

unchanged in their value, so they were simply transferred to the complete magic square Fig. 39. The second quadruple cell in Fig. 37 contains the numbers 7-12-9-6, and as the second cell in Fig. 38 contains the number 48, this number was added to each of the last mentioned four terms, converting them respectively into 55-60-57 and 54, which numbers were inscribed into the corresponding cells of Fig. 39, and so on throughout.

Attention may here be called to the "figure of equilibrium" shown in Fig. 38 by circles and its quadruple reappearance in Fig. 39 which is a complete "regular" and "continuous"  $8 \times 8$  magic square, having many unique summations.

The writer wishes to express his gratitude to his friend, and fellow student, Mr. W. S. Andrews, of Schenectady, New York, for having executed the diagrams illustrating this article and other incidental assistance. It is exceedingly doubtful whether this contribution to the literature of magic squares would ever have seen the light of day without his generous aid.

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### OVERLAPPING MAGIC SQUARES.\*

A peculiar species of Compound Squares may be called overlapping magic squares. In these the division is not made as usual by some factor of the root into four, nine, sixteen or more subsquares of equal area, but into several subsquares or panels not all of the same size, some lying contiguous, while others overlap. The simplest specimens have two minor squares of equal measure apart in opposite corners, and in the other corners two major squares which overlap at the center, having as common territory a middle square  $2 \times 2$ ,  $3 \times 3$ , or larger, or only a single cell. Such division can be made whether the root of the square is a composite or a prime number, as 4-5-9; 4-6-10; 5-6-11; 6-9-15; 8-12-20 etc. The natural series 1 to  $n^2$  may be entered in such manner that each subsquare shall be magic by itself, and the whole square also magic to a higher or lower degree. For example the 9-square admits of division into two minor squares  $4 \times 4$ , and two major squares  $5 \times 5$  which overlap in the center having one cell in common. For convenience, the process of construction may begin with an orderly arrangement of materials.

\* The diagrams have been drawn by Mr. W. S. Andrews of Schenectady, New York.

The series 1 to 81 is given in Fig. 1, which may be termed a *primitive square*. The nine natural grades of nine terms each, appear in direct order on horizontal lines. It is evident that any natural series 1 to  $n^2$  when thus arranged will exhibit  $n$  distinct grades of  $n$  terms each, the common difference being unity in the horizontal direction,  $n$  vertically,  $n+1$  on direct diagonals, and  $n-1$  on transverse diagonals. This primitive square is therefore something more than a mere assemblage of numbers, for, on dividing it as proposed, there is seen in each section a set of terms which may be handled as regular grades, and with a little manipulation may become magical. The whole square with all its component parts may be tilted over to right or left  $45^\circ$ , so that all grades will be turned into a diagonal direction, and all diagonals will become rectangular rows, and presto,

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81

Fig. 1.

the magic square appears in short order. The principle has been admirably presented and employed in various connections by Mr. W. S. Andrews in his recent treatise on *Magic Squares and Cubes*, and the process is beautifully illustrated on pp. 17 and 113 of that work. It is a well-known fact that the primitive square gives in its middle rows an average and equal summation; it is also a fact not so generally recognized, or so distinctly stated, that *all* the diagonal rows are already correct for a magic square. Thus in this 9-square the direct diagonal, 1, 11, 21, 31 etc. to 81 is a mathematical series,  $4\frac{1}{2}$  normal couplets = 369. Also the parallel partial diagonal 2, 12, 22, 32, etc. to 72, eight terms, and 73 to complete it, = 369. So of all the broken diagonals of that system; so also of all the nine transverse diagonals; each contains  $4\frac{1}{2}$  normal couplets or the value

