides, and divide the remainders by ε , we shall have, in the case when $\nu - \pi$ is a maximum,

$$\cos 2\nu = \frac{\cos 2\pi}{\cos (n - \pi)}.$$

If we combine this equation with the two equations that express, in general, the relations of n to ν , and of n to π , we shall have three equations sufficient to determine the three arches, n , ν and π , in the case when $\nu - \pi$ is a maximum. But, as one of the equations is transcendental, this could only be done by the method of infinite series, and would lead into very perplexed calculations. We may, however, by an easy formula, determine a limit, which the quantity $\nu - \pi$, when greatest of all, cannot exceed: this will be sufficient for the purpose we have at present in view.

FROM the equation $\text{cof } 2\nu = \frac{\text{cof } 2\pi}{\text{cof } (n - \pi)}$, we readily obtain

$$\text{fin } 2\nu = \frac{\sqrt{(\text{cof}^2 (n - \pi) - \text{cof}^2 2\pi)}}{\text{cof } (n - \pi)} = \frac{\sqrt{\text{fin}^2 2\pi - \text{fin}^2 (n - \pi)}}{\text{cof } (n - \pi)};$$

but $\text{fin } (n - \pi) = \frac{\epsilon}{2} \text{fin } 2\pi$, therefore, by substitution,

$$\text{fin } 2\nu = \frac{\text{fin } 2\pi}{\text{cof } (n - \pi)} \times \sqrt{1 - \frac{\epsilon^2}{4}}.$$

Multiply both sides by $\frac{\epsilon}{2}$; and substitute, for $\frac{\epsilon}{2} \text{fin } 2\nu$, its equal $n - \nu$; and for $\frac{\epsilon}{2} \text{fin } 2\pi$, its equal $\text{fin } (n - \pi)$, and we shall have,

$$n - \nu = \frac{\text{fin } (n - \pi)}{\text{cof } (n - \pi)} \times \sqrt{1 - \frac{\epsilon^2}{4}}.$$

Subtract both sides of this equation from $n - \pi$, and, remarking that $\tan (n - \pi) = \frac{\text{fin } (n - \pi)}{\text{cof } (n - \pi)}$, there will result,

$$\nu - \pi = n - \pi - \tan (n - \pi) \times \sqrt{1 - \frac{\epsilon^2}{4}}.$$

To simplify this formula, I put $n - \pi = \epsilon$, and $a = \sqrt{1 - \frac{\epsilon^2}{4}}$; and so, (in the case when $\nu - \pi$ is a maximum) we have

$$\nu - \pi = \epsilon - a \tan \epsilon.$$

LET us now consider the function $\epsilon - a \tan \epsilon$: because $a = \sqrt{1 - \frac{\epsilon^2}{4}}$ is less than unity, it is evident that we may take the arch ϵ so small, that $a \tan \epsilon$ shall be less than ϵ ; and, that there is a value of ϵ such that $\epsilon = a \tan \epsilon$: it is also manifest, that between the limits $\epsilon = 0$ and $\epsilon = a \tan \epsilon$, (in both which cases $\epsilon - a \tan \epsilon = 0$) the function $\epsilon - a \tan \epsilon$ attains a maximum value.

If, therefore, we can prove, that the arch $n - \pi$ is always between the limits $\epsilon = 0$ and $\epsilon = a \tan \epsilon$; it will follow that the maximum value of the function $\epsilon - a \tan \epsilon$ is greater than the arch $\nu - \pi$, even when that arch is greatest of all.

Now,

Now, since $\sin(n - \pi) = \varepsilon \sin \pi \times \cos \pi$, it is evident that $\sin(n - \pi)$ can never be greater than $\frac{\varepsilon}{2}$. Also, if $\varphi = a \tan \varphi$, we shall have $\sin \varphi < a \tan \varphi$; therefore $\frac{\sin \varphi}{\tan \varphi} = \cos \varphi < a$: therefore, since $a = \sqrt{1 - \frac{\varepsilon^2}{4}}$, it is manifest that $\sin \varphi$ is greater than $\frac{\varepsilon}{2}$. Whence it is obvious, that $n - \pi$ is less than the arch φ , determined by the formula $\varphi = a \tan \varphi$.

It now remains that we determine the maximum value of the function $\varphi - a \tan \varphi$: for this purpose, let $y = \tan \varphi$, and since $\dot{\varphi} = \frac{\dot{y}}{1 + y^2}$, we shall have, by the usual method,

$$\frac{1}{1 + y^2} - a = 0;$$

whence $y = \tan \varphi = \sqrt{\left(\frac{1}{a} - 1\right)}$.

THEREFORE, if we take $\tan \varphi = \sqrt{\frac{1}{\sqrt{1 - \frac{\varepsilon^2}{4}}} - 1}$, the arch $\nu - \pi$ can never be greater than $\varphi - a \tan \varphi$.

If we take $\varepsilon = 1$, we shall have $\tan \varphi = \sqrt{\left(\frac{2}{\sqrt{3}} - 1\right)}$, whence $\varphi = 21^\circ 28' 14''$ nearly, and $\varphi - \frac{\sqrt{3}}{2} \tan \varphi = .03411$, which is the length of an arch of $1^\circ 57'$ nearly. It is therefore certain, that, even in the extreme case, when $\varepsilon = 1$, the arch π cannot differ from half the arch of eccentric anomaly, more than $1^\circ 57'$; a very small error, considering that the first supposition of $n = \nu$ is very wide of the truth. We may therefore conclude, that the series $\pi, \pi', \pi'', \&c.$ converges to the true length of half the arch of eccentric anomaly with uncommon rapidity.

9. WE have now shewn, that, by means of the finite equation $\sin(n - \nu) = e \sin \nu \times \cos \nu$, together with the formula $e = \varepsilon \times \frac{\sin(n - \nu)}{n - \nu}$, we may deduce a series of arches, converging very rapidly to half the arch of eccentric anomaly. The reasonings

fonings that have led us to this conclusion are quite general, and hold good in every state of the data of the problem. To have a complete and universal solution of this famous problem, it only remains, that we investigate a rule for computing the arch ν , from the equation $\sin(n - \nu) = e \times \sin \nu \times \cos \nu$, supposing n and e to be given quantities. This is what we are now to set about.

THE given equation, $\sin(n - \nu) = e \times \sin \nu \times \cos \nu$, is easily transformed into this, (Art. 6.)

$$e = \frac{\sin n}{\sin \nu} - \frac{\cos n}{\cos \nu}.$$

Let ADB be a semicircle, and let the diameter DC be drawn at right angles to AB: Take the arch AM = n , and in CM produced take CF =

$\frac{1}{e} \times CA$: From the point F

draw the straight line FGH, so that the part GH, inter-

cepted between AB and CD, in the angle DCB, may be equal to CA, the radius of the circle: and, lastly, through C draw CN, parallel to FH: I say, that the arch AN is equal to ν .

FROM the two triangles FCH, FCG, we have

$$\sin FHA : \sin FCA :: FC : FH = FC \times \frac{\sin AM}{\sin AN},$$

$$\sin FGD : \sin FCD :: FC : FG = FC \times \frac{\cos AM}{\cos AN}.$$

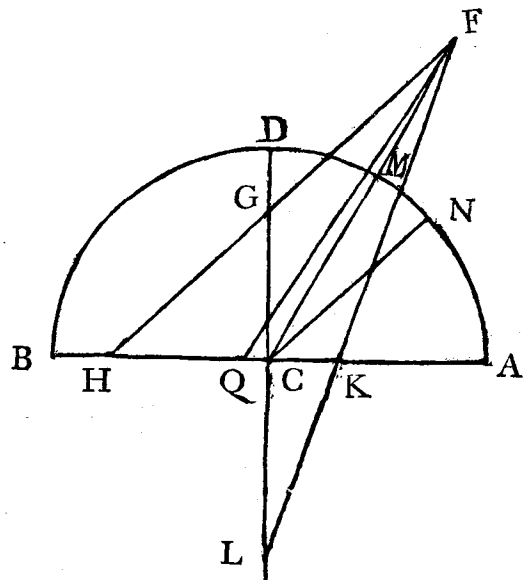
Now, FH — FG = HG = CA, therefore

$$CA = FC \times \left(\frac{\sin AM}{\sin AN} - \frac{\cos AM}{\cos AN} \right);$$

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consequently,



consequently, since $\frac{CA}{CF} = e$, we get

$$e = \frac{\sin AM}{\sin AN} = \frac{\cos AM}{\cos AN};$$

therefore, because $AM = n$, it is manifest that $AN = v$.

Thus, then, the resolution of the equation $\sin(n - v) = e \sin n \times \cos v$, coincides with that case of the general problem, "*De inclinationibus*" of the ancients, in which the two given lines are supposed to be straight lines, intersecting one another

at right angles. If we take $CF = \frac{1}{\varepsilon} \times CA$, the arch AN , found by the construction above, will be no other than the arch π , the first term in the series that we have already discussed: and, in like manner, if we take CF successively equal to $\frac{1}{\varepsilon} \times \frac{(n - \pi)}{\sin(n - \pi)} \times CA$;

$\frac{1}{\varepsilon} \times \frac{(n - \pi')}{\sin(n - \pi')} \times CA$; $\frac{1}{\varepsilon} \times \frac{(n - \pi'')}{\sin(n - \pi'')} \times CA$; and so on: we may find, by the same construction, the other terms π' , π'' , π''' , &c. of that series.

If, therefore, we are to rest satisfied with a geometrical construction, we may consider KEPLER's problem as already resolved. For, it is manifest, from what has been proved, that, by means of the known and elementary problem, "*De inclinationibus*," we may, in all cases, approximate to the arch of eccentric anomaly as nearly as may be required. It must be confessed, however, that a construction of this kind, let it be ever so ingenious or elegant, is of no use to the astronomer, who seeks for a rule by which to conduct his calculations, and who will not be satisfied with a speculation of the mind.