thereof = 369. The greater includes the less, and these features are prominent in the subsquares. By the expeditious plan indicated above we might obtain in each section some squares of fair magical quality, quite regular and symmetrical, but when paired they would not be equivalent, and it is obvious that the coupled squares must have an equal summation of rows, whatever may be their difference of complexion and constitution. The major squares are like those once famous Siamese twins, Eng and Chang, united by a vinculum, an organic part of each, through which vital currents must flow; the central cell containing the middle term 41, must be their bond of union, while it separates the other pair. The materials being parceled out and ready to hand, antecedents above and consequents below, an equitable allotment may be made of normal couplets to each square. Thus from N. W. section two grades may be taken as they stand horizontally, or vertically, or diagonally or any way symmetrically. The consequents belonging to those, found in S. E. section will furnish two grades more and complete the square. The other eight terms from above and their consequents from below will empty those compartments and supply the twin 4-square with an exact equivalent. Some elaborate and elegant specimens, magic to a high degree may be obtained from the following distribution:

1st grade 1, 3, 11, 13 (all odd), 2, 4, 10, 12 (all even);

2d grade 19, 21, 29, 31 and 20, 22, 28, 30.

Then from N. E. section two grades may be taken for one of the major squares; thus 5, 6, 7, 8, 9 and 23, 24, 25, 26, 27 leaving for the twin square, 14, 15, 16, 17, 18 and 32, 33, 34, 35, 36. To each we join the respective consequents of all those terms forming 4th and 5th grades, and they have an equal assignment. But each requires a middle grade, and the only material remaining is that whole middle grade of the 9-square. Evidently the middle portion, 39, 40, 41, 42, 43 must serve for both, and the 37, 38, and their partners 44, 45 must be left out as undesirable citizens. Each having received its quota may organize by any plan that will produce a magic and bring the middle grade near the corner, and especially the number 41 into a corner cell.

In the 5-square Fig. 2 we may begin anywhere, say the cell below the center and write the 1st grade, 14, 15, 16, 17, 18, by a uniform oblique step moving to the left and downward. From the end of this grade a new departure is found by counting two cells down or three cells up if more convenient, and the 2d grade, 32, 33, 34, 35, 36 goes in by the same step of the 1st grade. All the

grades follow the same rule. The leading terms 14, 32, 39, 46, 64 may be placed in advance, as they go by a uniform step of their own, analogous to that of the grades; then there will be no need of any "break move," but each grade can form on its own leader wherever that may stand, making its proper circuit and returning to its starting point. The steps are so chosen and adjusted that every number finds its appointed cell unoccupied, each series often crossing the path of others but always avoiding collision. The resulting square is magic to a high degree. It has its twelve normal couplets arranged geometrically radiating around that unmatched middle term 41 in the central cell. In all rectangular rows and in all diagonals, entire and broken, the five numbers give by addition the constant  $S = 205$ . There are twenty such rows. Other remarkable traits might be mentioned.

50	39	33	16	67
34	17	68	46	40
64	47	41	35	18
42	36	14	65	48
15	66	49	43	32

Fig. 2.

23	45	58	73	6
55	70	5	31	44
13	30	41	52	69
38	51	77	12	27
76	9	24	37	59

Fig. 3.

For the twin square Fig. 3, as the repetition of some terms and omission of others may be thought a blemish, we will try that discarded middle grade, 37, 38, 41, 44, 45. The other grades must be reconstructed by borrowing a few numbers from N. W. section so as to conform to this in their sequence of differences, as Mr. Frierson has ably shown (Andrews, p. 152). Thus the new series in line 5-6-9-12-13, 23-24-27-30-31, 37-38-(41)-44-45 etc. has the differences 1 3 3 1 repeated throughout, and the larger grades will necessarily have the same, and the differences between the grades will be reciprocal, and thus the series of differences will be balanced geometrically on each side of the center, as well as the normal couplets. Therefore we proceed with confidence to construct the 5-square Fig. 3 by the same rule as used in Fig. 2, only applied in contrary directions, counting two cells to right and one upward. When completed it will be the reciprocal of Fig. 2 in pattern, equivalent in summation, having only the term 41 in common and possessing similar magical properties. It remains to be seen how those

disorganized grades in the N. W. Section can be made available for the two minor squares. Fortunately, the fragments allow this distribution:

Regular grades 1, 2, 3, 4,—irregular grades 7, 8, 10, 11  
19, 20, 21, 22 25, 26, 28, 29

These we proceed to enter in the twin squares Figs. 4 and 5. The familiar two-step is the only one available, and the last half of each grade must be reversed, or another appropriate permutation employed in order to secure the best results. Also the 4th grade comes in before the 3d. But these being consequents, may go in naturally, each diagonally opposite its antecedent. The squares thus made are magical to a very high degree. All rectangular and all diagonal rows to the number of sixteen have the constant  $S = 164$ . Each quadrate group of four numbers  $= 164$ . There are nine of these overlapping 2-squares. The corner numbers or two numbers taken on one side together with the two directly opposite  $= 164$ . The

7	29	71	57
72	56	8	28
11	25	75	53
74	54	10	26

Fig. 4.

1	22	78	63
79	62	2	21
4	19	81	60
80	61	3	20

Fig. 5.

corner numbers of any 3-squares  $= 164$ . There are four of these overlapping combinations arising from the peculiar distribution of the eight normal couplets.

These squares may pass through many changes by shifting whole rows from side to side, that is to say that we may choose any cell as starting point. In fact both of them have been thus changed when taking a position in the main square. The major squares shown in Figs. 2 and 3 pass through similar changes in order to bring the number 41 to a corner. With these four subsquares all in place we have the 9-square, shown in Fig. 6, containing the whole series 1 to 81. The twenty continuous rows have the constant  $S = 164 + 205 = 369$ . Besides the 4-squares in N. W. and S. E. there is a 4-square in each of the other corners overlapping the 5-square, not wholly magic but having eight normal couplets placed geometrically opposite, so that taken by fours symmetrically they  $= 164$ . The four corner numbers  $31 + 36 + 22 + 75 = 164$ .

This combination may be taken as typical of the odd squares which have a pair of subsquares overlapping by a single cell. Whatever peculiarities each individual may exhibit they must all conform to the requirement of equal summation in coupled subsquares; and for the distribution of values the plan of taking as a unit of measure the normal couplet of the general series is so efficacious and of so universal application that no other plan need be suggested. These principles apply also to the even squares which have no central cell but a block of four cells at the intersection of the axes. For example, the 14-square, Fig. 7, has two minor subsquares  $6 \times 6$ , and two major squares  $8 \times 8$ , with a middle square  $2 \times 2$ . This indicates a convenient subdivision of the whole area into 2-squares. Thus in N. W. Section we have sixteen blocks; it is a quasi-4-square, and

75	53	11	25	14	65	48	42	36
10	26	74	54	49	43	32	15	66
71	57	7	29	33	16	67	50	39
8	28	72	56	68	46	40	34	17
52	69	13	30	41	35	18	64	47
12	27	38	51	77	80	20	3	61
57	59	76	9	24	4	60	81	19
73	6	23	45	58	79	21	2	62
31	44	55	70	5	1	63	78	22

Fig. 6.

the compartments may be numbered from 1 to 16 following some approved pattern of the magic square, taking such point of departure as will bring 16 to the central block. This is called 1 for the S. E. section in which 2, 3 etc. to 16 are located as before. Now as these single numbers give a constant sum in every line, so will any mathematical series that may replace them in the same order as 1st, 2d, 3d terms etc. Thus in 1 the numbers 1, 2, 3, 4, in 2; 5, 6, 7, 8, and so on by current groups, will give correct results. In this case the numbers 1 to 18, and 19 to 36 with their consequents should be reserved for the twin minor squares. So that here in the N. W. section we begin with 37, 38, in 1 instead of 1, 2, leaving the 3, 4 spaces to be occupied by the consequents 159, 160. Then in 2 we continue 39, 40 (instead of 5, 6) and so on following the path of the

primary series, putting two terms into each 2-square, and arriving with 67, 68 at the middle square. Then the coupled terms go on 69, 70 = 71, 72 etc. by some magic step across the S. E. section reaching the new No. 16 with the terms 97, 98. This exhausts the antecedents. Each 2-square is half full. We may follow a reversed track putting in the consequents 99, 100 etc. returning to the starting point with 159, 160. It is evident that all the 2-squares are equivalent, and that each double row of four of them = 1576, but it does not follow that each single row will = 788. In fact they