10. The problem, "*De inclinationibus*," when the two lines given by position are supposed to be straight lines, is, in general, a solid problem. The geometrical construction cannot be effected, unless by the help of the conic sections; and the solution, by the modern algebra, leads to an equation of the fourth power,

power, or at least to an equation of the third power. It is known, however, that in one particular situation of the given point, the problem becomes more simple. This happens when the given point is situated any where in a straight line, bisecting the angle contained by the two lines given by position. In this case, the problem becomes a plane problem, and leads to an equation of the second degree only.

SINCE, then, the straight lines AB, CD, intersect one another at right angles, the plane case of the problem will happen, when the angle ACM is half a right angle; that is, when n is equal to an arch of 45° , or when the mean anomaly is a right angle. This case deserves to be particularly considered, on account of the simplicity of the solution it admits of.

WHEN $n = 45^\circ$, it is obvious, that $\sin n = \cos n = \frac{1}{\sqrt{2}}$: and the general equation becomes

$$e = \frac{1}{\sqrt{2}} \times \left(\frac{1}{\sin \nu} - \frac{1}{\cos \nu} \right) = \frac{1}{\sqrt{2}} \times \frac{\cos \nu - \sin \nu}{\sin \nu \cos \nu};$$

squaring both sides,

$$e^2 = \frac{1}{2} \times \frac{\cos^2 \nu + \sin^2 \nu - 2 \sin \nu \cos \nu}{\sin^2 \nu \cos^2 \nu} = \frac{1}{2} \times \frac{1 - 2 \sin \nu \cos \nu}{\sin^2 \nu \cos^2 \nu},$$

but $\sin 2\nu = 2 \sin \nu \cos \nu$: therefore

$$e^2 = 2 \times \frac{1 - \sin 2\nu}{\sin^2 2\nu}.$$

HAVING reduced this formula, we shall find

$$\sin^2 2\nu + \frac{2}{e^2} \sin 2\nu = \frac{2}{e^2}: \text{ whence } \sin 2\nu = \frac{\pm \sqrt{2e^2 + 1} - 1}{e^2}.$$

If we regard the problem, "*De inclinationibus*," independently of any application, and suppose that e may have all possible degrees of magnitude, both these values of $\sin 2\nu$ may give solutions of the problem. One value, viz. $\sin 2\nu = \frac{\sqrt{2e^2 + 1} - 1}{e^2}$,

being always less than 1, will give a solution, whatever be the magnitude of e : it will even give two solutions; because $\sin 2\nu$,

E e 2

being

being determined in magnitude only, will correspond to two arches, the one less, and the other greater than 90° . Of these two arches, the former will give the position of the line FH, having the part GH, inscribed in the angle DCB, equal to CA; and the latter will give the position of the line FKL, having the part KL, inscribed in the angle ACL, equal to CA.

THE other value of $\sin 2\nu$, viz. $\sin 2\nu = \frac{-\sqrt{2e^2 + 1} - 1}{e^2}$, will give no solution when e is not greater than 1; because, in that case, $\frac{\sqrt{2e^2 + 1} + 1}{e^2}$ is greater than 1: But, for other values of e , when $\frac{\sqrt{2e^2 + 1} + 1}{e^2}$ is less than 1, this value of $\sin 2\nu$ will also determine the position of two straight lines, both lying in the angle ACD, that will satisfy the problem.

IN the particular application we have in view, e being never greater than 1, we must compute $\sin 2\nu$ by the formula $\sin 2\nu = \frac{\sqrt{2e^2 + 1} - 1}{e^2}$: and all ambiguity will be taken away, by the consideration that 2ν , or the arch of eccentric anomaly, must be less than the arch of mean anomaly.

THE formula $\sin 2\nu = \frac{\sqrt{2e^2 + 1} - 1}{e^2}$ is not very convenient in practice: we shall obtain another method of computing $\sin 2\nu$, greatly preferable in this respect, in the following manner.

RESUME the formula,

$$e^2 = 2 \times \frac{1 - \sin 2\nu}{\sin^2 2\nu}.$$

Suppose $\sin 2\nu = \frac{2 \cos A}{1 + \cos A}$: then, by substitution, we shall find

$$e^2 = \frac{1}{2} \times \frac{(1 - \cos A) \times (1 + \cos A)}{\cos^2 A} = \frac{\sin^2 A}{2 \cos^2 A} = \frac{1}{2} \times \tan^2 A:$$

therefore $\tan A = e \times \sqrt{2} = e \times \sec 45^\circ$.

Also,

Also, since $\sin 2\nu = \frac{2 \operatorname{cof} A}{1 + \operatorname{cof} A}$; we have $1 - \sin 2\nu =$

$$\frac{1 - \operatorname{cof} A}{1 + \operatorname{cof} A} = \frac{\sin^2 \frac{A}{2}}{\operatorname{cof}^2 \frac{A}{2}} = \tan^2 \frac{A}{2} : \text{now let } 2\nu = \mu = 90^\circ - 2\lambda;$$

then $1 - \sin 2\nu = 1 - \operatorname{cof} 2\lambda = 2 \sin^2 \lambda$: therefore $\sin \lambda = \tan \frac{A}{2} \times \frac{1}{\sqrt{2}} = \tan \frac{A}{2} \times \sin 45^\circ$.

INVERTING this analysis, we derive the following rule for computing the eccentric anomaly, when the mean anomaly is a right angle: Take $\tan A = e \times \sec 45^\circ$; then $\sin \lambda = \tan \frac{A}{2} \times \sin 45^\circ$, and $\mu = 90^\circ - 2\lambda$. This rule would be rigorous and exact, if we could give to e the value it has in the

formula $e = \varepsilon \times \frac{\sin \frac{m - \mu}{2}}{\frac{1}{2}(m - \mu)}$: but as this cannot be done, we

must be content with approximating to the arch sought as nearly as our purpose may require. We will, therefore, by means of the rule, compute a series of arches, $p, p', p'', \&c.$ by successively

substituting for e , the values $\varepsilon, \varepsilon \times \frac{\sin \frac{m - p}{2}}{\frac{1}{2}(m - p)}, \varepsilon \times \frac{\sin \frac{m - p'}{2}}{\frac{1}{2}(m - p')}, \&c.$

and the series $p, p', p'', \&c.$ will converge very quickly to the exact value of the arch of eccentric anomaly, erring alternately in defect and in excess. For the arches here denoted by $p, p', p'', \&c.$ are manifestly no other than the double of the arches formerly denoted by $\pi, \pi', \pi'', \&c.$ respectively.

THE case of the general problem that we have here resolved, may be thus enunciated: "To draw a straight line from an
" eccentric

“eccentric point in the diameter of a femicircle, that shall divide the femicircle into two equal parts.”

II. LET us now proceed to consider the resolution of the formula,

$$e = \frac{\sin n}{\sin v} - \frac{\cos n}{\cos v},$$

supposing n to be any angle whatever not greater than a right angle.

FROM the point F , draw the straight line FKL , so that the part KL , inscribed in the angle ACL , may be equal to CA ; and put $q = \text{angle } FKA$. Then, by proceeding in a manner similar to what is done in Art. 9. we shall easily derive this equation,

$$e = \frac{\cos n}{\cos q} - \frac{\sin n}{\sin q}.$$

FURTHER, let the straight line FQ be drawn to bisect the angle HFK : put $\phi = \text{angle } FQA$, and $\psi = \text{angle } KFQ = \text{angle } QFH$: then, since it has already been shewn, that $v = \text{angle } FHA$, it is obvious that $v = \phi - \psi$ and $q = \phi + \psi$: whence we shall have these two equations, for determining the two angles ϕ and ψ ,

$$e = \frac{\sin n}{\sin (\phi - \psi)} - \frac{\cos n}{\cos (\phi - \psi)};$$

$$e = \frac{\cos n}{\cos (\phi + \psi)} - \frac{\sin n}{\sin (\phi + \psi)}.$$

By taking away the denominators, and remarking, that $\sin (\phi + \psi) \times \cos (\phi + \psi) = \frac{1}{2} \sin (2\phi + 2\psi)$ and $\sin (\phi - \psi) \times \cos (\phi - \psi) = \frac{1}{2} \sin (2\phi - 2\psi)$, there will result

$$\frac{e}{2} \sin (2\phi - 2\psi) = \sin n \times \cos (\phi - \psi) - \cos n \times \sin (\phi - \psi), \text{ and}$$

$$\frac{e}{2} \sin (2\phi + 2\psi) = \cos n \times \sin (\phi + \psi) - \sin n \times \cos (\phi + \psi).$$

TAKE the sum and difference of these two equations, and, for the sums and differences of the sines and co-sines, having the

the same co-efficient, substitute the products that are equal to them; we shall have,

$$e \sin 2\phi \cos 2\psi = 2 \sin n \sin \phi \sin \psi + 2 \cos n \cos \phi \sin \psi, \text{ and} \\ e \cos 2\phi \sin 2\psi = -2 \sin n \cos \phi \cos \psi + 2 \cos n \sin \phi \cos \psi.$$

Now, $\sin n \sin \phi + \cos n \cos \phi = \cos(\phi - n)$, and
 $-\sin n \cos \phi + \cos n \sin \phi = \sin(\phi - n)$: therefore,

$$\left. \begin{aligned} e \sin 2\phi \cos 2\psi &= 2 \cos(\phi - n) \sin \psi \\ e \cos 2\phi \sin 2\psi &= 2 \sin(\phi - n) \cos \psi \end{aligned} \right\} (A)$$

If in the second of the equations (A), we write $2 \sin \psi \cos \psi$ for $\sin 2\psi$, and divide by $\cos \psi$, we shall obtain $e \cos 2\phi \sin \psi = \sin(\phi - n)$:

whence $\sin \psi = \frac{1}{e} \times \frac{\sin(\phi - n)}{\cos 2\phi}$. But $\cos 2\psi = 1 - 2 \sin^2 \psi =$

$1 - \frac{2}{e^2} \times \frac{\sin^2(\phi - n)}{\cos^2 2\phi}$: Substitute these values of $\sin \psi$ and

$\cos 2\psi$ in the first of the equations (A), and, after having reduced, there will result,

$$e^2 \sin 2\phi \cos^2 2\phi - 2 \sin^2(\phi - n) \sin 2\phi = 2 \sin(\phi - n) \cos(\phi - n) \cos 2\phi.$$

Now, $2 \sin^2(\phi - n) = 1 - \cos(2\phi - 2n)$, and $2 \sin(\phi - n) \times \cos(\phi - n) = \sin(2\phi - 2n)$: therefore, by properly ordering the terms,

$$\sin 2\phi - e^2 \sin 2\phi \cos^2 2\phi = \sin 2\phi \cos(2\phi - 2n) - \cos 2\phi \sin(2\phi - 2n).$$

But $\sin 2\phi \cos(2\phi - 2n) - \cos 2\phi \sin(2\phi - 2n) = \sin 2n$: therefore, if we put $x = \sin 2\phi$, since $\cos^2 2\phi = 1 - x^2$, our equation will finally become

$$x^3 + \left(\frac{1}{e^2} - 1\right)x = \frac{\sin 2n}{e^2},$$

which equation will serve to determine $x = \sin 2\phi$.

We have still to find the angle ψ : for this purpose I resume the equations (A), and multiplying cross-wise, and dividing by $2e$, there results,

$$\sin 2\phi \sin(\phi - n) \cos 2\psi \cos \psi = \cos 2\phi \cos(\phi - n) \sin 2\psi \sin \psi.$$

For $\sin 2\psi$ write $2 \sin \psi \cos \psi$: and for $2 \sin^2 \psi$ write $1 - \cos 2\psi$; then, having properly disposed the terms, we shall find

$$(\sin 2\phi \sin(\phi - n) + \cos 2\phi \cos(\phi - n)) \cos 2\psi = \cos 2\phi \cos(\phi - n).$$

Now

Now, $\sin 2\phi \sin (\phi - n) + \cos 2\phi \cos (\phi - n) = \cos (\phi + n)$:
therefore

$$\cos (\phi + n) \times \cos 2\psi = \cos 2\phi \times \cos (\phi - n),$$

whence, as ϕ is now known, $\cos 2\psi$ will be found by this proportion,

$$\cos (\phi + n) : \cos (\phi - n) :: \cos 2\phi : \cos 2\psi.$$

HAVING thus found both the angles ϕ and ψ , their difference will give the angle ν which is sought.

IN order to render the method of computation, derived from the preceding analysis, as simple and as commodious for practice as the nature of the subject will permit, we shall change the letters ϕ and ψ to denote the double of what they have hitherto done; and we shall also put m for its equal $2n$: This being observed, we have the following rule:

1. Let there be formed this cubic equation

$$x^3 + \left(\frac{1}{e^2} - 1\right)x = \frac{\sin m}{e^2},$$

from which x is to be found: then $x = \sin \phi$.

2. STATE this proportion,

$$\cos \frac{\phi + m}{2} : \cos \frac{\phi - m}{2} :: \cos \phi : \cos \psi,$$

by which the angle ψ will be found.

THEN $\phi - \psi = 2\nu =$ arch of eccentric anomaly.

THIS rule would be rigorous and exact if we could give to e

the value it has in the formula $e = \varepsilon \times \frac{\sin \frac{m - \mu}{2}}{\frac{1}{2}(m - \mu)}$: and if, by

means of the rule, we compute a series of arches, $p, p', p'', \&c.$

by successively substituting for e , the values $\varepsilon, \varepsilon \times \frac{\sin \frac{m - p}{2}}{\frac{1}{2}(m - \mu)}$,

$\varepsilon \times$

$\varepsilon \times \frac{\sin \frac{m - p'}{2}}{\frac{1}{2}(m - p')}$, &c.; the series $p, p', p'',$ &c. will converge very

quickly to the exact value of the arch of eccentric anomaly, erring alternately in defect and in excess. For the arches that we here denote by $p, p', p'',$ &c. are manifestly no other than the doubles of the arches formerly denoted by $\pi, \pi', \pi'',$ &c. respectively.

12. It is to be remarked, that, since ε is never greater than 1, the cubic equation in the rule above, is of the form which admits only one real root, so that it may either be resolved by CARDAN'S rule, or by the ordinary methods of approximation; its root is always positive.

It is to be remarked too, that in the same cubic equation, $x = 1$, or $\sin \phi = 1$ when $\sin m = 1$; and, consequently, $\phi = 90^\circ$ when $m = 90^\circ$. Hence, it is easy to infer, that the arch ϕ is less, or greater than a quadrant, or equal to it, according as the arch m is less, or greater than a quadrant, or equal to it. This remark is necessary to determine ϕ , when its sine x is given, on account of the ambiguity of the sines.

When $m = 90$, or $\sin m = 1$, we shall manifestly have the case of the problem that we before (Art. 10.) considered separately. But though we have here $\sin \phi = 1$, and $\phi = 90^\circ$, we shall in vain look for a solution of this case from the general rule, (Art. 11.): because the first and third terms in the proportion for computing $\cos \psi$ become evanescent. We may, however, deduce the rule of calculation of Art. 10. from the general investigation of Art. 11. in the following manner:

RESUME the first of the equations (A), writing ϕ, ψ and m , for $2\phi, 2\psi$ and $2n$, according to the change made in the notation: viz.

$$\varepsilon \sin \phi \times \cos \psi = 2 \cos \frac{\phi - m}{2} \times \sin \frac{\psi}{2}.$$

Suppose $\text{cof } \psi = \frac{2 \text{ cof } A}{1 + \text{cof } A}$: then, since $1 - \text{cof } \psi = 2 \sin^2 \frac{\psi}{2}$

we have $2 \sin^2 \frac{\psi}{2} = \frac{1 - \text{cof } A}{1 + \text{cof } A} = \frac{\sin^2 \frac{A}{2}}{\text{cof}^2 \frac{A}{2}} = \tan^2 \frac{A}{2}$, therefore

$\sin \frac{\psi}{2} = \tan \frac{A}{2} \times \frac{1}{\sqrt{2}} = \tan \frac{A}{2} \times \sin 45^\circ$: We shall also have

$2 \sin^2 \frac{\psi}{2} = \frac{1 - \text{cof } A}{1 + \text{cof } A} = \frac{(1 - \text{cof } A)^2}{1 - \text{cof}^2 A} = \frac{(1 - \text{cof } A)^2}{\sin^2 A}$; whence

$\sin \frac{\psi}{2} = \frac{1 - \text{cof } A}{\sin A} \times \frac{1}{\sqrt{2}}$: Substitute now, for $\text{cof } \psi$, its value

$\frac{2 \text{ cof } A}{1 + \text{cof } A}$; and for $\sin \frac{\psi}{2}$, its value $\frac{1 - \text{cof } A}{\sin A} \times \frac{1}{\sqrt{2}}$; and we shall easily obtain,

$$e \sqrt{2} \times \sin \phi \times \text{cof } A = \text{cof} \frac{\phi - m}{2} \times \sin A,$$

$$\text{whence } \frac{\sin A}{\text{cof } A} = \tan A = e \times \frac{\sin \phi}{\text{cof} \frac{\phi - m}{2}} \times \sqrt{2}.$$

Hence we have another general rule for computing ψ when ϕ is given, which is this: Take $\tan A = e \times \frac{\sin \phi}{\text{cof} \frac{\phi - m}{2}} \times \sec 45^\circ$;

then $\sin \frac{\psi}{2} = \tan \frac{A}{2} \times 45^\circ$.

TAKING $m = 90^\circ$, and $\phi = 90^\circ$, this rule coincides with that already given in Art. 10.: And it may be of use, not only when m is exactly a right angle, but also when it is very nearly so.

THE detail with which we have discussed the rules and formulas of computation that have been deduced from the analysis, leaves no difficulty in applying them to practice. These rules
are

are sufficient, in all cases whatever, for computing the eccentric anomaly, when the mean anomaly is given. They embrace the problem in its fullest extent, and, in point of universality, nothing more is to be wished for.

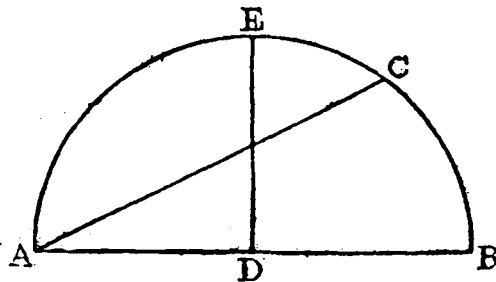
THUS, then, we have a general and direct method of determining the motion of a body describing an elliptic orbit, whether the eccentricity of the orbit be small or great. The method is so extensive, as even to comprehend the case, when the elliptic orbit, having become indefinitely flattened, the motion of the body is no longer in a curve, but in a straight line, tending to the centre of forces. (*Vide Prin. Math. lib. I. sect. 7. prop. 32. et 36.*)

13. IN order to illustrate the method of computation required in the rules that have been investigated, I shall now subjoin two examples. I have selected, for this purpose, two problems relating to the circle, taken from a work of M. EULER, (*Int. in Analys. Inf. lib. xi. cap. 22. prob. 4. et 5.*) where they are resolved by the method of *trial and error*.

EXAMPLE I. Prob. To draw a chord, AC, from the extremity of the diameter of a semicircle, that shall divide the semicircle into two equal parts.

TAKE D, the centre of the circle, and draw DE perpendicular to AB: It is manifest, that the sector BDE will be equal to the sector BAC; and that BE, being the mean anomaly, BC will be the anomaly of the eccentric. We have here, then, $m = 90^\circ$

and $e = 1$: and we must compute by the rule in Art. 10.



F f 2

1. To

1. To compute p , the first term in the series, approximating to BC: Since $e = \varepsilon = 1$, we have

$$\tan A = \sec 45^\circ = 10.1505150,$$

therefore $A = 54^\circ 44'$.