47	149	65	131	56	142	44	154	7	18	193	4	185	184
48	150	66	132	55	141	43	153	186	6	187	194	1	17
57	139	39	157	50	148	62	136	9	15	183	8	181	195
58	140	40	158	49	147	61	135	188	16	13	190	182	2
145	51	133	63	138	60	160	38	12	136	10	3	191	179
146	52	134	64	137	59	159	37	189	180	5	192	11	14
143	53	155	41	152	46	130	68	108	90	103	93	115	81
144	54	156	42	151	45	129	67	107	89	104	94	116	82
25	36	175	22	167	166	99	97	121	75	126	72	114	84
168	24	169	176	19	35	100	98	122	76	125	71	113	83
27	33	165	26	163	177	73	123	85	111	96	102	78	120
170	34	31	172	164	20	74	124	86	112	95	101	77	119
30	178	28	21	173	161	91	105	79	117	70	128	88	110
171	162	23	174	29	32	92	106	80	118	69	127	87	109

Fig. 7.

do so, but that is due to the position of each block as direct or reversed or inverted according to a chart or theorem employed in work of this kind. The sixteen rectangular rows, the two entire diagonals and those which pass through the centers of the  $2 \times 2$  blocks sum up correctly. There are also many bent diagonals and zigzag rows of eight numbers that = 788. Each quarter of the square = 1576 and any overlapping 4-square made by four of the blocks gives the same total. The minor squares are *inlaid*. Thus in the N. E. square if the twenty numbers around the central block be dropped out and the three at each angle be brought together around

the block we shall have a 4-square magical to a high degree. In fact this is only reversing the process of construction.

Fig. 8 is a 15-square which develops the overlapping principle to an unusual extent. There are two minor squares  $6 \times 6$ , and two major squares  $9 \times 9$  with a middle square  $3 \times 3$  in common. The whole area might have been cut up into 3-squares. The present division was an experiment that turned out remarkably well. The general series, 1 to 225 is thus apportioned. For N. W. 6-square the numbers 1 to 18 and 208 to 225; for S. E. 19 to 36 and 190 to

225	216	3	222	5	7	73	143	75	141	77	139	79	152	138
10	1	223	4	221	219	153	83	151	85	149	87	147	88	74
6	220	11	18	212	211	89	129	91	127	93	136	126	81	145
218	8	213	210	12	17	137	97	135	99	133	100	90	82	144
2	224	14	15	215	208	101	119	103	124	118	95	131	150	76
217	9	214	209	13	16	125	107	123	108	102	96	130	84	142
77	149	71	155	69	157	112	117	110	105	121	134	92	148	78
52	174	64	162	70	156	111	113	115	106	120	98	128	86	140
181	45	180	46	186	40	116	109	114	122	104	132	94	146	80
53	173	66	160	169	154	37	167	39	29	36	194	193	24	202
178	48	163	63	72	58	159	59	187	135	192	30	35	20	206
55	171	169	158	38	161	44	159	62	32	33	197	190	200	26
176	50	68	57	188	65	182	67	164	196	191	31	34	199	27
184	165	41	172	43	170	47	146	49	21	204	23	25	207	198
61	42	185	54	183	56	179	80	177	205	22	203	201	28	19

Fig. 8.