Then $\sin \lambda = \tan \frac{A}{2} \times \sin 45^\circ$; and $p = 90^\circ - 2\lambda$.

$$\text{Now } \log. \tan \frac{A}{2} = 9.7140051$$

$$\log. \sin 45^\circ = 9.8494850$$

$$\log. \sin \lambda = 9.5634901 = \sin 21^\circ 28'.$$

Therefore $\lambda = 21^\circ 28''$, and $p = 90^\circ - 2\lambda = 47^\circ 4'$; this value of p is less than BC.

$$\begin{aligned} 2. \text{ To compute the second term } p', \text{ we have } e &= \varepsilon \times \frac{\sin \frac{m-p}{2}}{\frac{1}{2}(m-p)} \\ &= \frac{\sin 21^\circ 28'}{\sin 21^\circ 28'} \text{ and } \tan A = e \times \sec 45^\circ = \frac{\sin 21^\circ 28'}{\sin 21^\circ 28'} \times \sec 45^\circ. \end{aligned}$$

$$\log. \sec 45^\circ = 10.1505150,$$

$$\log. \sin 21^\circ 28' = 9.5634335$$

$$\text{sum} - 10 = 9.7139485$$

$$\text{add constant log.} = 3.5362739$$

$$13.2502224$$

$$\text{subtract log. } 1288' (= 21^\circ 28') = 3.1099159$$

$$\log. \tan A = 10.1403065$$

therefore $A = 54^\circ 5' 54''$.

Then,

Then, $\sin \lambda = \tan \frac{A}{2} \times \sin 45^\circ$, and $p' = 90^\circ - 2\lambda$,

$$\log. \tan \frac{A}{2} = 9.7080866$$

$$\log. \sin 45^\circ = 9.8494850$$

$$\log. \sin \lambda = 9.5575716 = \log. \sin 21^\circ 9' 53'',$$

therefore, $2\lambda = 42^\circ 19' 46''$, and $p' = 47^\circ 40' 14''$; this value of p' is greater than BC.

3. For the third term p'' , we have $e = \epsilon \times \frac{\sin \frac{m-p'}{2}}{\frac{1}{2}(m-p')} =$

$$\frac{\sin 21^\circ 10'}{\text{arc } 21^\circ 10'} : \text{and } \tan A = e \times \sec 45^\circ = \frac{\sin 21^\circ 10'}{\text{arc } 21^\circ 10'} \times \sec 45^\circ.$$

$$\log. \secant 45^\circ = 10.1505150$$

$$\log. \sin 21^\circ 10' = 9.5576060$$

$$\text{sum} - 10 = 9.7081210$$

$$\text{add const. log.} = 3.5362739$$

$$13.2443949$$

$$\text{subtract log. } 1270' = 3.1038037$$

$$\log. \tan A = 10.1405912$$

therefore $A = 54^\circ 6' 58''$.

Now $\sin \lambda = \tan \frac{A}{2} \times \sin 45^\circ$, and $p'' = 90^\circ - 2\lambda$;

$$\log. \tan \frac{A}{2} = 9.7082530$$

$$\log. \sin 45^\circ = 9.8494850$$

$$\log. \sin \lambda = 9.5577380 = \log. \sin 21^\circ 10' 24''.$$

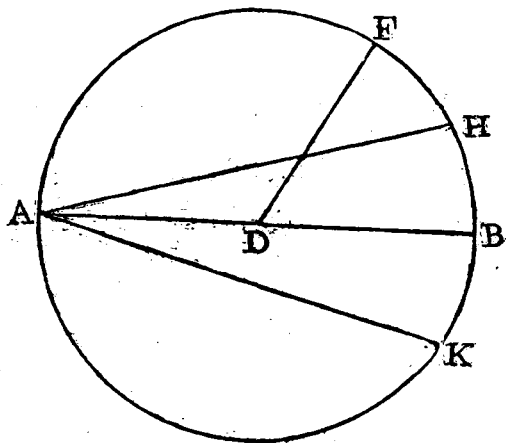
Therefore $2\lambda = 42^\circ 20' 48''$, and $p'' = 47^\circ 39' 12''$; this value of p'' is less than BC.

THE

THE more accurate value of BC , according to the computation of M. EULER, is $47^{\circ} 39' 12'' 46'''$: so that the last approximation is already almost exact. This example is well fitted to illustrate the convergency of the series, $p, p', p'', \&c.$ even in the most unfavourable circumstances.

EXAMPLE 2. Prob. From a given point A , in the circumference of a circle, to draw two chords, AH and AK , that shall divide the circle into three equal parts.

DRAW the diameter AB , and, from the centre D , draw the radius DF , making the angle $BDF = 60^{\circ}$. Because the segment AH is one-third part of the whole circle, it will be two-third parts of the semicircle: therefore the sector HAB will be one third part of the semicircle, and will be equal to the sector BDF . Wherefore it is evident, that BF , being the mean anomaly, BH will be the anomaly of the eccentric: so that in this case we have $m = 60$, and $\epsilon = 1$.



1. To compute the first term p : we have $e = \epsilon = 1$, and the cubic equation becomes simply, $x^3 = \sin m$; whence $x = \sin \phi = \sqrt[3]{\sin m} = \sqrt[3]{\sin 60^{\circ}}$: therefore

$$\begin{aligned} \log. \sin \phi &= 9.9791769, \\ \text{and } \phi &= 72^{\circ} 24'. \end{aligned}$$

$$\text{Now } \cos \frac{\phi + m}{2} : \cos \frac{\phi - m}{2} :: \cos \phi : \cos \psi,$$

cos

$$\begin{aligned}\log. \operatorname{cof} \phi &= 9.4805385 \\ \log. \operatorname{cof} \frac{\phi - m}{2} &= 9.9974523 \\ &\quad \underline{19.4779908} \\ \log. \operatorname{cof} \frac{\phi + m}{2} &= 9.6058923 \\ &\quad \underline{9.8720985} \\ \log. \operatorname{cof} \psi &= 9.8720985\end{aligned}$$

therefore $\psi = 41^\circ 51'$,

consequently $p = \phi - \psi = 30^\circ 33'$, which is less than BH.

2. For the next term p' , we have $e = e \times \frac{\sin \frac{m - p}{2}}{\frac{1}{2}(m - p)} =$

$$\frac{\sin 14^\circ 43' 30''}{\operatorname{arc} 14^\circ 43' 30''}; \text{ and } x^3 + \left(\frac{1}{e^2} - 1\right)x = \frac{\sin m}{e^2} :$$

$$\log. \sin 14^\circ 43' 30'' = 9.4051412$$

$$\text{add const. log.} = 3.5362739$$

$$\text{sum} = 2.9414151$$

$$\text{subtract log. } 883'.5 = 2.9462066$$

$$\log. e = 1.9952085$$

$$\log. \frac{1}{e} = 0.0047915$$

2.

$$\log. \frac{1}{e^2} = 0.0095830, \text{ and } \frac{1}{e^2} = 1.022410,$$

also $\frac{1}{e^2} - 1 = .022410 = a.$ Now, $\log. \sin m = 9.9375306,$

therefore $\log. \frac{\sin m}{e^2} = 9.9471136,$ and $\frac{\sin m}{e^2} = .885347 = b.$

The

The cubic equation therefore becomes $x^3 + ax = b$: and it is manifest that x is nearly $= \sin 72^\circ 24'$; and, having corrected this value, by the ordinary method, I find,

$$x = \sin \phi = .9524420$$

therefore $\phi = 72^\circ 15' 31''$.

$$\text{But, } \operatorname{cof} \frac{\phi + m}{2} : \operatorname{cof} \frac{\phi - m}{2} :: \operatorname{cof} \phi : \operatorname{cof} \psi,$$

$$\log. \operatorname{cof} \phi = 9.4839026$$

$$\log. \operatorname{cof} \frac{\phi - m}{2} = 9.9975102$$

$$19.4814128$$

$$\log. \operatorname{cof} \frac{\phi + m}{2} = 9.6071052$$

$$\log. \operatorname{cof} \psi = 9.8743076.$$

Therefore $\psi = 41^\circ 31' 20''$, and

consequently $p' = \phi - \psi = 30^\circ 44' 11''$, which is greater than BH.

M. EULER finds the arch $AH = 129^\circ 16' 27''$; therefore, $BH = 30^\circ 43' 33''$: so that the second approximation p' , differs little more than half a minute of a degree from the truth.

14. THE only cases of KEPLER's problem that are interesting to the astronomical observer, are the two extreme cases, when the eccentricity is very small, and when it is very great, approaching to unity. The former of these two cases is that of the planets, all of which describe orbits very little eccentric, and nearly circular; the latter is that of the comets, which, on the contrary, move in very eccentric orbits. The principal object of this paper is accomplished in what has already been done; but it will be no improper sequel, to apply the general method to the two cases just mentioned.

THE supposition that the eccentricity is small, contributes greatly to remove the chief difficulties that occur in the solution
of

of KEPLER's problem. Indeed, it is only this particular case of the general problem, that we can consider to have been resolved, hitherto, in a satisfactory manner. We already possess many excellent solutions of this case, some of them deduced from the most elementary principles, and others obtained by the aid of the higher calculus. To these another may be added, derived as a corollary from the general solution contained in this paper, and which will be found not unworthy of the notice of astronomers.

BECAUSE the two arches denoted by μ and p are the double of the arches ν and π , the arch $\mu - p$, which is the error of p , considered as an approximation to the eccentric anomaly, will be double of the arch $\nu - \pi$: therefore, it is obvious, (Art. 8.) that the arch $\mu - p$ will never be greater than

$$2 \left(\varepsilon - \tan \varepsilon \sqrt{1 - \frac{\varepsilon^2}{4}} \right),$$

$$\text{taking } \tan \varepsilon = \sqrt{\frac{1}{\sqrt{1 - \frac{\varepsilon^2}{4}}} - 1}.$$

THE slightest attention to the nature of this expression, is sufficient to evince, that it decreases very rapidly as ε decreases. If we evolve it into a series, proceeding according to the powers of ε , that series will contain only the third and higher odd powers. Therefore, when ε is small, as it is in the case of the planets, the amount of the above formula will be inconsiderable. It may even be so inconsiderable, that the error of p will be of no account in practice, and the first approximation will give the eccentric anomaly with the requisite exactness.

By means of this formula, I have computed the limit of the error of p , for all the planetary orbits, and have arranged the results in the following table:

	Value of ϵ	Limit of the error of p
Mercury,	.205513	1' 46".0
Venus,	.006885	0.0
Earth,	.016814	0.0
Mars,	.093088	9.8
Jupiter,	.048077	1.3
Saturn,	.056223	2.1
Georgium Sidus,	.046683	1.2

The inspection of this table shews, that the error of the first approximation, obtained by the supposition of $e = \epsilon$, in all the planetary orbits, is a very small quantity, and such as may be neglected on most occasions.

It is to be recollected, that the rule we have investigated for computing the eccentric anomaly, would give a rigorous result, provided the exact value of e were known. But, as that value cannot be deduced directly from the data, the repetition of the calculation is necessary to correct the first assumed value of e , and to make it approach nearer and nearer to the true value. The method of proceeding that is directed above, viz. to assume at first $e = \epsilon$, and from thence to deduce a series of approximations to the arch sought, is perhaps the only one that will apply universally, and in all circumstances of the problem. But it is to be observed, that the reasoning in Art. 4, 5, and 6. will remain the same, provided only that the first assumed value of e be greater than its true value: and, if the first assumed value of e be less than its true value, the reasoning will not thereby be essentially altered; all the change that will take place, is, that the error of the approximations will now be alternately in excess and in defect. Therefore, in applying the general method, if we can take hold of any circumstances, peculiar to the particular

cular case, from which we can directly deduce a nearer value of e than the supposition of $e = \varepsilon$, we will avail ourselves of such circumstances, and will thereby obtain a series of approximations, converging faster to the eccentric anomaly.

RESUME the general equation of the arches of eccentric and mean anomalies (Art. 2.) viz.

$$m - \mu = \varepsilon \sin \mu :$$

it is evident, that the less ε is, the less will be the difference of the two arches m and μ ; and that, when ε is small, the quantity $\varepsilon \sin \mu$ will be nearly equal to the quantity $\varepsilon \sin m$. Therefore, if we take an arch τ , such that $2\tau = \varepsilon \sin m$, it is ob-

vious that we shall have $\frac{\sin \tau}{\tau} = \frac{\sin \frac{m - \mu}{2}}{\frac{1}{2}(m - \mu)}$ nearly, and, conse-

quently, $e = \varepsilon \times \frac{\sin \tau}{\tau}$ nearly. To speak more correctly, the error of the assumption $e = \varepsilon$, will be of the same order with the third power of the eccentricity; but the error of the assumption, $e = \varepsilon \frac{\sin \tau}{\tau}$, will be of the same order with the fourth power of the eccentricity*.

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* SINCE $m - \mu = \varepsilon \sin \mu$, we have, in series,

$$\sin \frac{m - \mu}{2} = \frac{\varepsilon \sin m}{2} - \frac{1}{6} \cdot \frac{\varepsilon^3 \sin^3 \mu}{2^3} + \frac{3}{40} \cdot \frac{\varepsilon^5 \sin^5 \mu}{2^5} - \&c.$$

$$\text{Therefore } \frac{\sin \frac{m - \mu}{2}}{\frac{1}{2}(m - \mu)} = 1 - \frac{\varepsilon^2}{24} \cdot \sin^2 \mu + \frac{3\varepsilon^4}{640} \sin^4 \mu - \&c.$$

$$\text{and } e = \varepsilon \times \frac{\sin \frac{m - \mu}{2}}{\frac{1}{2}(m - \mu)} = \varepsilon - \frac{\varepsilon^3}{24} \sin^2 \mu + \frac{3\varepsilon^5}{640} \sin^4 \mu - \&c.$$

so that the error of the supposition $e = \varepsilon$, is manifestly of the order ε^3 .

AGAIN, from the equation $m - \mu = \varepsilon \sin \mu$, we easily derive $\sin \mu = \sin m - \frac{\varepsilon}{2} \sin 2m$, neglecting the terms above the first order: and, substituting

this

It has been shewn above, that the error of the first approximation, derived from the assumption $e = \epsilon$, is almost of no account, as to any real practical purpose, in the orbits of all the planets, excepting Mercury: and much more will this conclusion be true of the more exact assumption $e = \epsilon \frac{\sin \tau}{\tau}$.

LET us now consider the cubic equation which the rule requires to be resolved: the equation is

$$x^3 + \left(\frac{1}{e^2} - 1\right)x = \frac{\sin m}{e^2},$$

and, in the case we are now occupied with, e is small, being nearly equal to the eccentricity. Multiply all the terms of the equation by e^2 ; write $\sin \phi$ for x , and $\cos^2 \phi$ for $1 - x^2$; and we shall easily obtain,

$$\sin \phi = \sin m + e^2 \sin \phi \cos^2 \phi.$$

In this formula it is clear, that the term $e^2 \sin \phi \cos^2 \phi$ is inconsiderable in comparison of the other two: therefore $\sin \phi = \sin m$ nearly; and, consequently, the two arches ϕ and m will differ but little from one another. From this consideration, we readily derive a series of approximations, ϕ' , ϕ'' , ϕ''' , &c. converging very fast to the exact value of ϕ ; viz.

$$\sin \phi' = \sin m + e^2 \sin m \cos^2 m$$

$$\sin \phi'' = \sin m + e^2 \sin \phi' \cos^2 \phi'$$

$$\sin \phi''' = \sin m + e^2 \sin \phi'' \cos^2 \phi'',$$

and so on. The error of the first approximation ϕ' , is of the order

this value of $\sin \mu$ in the series for e , we have, (neglecting the terms above the order ϵ^4),

$$e = \epsilon - \frac{\epsilon^3}{24} \sin^2 m + \frac{\epsilon^4}{24} \sin m \times \sin 2m;$$

and this value of e is exact, as far as the order ϵ^4 inclusively.

BUT the assumption $e = \epsilon \times \frac{\sin \tau}{\tau}$, where $\tau = \frac{\epsilon \sin m}{2}$, being thrown into a series, we get,

$$e = \epsilon - \frac{\epsilon^3}{24} \sin^2 m + \frac{\epsilon^5}{640} \sin^4 m - \&c.$$

and therefore the error of this assumption is of the order ϵ^4 .

order e^4 or ϵ^4 ; that of the second approximation is of the order ϵ^5 ; and in none of the planetary orbits will it be necessary to push the approximations further than the second term of the series.

THE method of finding the arch ϕ , that has just been explained, is very commodious in practice; because the value of $e^2 \sin \phi \cos^2 \phi$ is easily computed by the common tables, when a known arch is substituted for ϕ . But we may, with advantage, apply the method of infinite series to the resolution of the equation. If we put $\beta^2 = 1 - e^2$, and $z = \frac{\sin m}{\beta^2} \times \frac{e}{\beta}$, we have

$$\sin \phi = \frac{\sin m}{\beta^2} \times (1 - z^2 + 3 z^4 - 12 z^6 + 55 z^8 - \&c.).$$

THE rule requires still another operation, viz. to find the arch ψ . For this purpose we have the proportion

$$\cos \frac{\phi + m}{2} : \cos \frac{\phi - m}{2} :: \cos \phi : \cos \psi.$$

But, in the case we are now considering, the arch ψ is always small: and, on this account, the proportion above is of little use in practice, when any degree of accuracy is required. The reason is, that the common tables are not computed to a sufficient number of figures for determining small angles from their cosines. We will, therefore, prefer the other method of computing ψ , given in Art. 12. which is not liable to the same inconvenience.

THE observations we have now made, lead us to the following rule, for computing the anomaly of the eccentric in the orbits of the planets:

1. COMPUTE the arch τ from the formula $2\tau = \epsilon \times \sin m$.
2. COMPUTE $e = \epsilon \times \frac{\sin \tau}{\tau}$; and determine the arch ϕ from the equation $\sin \phi = \sin m + e^2 \sin \phi \cos^2 \phi$.
3. COMPUTE $\tan A = e \times \frac{\sin \phi}{\cos \frac{\phi - m}{2}} \times \sec 45^\circ$; and $\sin \frac{\psi}{2} = \tan \frac{A}{2} \times \sin 45^\circ$.

THEN $\mu = \phi - \psi$.

It

It is to be kept in mind, that the arch found by this rule is only the first term of a series of approximations, converging very fast to the eccentric anomaly: and that, by a repetition of the calculation, a result may be obtained, that will satisfy the most scrupulous accuracy. But the rule may be considered as exact, as to any real practice, for the orbits of all the planets, excepting Mercury: and, even in the orbit of Mercury, the error will never exceed a few seconds. Let it be observed, further, that the error of the rule is chiefly in the arch ψ : for the error of ψ is of the same order with the error of e ; whereas the error of ϕ is of the same order with the error of e^2 .

EXAMPLE. Let it be required to find the eccentric anomaly, corresponding to the mean anomaly $64^\circ 37' 8''.5$ in the orbit of Mars, supposing the eccentricity to be $= .093088$.

We have here $m = 64^\circ 37' 8''.5$, and $\varepsilon = .093088$.

1. To compute τ from the formula $2\tau = \varepsilon \sin m$;

$$\log. \varepsilon = \overline{2.9688937}$$

$$\log. \sin m = 9.9559089$$

$$\text{const log.} = \underline{3.5362739}$$

$$\log. 2\tau \text{ in minutes} = 2.4610765; 2\tau = 289'; \text{ and } \tau = 2^\circ 24'.$$

2. THEN $e = \varepsilon \frac{\sin \tau}{\tau}$; and $\sin \phi' = \sin m + e^2 \sin m \cos^2 m$.

$$\sin \tau = 8.6219616$$

$$\log. \varepsilon = \overline{2.9688937}$$

$$\text{const. log.} = \underline{3.5362739}$$

$$\text{sum} - 10 = 1.1271292$$

$$\text{subtract log. } 144' = \underline{2.1583625}$$

$$\log. e = \overline{2.9687667}$$

$$\log. e^2 = \overline{3.9375334}$$

$$\log. \sin m = 9.9559089$$

$$2 \log. \cos m = \underline{19.2642510}$$

$$\log. e^2 \sin m \cos^2 m = \overline{3.1576933}, \text{ and } e^2 \sin m \cos^2 m = .0014378.$$

To

To this adding nat. $\sin m = .9034776$, we have nat. $\sin \phi' = .9049154$, and $\phi' = 64^\circ 49'$. To have a more correct value of ϕ , repeat the calculation,

$$\begin{aligned} \text{Log. } e^2 &= \overline{3.9375334} \\ \log. \sin \phi' &= 9.9566250 \\ 2 \times \log. \cos \phi' &= \underline{19.2578320} \\ \log. e^2 \sin \phi' \cos^2 \phi' &= \overline{3.1519904}, \text{ whence nat. } \sin \phi = .9048966, \\ \text{therefore } \phi &= 64^\circ 48' 33''. \end{aligned}$$

3. To find ψ , we have $\tan A = e \times \frac{\sin \phi}{\cos \frac{\phi - m}{2}} \times \sec 45^\circ$;

$$\begin{aligned} \log. e &= \overline{2.9687667} \\ \log. \sin \phi &= 9.9565982 \\ \log. \sec 45^\circ &= \underline{10.1505150} \\ \text{Sum} &= 19.0758799 \\ \log. \cos \frac{\phi - m}{2} &= 9.9999994 \\ \log. \tan A &= \overline{9.0758805} = \log. \tan 6^\circ 47' 29'' \end{aligned}$$

But $\sin \frac{\psi}{2} = \tan \frac{A}{2} \times \sin 45^\circ$, therefore

$$\begin{aligned} \log. \tan \frac{A}{2} &= 8.7733146 \\ \log. \sin 45^\circ &= \underline{9.8494850} \end{aligned}$$

$$\log. \sin \frac{\psi}{2} = 8.6227996 = \log. \sin 2^\circ 24' 16''.5. \text{ Therefore}$$

$\frac{\psi}{2} = 2^\circ 24' 16''.5$, and $\psi = 4^\circ 48' 33''$, so that $\mu = \phi - \psi = 60^\circ$, the eccentric anomaly required.

15. It remains now to consider the case of the problem, applicable to comets, where the eccentricity is very great, or nearly equal to unity. Here the general solution may be used, and, even without any new simplification, arising from the peculiarities of the case, will bring out an accurate result with very little trouble.

trouble. I shall give an example in the orbit of the famous comet of 1682.

THE comet of 1682, which re-appeared in 1759, according to the prediction of Dr HALLEY, is the only one of which the period is known with any tolerable degree of certainty. M. de la LANDE has fixed the period of this comet at 28070 days: computing from this the mean distance from the sun, by the law of KEPLER, that the cubes of the mean distances are as the squares of the periodic times, we shall find, that half the greater axis of the ellipse described by the comet, is 18.07575, the mean distance of the earth from the sun being unity. According to the determination of the same astronomer, the perihelion distance, estimated in parts of the same unit, is 0.5835; consequently, the distance of the focus of the ellipse from the centre, is 17.49225. Therefore, in the orbit of this comet, the eccentricity, or the quantity ϵ , is equal to $\frac{17.49225}{18.07575} = 0.96772$

nearly: and we can now assign the true place of the comet in the orbit, as well as its true distance from the sun, at any given distance of time, from the passage over the perihelion or aphelion.

EXAMPLE. Let it be required to find the anomaly of the eccentric of the comet 1759, (from which the true place and true distance from the sun are derived by easy and known rules), 16 days, 4 hours, 44' before or after the passage over the perihelion.

The mean anomaly, corresponding to the given time, is $0^{\circ} 12' 27''.83$, reckoned from the perihelion; but as our method requires the mean anomaly to be reckoned from the aphelion, we have $m = 179^{\circ} 47' 32''.17$.

1. To compute the first approximation to the eccentric anomaly sought, we have $e = \epsilon = 0.96772$; hence $\frac{1}{e^2} = 1.0678$;

$$\frac{1}{e^2} - 1 = 0.0678 = a; \quad \frac{\sin m}{e^2} = .0038715 = b.$$

Then,

Then, to find a value of x , from the equation, $x^3 + a x = b$: since b is small in comparison of a , it is manifest that x must be a very small fraction; and, consequently, x^3 inconsiderable in respect of $a x$: therefore $x = \frac{b}{a} = .05$ nearly: and, having corrected this value by the common method, we shall find,

$$x = \sin \phi = .0547 = \sin 3^\circ 8', \text{ and so } \phi = 176^\circ 52'.$$

As the first approximation which we are now computing, cannot be exact, even to the nearest minute, it would be fruitless to push the calculation to a great degree of accuracy. For the same reason, I here use the proportion in the general rule, because it requires but one operation for finding ψ , viz. $\cos \frac{\phi + m}{2}$:

$$\cos \frac{\phi - m}{2} :: \cos \phi : \cos \psi.$$

HENCE ψ is found $= 3^\circ 1'$, and therefore $p = \phi - \psi = 173^\circ 51'$; this is the first approximation to the eccentric anomaly, reckoned from the aphelion, and is too small.

adly, To compute the second approximation, we have,

$$e = \varepsilon \times \frac{\sin \frac{m - p}{2}}{\frac{1}{2}(m - p)} = \varepsilon \times \frac{\sin 2^\circ 58'}{\text{arc } 2^\circ 58'}; \text{ whence } \frac{1}{e^2} = 1.06878;$$

$\frac{1}{e^2} - 1 = .06878 = a$; $\frac{\sin m}{e^2} = .0038749 = b$. Therefore, $x^3 + a x = b$: but we know, that a near value of x is .0547: and, having corrected this value, we shall find, $x = \sin \phi = .0540430$; therefore, $\sin \phi = .0540430 = \sin 3^\circ 5' 52''$, and $\phi = 176^\circ 54' 8''$.

As ψ is here a small angle, we must use the method of Art. 12. to find it with the requisite exactness: this gives $\tan A = e \times \frac{\sin \phi}{\cos \frac{\phi - m}{2}} \times \sec 45^\circ$, and $\sin \frac{\psi}{2} = \tan \frac{A}{2} \times \sin 45^\circ$; there-

fore $\log. \tan A = 8.8689481$, and $\psi = 2^\circ 59' 32''$,

wherefore $p' = \phi - \psi = 173^\circ 54' 36''$, which is greater than the anomaly of the eccentric, reckoning from the aphelion, but very near to it: the error not exceeding three or four seconds. The eccentric anomaly, reckoned from the perihelion, is therefore nearly $6^\circ 5' 24''$, but greater than this arch.

16. In this method of calculating, though any degree of accuracy may be obtained, yet when the distance from the perihelion is very small, the computation may run out to considerable length. In seeking a remedy for this inconvenience, I saw that much advantage might be obtained, by comparing the motion in an eccentric ellipse with the motion in a parabola, since at the perihelion they so nearly agree. The result of this comparison will perhaps be thought to make an useful addition to the general methods explained above.

LET there be proposed this geometrical problem: An eccentric ellipse being given, and likewise a parabola, having the same perihelion distance with the ellipse; it is required to draw a radius vector in the ellipse, to cut off a sector that shall be equal to a given sector of the parabola.

IF this problem can be resolved, the application of it to the present research will be easy.

LET the radii vectores that cut off the equal sectors from the ellipse and the parabola, be respectively denoted by ρ and r ; and let v and z be the angles that ρ and r make with the axes of the curves, reckoned from the perihelia: then, considering the two sectors as variable quantities, we shall have

$$\rho^2 \dot{v} = r^2 \dot{z},$$

for these are obviously the doubles of the fluxions of the two sectors.

LET a be the mean distance of the ellipse, and ϵ the eccentricity: then, from the known property of the curve,

$$\rho = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos v}.$$

SUPPOSE

SUPPOSE $x = \tan \frac{v}{2}$; then $\text{cof } v = \frac{1 - x^2}{1 + x^2}$: and, if we put p to denote the perihelion distance, $= a (1 - \varepsilon)$, and $\lambda = \frac{1 - \varepsilon}{1 + \varepsilon}$, we shall obtain, by substitution,

$$\varepsilon = \frac{p (1 + \varepsilon) (1 + x^2)}{(1 + \varepsilon) + (1 - \varepsilon) x^2} = p \times \frac{1 + x^2}{1 + \lambda x^2}.$$

Further, from the equation $x = \tan \frac{v}{2}$, we get, $\dot{v} = \frac{2 \dot{x}}{1 + x^2}$:

Therefore,

$$\varepsilon^2 \dot{v} = 2 p^2 \times \frac{\dot{x} (1 + x^2)}{(1 + \lambda x^2)^2}.$$

AGAIN, the perihelion distance of the parabola being, by the supposition, equal to the perihelion distance of the ellipse, we have, from the nature of the curve,

$$r = \frac{2 p}{1 + \text{cof } z} = \frac{p}{\text{cof}^2 \frac{z}{2}}.$$

LET $y = \tan \frac{z}{2}$: then $\frac{1}{\text{cof}^2 \frac{z}{2}} = 1 + y^2$ and $z = \frac{2 y}{1 + y^2}$: con-

sequently,

$$r^2 \dot{z} = 2 p^2 \times \dot{y} (1 + y^2).$$

Now, equating the values of $\varepsilon^2 \dot{v}$ and $r^2 \dot{z}$, that have just been obtained, and, omitting the common multiplier, $2 p^2$, there will result,

$$\dot{y} (1 + y^2) = \frac{\dot{x} (1 + x^2)}{(1 + \lambda x^2)^2}.$$

and taking the fluents

$$y + \frac{y^3}{3} = \left\{ x + \frac{x^3}{3} \right\} - 2\lambda \times \left\{ \frac{x^3}{3} + \frac{x^5}{5} \right\} + 3\lambda^2 \left\{ \frac{x^5}{5} + \frac{x^7}{7} \right\} - 4\lambda^3 \left\{ \frac{x^7}{7} + \frac{x^9}{9} \right\} + \&c.$$

It is manifest, that this fluent requires no correction; because

H h 2

the

the two angles, v and z , are supposed to begin to flow together at the perihelia of the curves.

17. ACCORDING to the supposition, the eccentricity ε is nearly equal to unity, and consequently, $\lambda = \frac{1 - \varepsilon}{1 + \varepsilon}$ will be a small fraction: but, as $x \left(= \tan \frac{v}{2} \right)$ increases from 0 to ∞ , it is obvious, that the fluent, obtained above, will be of use in computation only to a certain limit, however small λ may be supposed to be. For the part of the fluent depending on λ manifestly converges by the powers of the quantity λx^2 ; and therefore, as long as λx^2 is a small fraction, so long only can we compute x when y is given, by means of the fluent: but when x has passed that limit, the fluent, in the form here given to it, is no longer of any use in computation.

BUT the fluent, although limited in its application by the consideration just explained, will enable us to compute x when y is given, and to determine the angle v of true anomaly in the ellipse, by means of the angle z in the parabola, for a considerable portion of the elliptic orbit lying adjacent to the perihelion, on either side. We may therefore deduce from it a series that will serve to compute the true place of a comet in the portion of its orbit which it describes during one apparition.

IN order to determine the angle v by means of the angle z , we must first find a value of x in terms of y : and, to avoid too complex calculations, we shall neglect the terms multiplied by the powers of λ , higher than the square. The extreme smallness of λ , in the orbits of all the comets, permits to simplify the calculation in this manner, and, nevertheless, to obtain a result, that will be sufficiently exact during the time of one apparition.

NEGLECTING, then, the terms multiplied by the powers of λ higher than the square, we have

$$y + \frac{y^3}{3} = \left\{ x + \frac{x^3}{3} \right\} - 2\lambda \left\{ \frac{x^3}{3} + \frac{x^5}{5} \right\} + 3\lambda^2 \left\{ \frac{x^5}{5} + \frac{x^7}{7} \right\}.$$

Assume

Assume $x = y + A \lambda y^3 + B \lambda^2 y^5$,

A and B being indeterminate quantities, not depending on λ : Then, neglecting the quantities which the degree of exactness prescribed permits us to neglect, we shall find,

$$x^3 = y^3 + 3 A \lambda y^5 + 3 (B + A^2) \lambda^2 y^7,$$

$$x^5 = y^5 + 5 A \lambda y^7,$$

$$x^7 = y^7.$$

If now we substitute these values in the equation between x and y , and omit the terms common to both sides, there will result,

$$\begin{aligned} 0 &= (A + A y^2) \times \lambda y^3 + (B + B y^2 + A^2 y^2) \times \lambda^2 y^5, \\ &- 2 \left(\frac{1}{3} + \frac{1}{5} y^2 \right) \times \lambda y^3 - 2 (A + A y^2) \times \lambda^2 y^5, \\ &+ 3 \left(\frac{1}{5} + \frac{1}{7} y^2 \right) \times \lambda^2 y^5. \end{aligned}$$

Hence

$$A = \frac{2}{3} \times \frac{1}{1 + y^2} + \frac{2}{5} \times \frac{y^2}{1 + y^2};$$

$$B = \frac{11}{15} \times \frac{1}{1 + y^2} + \frac{13}{35} \times \frac{y^2}{1 + y^2} - A^2 \times \frac{y^2}{1 + y^2}.$$

But, since $y = \tan \frac{z}{2}$; therefore $\frac{1}{1 + y^2} = \cos^2 \frac{z}{2} = 1 - \sin^2 \frac{z}{2}$,

and $\frac{y^2}{1 + y^2} = \sin^2 \frac{z}{2}$: consequently

$$A = \frac{2}{3} - \frac{4}{15} \sin^2 \frac{z}{2},$$

$$B = \frac{11}{15} - \frac{254}{315} \sin^2 \frac{z}{2} + \frac{16}{45} \sin^4 \frac{z}{2} - \frac{16}{225} \sin^6 \frac{z}{2}.$$

18. SUPPOSE $v = z + w$; w expressing the difference of the two angles v and z , which, it is obvious, depends on λ , and is to be reckoned of the same order with that quantity: Then $x = \tan \frac{v}{2} = \tan \frac{z + w}{2}$: but $y = \tan \frac{z}{2}$; therefore, according to TAYLOR'S theorem, rejecting the quantities that ought to be rejected,

rejected,

$$x = \tan \frac{z + w}{2} = y + \frac{\dot{y}}{z} \times w + \frac{1}{2} \cdot \frac{\ddot{y}}{z^2} \times w^2.$$

Assume now $w = C \times \lambda + D \times \lambda^2$; C and D being indeterminate quantities, not depending on λ ; then, by substitution,

$$x = y + \frac{\dot{y}}{z} \times C \lambda + \left\{ \frac{\dot{y}}{z} D + \frac{1}{2} \cdot \frac{\ddot{y}}{z^2} \times C^2 \right\} \times \lambda^2.$$

But $\frac{\dot{y}}{z} = \frac{1 + y^2}{2}$; and $\frac{1}{2} \cdot \frac{\ddot{y}}{z^2} = \frac{y(1 + y^2)}{4}$: therefore,

$$x = y + \frac{C}{2} \times (1 + y^2) \times \lambda + \left\{ \frac{D}{2} + \frac{C^2}{4} y \right\} \times (1 + y^2) \times \lambda^2.$$

We must now determine C and D, so that this value of x , and its value already found, may be identical: Thus we have,

$$\frac{C}{2} = A \times \frac{y^2}{1 + y^2} \times y,$$

$$\frac{D}{2} + \frac{C^2}{4} \times y = B \times \frac{y^2}{1 + y^2} \times y^3;$$

therefore,

$$C = 2 A \times \sin^2 \frac{z}{2} \times y,$$

$$D = \left\{ 2 B \sin^2 \frac{z}{2} - 2 A^2 \sin^4 \frac{z}{2} \right\} \times y^3,$$

and, taking the values of A and B, (Art. 17.)

$$C = \left\{ \frac{4}{3} \sin^2 \frac{z}{2} - \frac{8}{15} \sin^4 \frac{z}{2} \right\} \times y;$$

$$D = \left\{ \frac{22}{15} \sin^2 \frac{z}{2} - \frac{788}{315} \sin^4 \frac{z}{2} + \frac{64}{45} \sin^6 \frac{z}{2} - \frac{64}{225} \sin^8 \frac{z}{2} \right\} \times y^3,$$

and finally, if we substitute for the powers of $\sin \frac{z}{2}$, their values in the cofines of the multiples of the arch, we shall find,

$$C = \frac{1}{15} \times \{ 7 - 6 \cos z - \cos 2z \} \times y,$$

$$D = \frac{1}{3150} \{ 510 - 78 \cos z - 341 \cos 2z - 84 \cos 3z - 7 \cos 4z \} y^3$$

Having

Having thus found C and D, we have $v = z + w = z + C\lambda + D\lambda^2$: that is,

$$v = z + \frac{1}{15} \{7 - 6 \operatorname{cof} z - \operatorname{cof} 2z\} \lambda \tan \frac{z}{2},$$

$$+ \frac{1}{3150} \{510 - 78 \operatorname{cof} z - 341 \operatorname{cof} 2z - 84 \operatorname{cof} 3z - 7 \operatorname{cof} 4z\} \lambda^2 \tan^3 \frac{z}{2}.$$

19. Having now resolved the problem that was proposed, it remains to apply it to find the true place of a comet in an eccentric orbit. For this purpose, nothing more is wanting, than to be able to determine the angle z in the parabola, at any given instant of time, reckoning from the passage over the perihelion. We shall here suppose, as a matter already known and demonstrated, the theory that is commonly given of a body describing a parabolic trajectory round the sun, placed in the focus: and we shall also make use of the astronomical tables that have been computed, for finding the true place in that trajectory when the time is given. It would indeed be easy for us to deduce the whole of that theory, and to explain the construction of the tables, from the fluxional equation,

$$r^2 \dot{z} = 2p^2 \times \dot{y} (1 + y^2)$$

obtained above: but this would only be to repeat what is already familiar to astronomers.

SUPPOSE, then, a body to describe the given parabola, by its gravitation to the sun placed in the focus; and let us compare the motion of the body in the parabola, with the motion of the comet in the eccentric orbit: If two bodies describe different conic sections by the action of a central force, tending to the foci of the curves, and varying inversely as the square of the distance, it is demonstrated, by the writers on central forces, (*Vide* NEWT. *Prin. Math. lib. 1. prop. 14.*) that they will describe areas, in the same time, that are in the subduplicate ratio of the two parameters: Therefore, the area described by the body in the parabola, in any given time, will be to the area described

by

by the comet, in the same time as the square root of the parameter of the parabola to the square root of the parameter of the ellipse; that is, as $\sqrt{2p}$ to $\sqrt{a(1-\epsilon^2)}$; or as $\sqrt{2p}$ to $\sqrt{p(1+\epsilon)}$; or as $\sqrt{2}$ to $\sqrt{1+\epsilon}$; or, finally, as $\sqrt{1+\lambda}$ to 1. But the area, described by the comet in the eccentric ellipse, that is, the sector of the ellipse cut off by the radius vector ρ , is equal to the sector of the parabola, cut off by the radius vector r : Therefore, the sector cut off by a radius vector drawn to the body, describing the parabola by the solar force, will be to the sector cut off by the radius vector r , in the proportion of $\sqrt{1+\lambda}$ to 1. Now, in the table of the motion in a parabola, the arguments of the true anomaly are no other than the areas cut off by the radii vectores; or, which is the same thing, they are numbers proportional to those areas: Therefore, if the argument of the true anomaly of the body in the parabolic trajectory, found for the given instant of time, be diminished in the proportion of $\sqrt{1+\lambda}$ to 1, we shall have the argument, which corresponds in the table, to the angle z required.

WE have, therefore, this rule for finding the angle z at any given time by means of the table of the motion in a parabola*: Let d denote the interval between the given time and the passage over the perihelion, expressed in days; then z will be the angle in the table that corresponds to the argument, $\frac{d}{p^{\frac{1}{2}}} \times \frac{1}{\sqrt{1+\lambda}}$.

Having thus found the angle z in the parabola for the given time, we must apply to it the *equation* in the formula of Art. 18. to have the true anomaly in the eccentric orbit †.

5. IN

* *Vide* Table Generale du Mouvement des Cometes, Astronomie de LA LANDE, tom. iii. p. 335. 2d edit.

† It may be remarked, that the angle z is always less than the angle v , and that the equation to be applied to z is always additive.

20. IN calculating the place of a comet, as seen from the earth, the astronomer has occasion to compute, not only the heliocentric

IF we compare the motion of the comet in the eccentric orbit, immediately to the motion of the body in the parabolic trajectory, it is obvious, that the angular velocity of the former is less than the angular velocity of the latter, at the perihelia of the curves : therefore, supposing the two bodies to pass over the perihelia together, the body in the parabola will advance before the comet. But, as the radii vectores in the ellipse increase at a slower rate than the radii vectores in the parabola, the angular velocity in the ellipse will increase at a faster rate than the angular velocity in the parabola, in order that the areas described in the same time may preserve their just proportion. Hence it is clear, that the angular velocity of the comet will, in the first place, become equal to the angular velocity of the body in the parabola, after which the former body will gain upon the latter ; the difference of the true anomalies will become less and less, and will at last vanish, the two heliocentric places exactly coinciding.

IF we denote by v the true anomaly, common to both the ellipse and parabola, when the heliocentric places coincide ; and if $x = \tan \frac{v}{2}$, it will not be difficult to deduce, from the reasoning above, the following equation for determining x , viz.

$$\frac{1}{\sqrt{1+\lambda}} \times \left\{ x + \frac{x^3}{3} \right\} = \left\{ x + \frac{x^3}{3} \right\} - 2\lambda \left\{ \frac{x^3}{3} + \frac{x^5}{5} \right\} + 3\lambda^2 \left\{ \frac{x^5}{5} + \frac{x^7}{7} \right\} - \&c.$$

which is easily reduced to this,

$$0 = \frac{1}{\lambda} \times \left(1 - \frac{1}{\sqrt{1+\lambda}} \right) \times \left\{ 1 + \frac{x^2}{3} \right\} - 2 \left\{ \frac{x^2}{3} + \frac{x^4}{5} \right\} + 3\lambda \left\{ \frac{x^4}{5} + \frac{x^6}{7} \right\} - \&c.$$

and, if we neglect the quantities multiplied by λ and its powers, we shall have simply,

$$0 = \frac{1}{2} - \frac{x^2}{2} - \frac{2x^4}{5},$$

whence $x = \sqrt{\frac{\sqrt{105}-5}{8}} = 0.8098 = \tan 39^\circ$, nearly.

THEREFORE $v = 78^\circ$: and, whatever be the eccentricity of the orbit, provided it be very great, the heliocentric place in the ellipse will, at this distance from the perihelion, coincide with the heliocentric place in the parabola : nearer the perihelion, the true anomaly in the ellipse will be less than the true anomaly in the parabola : and, more remote from the perihelion, the true anomaly in the ellipse will be greater than the true anomaly in the parabola. I need not remark, that this conclusion is not to be understood with the utmost rigour ; for we have arrived at it by neglecting the quantities multiplied by the small fraction λ and its powers.

tric place, but also the distance of the comet from the sun, or the radius vector of the orbit. The formula $\varrho = p \times \frac{1+x^2}{1+\lambda x^2}$ found in Art. 16. will furnish a convenient rule for this purpose :

For, if we make $\tan u = x \times \sqrt{\lambda} = \tan \frac{v}{2} \times \sqrt{\lambda}$; we shall have

$\operatorname{cof}^2 u = \frac{1}{1+\lambda x^2}$, and $\frac{1}{\operatorname{cof}^2 \frac{v}{2}} = 1+x^2$; and, therefore, also

$$\varrho = \frac{\operatorname{cof}^2 u}{\operatorname{cof}^2 \frac{v}{2}} \times p, \text{ whence } \varrho \text{ may be found.}$$

X.

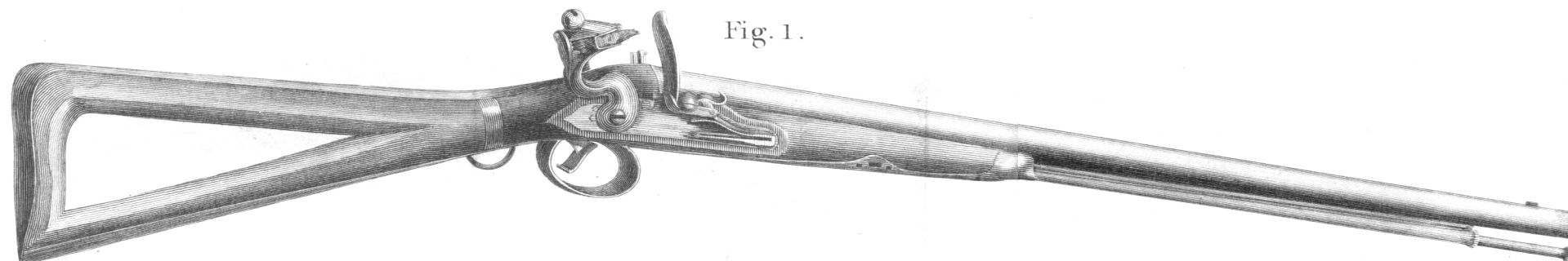


Fig. 1.

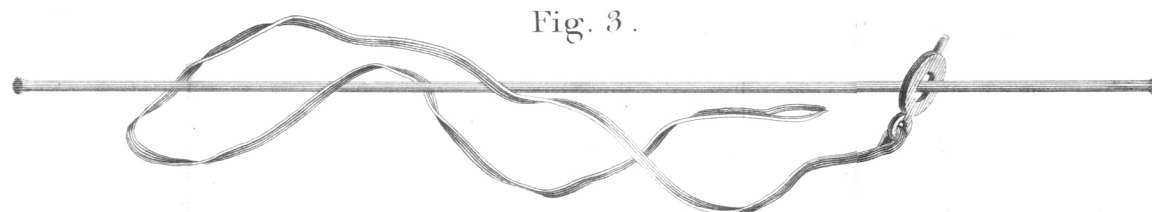


Fig. 3.

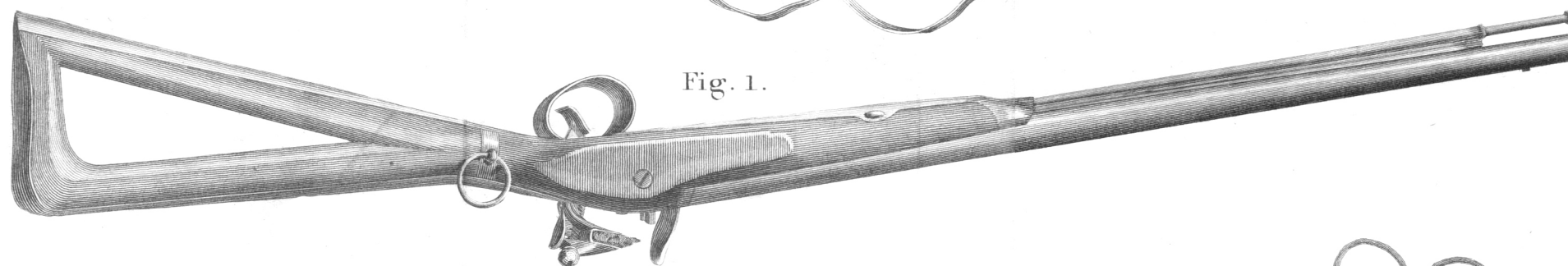


Fig. 1.

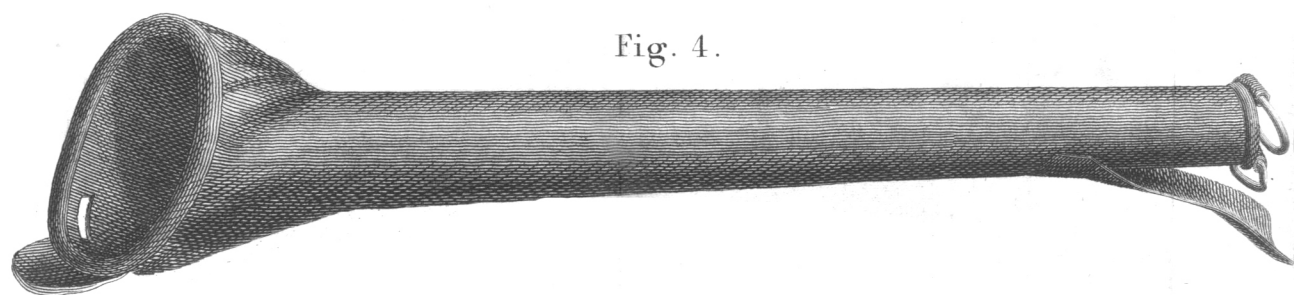


Fig. 4.

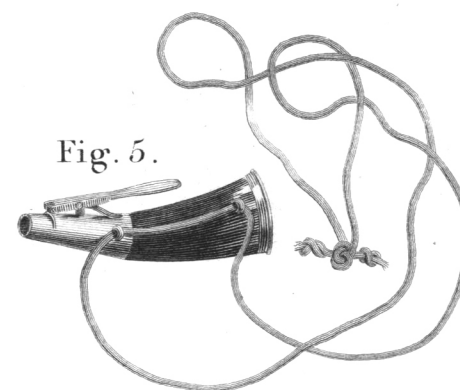


Fig. 5.

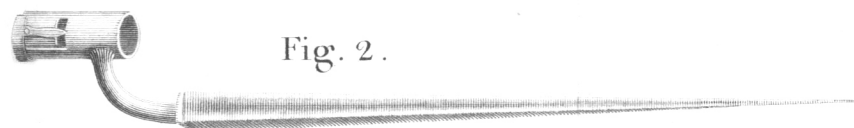


Fig. 2.

Scale of 0 1 2 3 4 5 6 7 8 9 10 11 12 Inches.