207; that is just eighteen normal couplets to each. For S. W. 9-square the numbers 37 to 72 and 154 to 189; for N. E. 73 to 108 and 118 to 153; for the middle square, 109 to 117. Figs. 9 and 10 show the method of construction. The nine middle terms are first arranged as a 3-square, and around this are placed by a well-known process (Andrews, p. 47) eight normal couplets 101 + 125 etc. forming a border and making a 5-square. By a similar process this is enlarged to a 7-square, and this again to a 9-square, Fig. 9. Each of these concentric, or bordered, or overlapping squares is magic by itself. The twin square N. E. is made by the same process with



the same 3-square as nucleus. In order to bring this nucleus to the corner of each so that they may coalesce with a bond of union, both of the squares are turned inside out. That is, whole rows are carried from bottom to top and from left to right. Such transposition does not affect the value of any rectangular row, but it does affect the diagonals. In this case the corner numbers, 74, 138 and 152 become grouped around the other corner 88, each of the couplets having the same diagonal position as before. Thus we obtain a 7-square with double border or panel on the North and East, still magic. This 7-square may now be moved down and out a little, from the border so as to give room to place its bottom row above, and its left column to the right, and we have a 5-square with panels of four rows. Again we move a little down and out

74	153	83	151	85	149	87	147	88
145	90	137	97	135	99	133	100	81
144	131	102	125	107	123	108	95	82
76	130	121	112	117	110	105	96	150
142	92	120	111	113	115	106	134	84
78	128	104	116	109	114	122	98	148
140	94	118	101	119	103	124	132	86
80	126	89	129	91	127	93	136	146
138	73	143	75	141	77	139	79	152

Fig. 9.

1	223	4	221	219	10
220	11	18	212	211	6
8	213	210	12	17	218
224	14	15	215	208	2
9	214	209	13	16	217
216	3	222	5	7	225

Fig. 10.

leaving space for the bottom and left rows of the 5-square and thus the 3-square advances to the required position, and the four squares still overlap and retain all of their magical properties. The twin square S. W. passes through analogous transformation. The minor squares were first built up as bordered 4-s as shown in Fig. 10 and then the single border was changed to double panel on two sides, but they might have gone in without change to fill the corners of the main square. As all this work was done by the aid of movable numbered blocks the various operations were more simple and rapid than any verbal description can be. The 15-square (Fig. 8) as a whole has the constant  $S = 1695$  in thirty rectangular rows and two diagonals, and possibly some other rows will give a correct result. If the double border of fifty-two normal couplets be re-

moved the remaining 11-square, 4-7-11 will be found made up of two 4-squares and two overlapping 7-squares with middle 3-square, all magic. Within this is a volunteer 7-square, of which we must not expect too much, but its six middle rows and two diagonals are correct, and the corner  $2 \times 2$  blocks pertaining to the 4-squares although not composed of actual couplets have the value thereof,  $224 + 228$ . However, without those blocks we have two overlapping 5-squares all right. By the way, these 4-squares have a very high degree of magic, like those shown in Fig. 6, with their 2-squares and 3-squares so curiously overlapping. Indeed, this recent study had its origin some years ago from observing these special features of the 4-square at its best state. The same traits were recognized in the 8's and other congeners; also some remarkable results found in the oddly-even squares when filled by current groups, as well as in the quartered squares, led gradually to the general scheme of overlapping squares as here presented. Other investigators may have been working consciously or unconsciously on similar lines, but perhaps not to a great extent. It will be observed that the sections of Fig. 8 have a resemblance to some curious modifications of the concentric square, devised by Mr. Frierson (Andrews, p. 183). This is not merely a chance coincidence, nor an imitation, but doubtless there was a suggestion of possibilities. Without raising any question of originality or priority of invention it may be claimed that here the purpose and the conditions of the combination were quite different, the materials more extensive, and the methods of construction probably not exactly the same.

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#### THE BAGPIPE NOT A HEBREW INSTRUMENT.

In the course of an interesting article on "Music in the Old Testament," written for *The Monist*, April, 1909, Professor Carl Heinrich Cornill, of Breslau, makes the following statement:<sup>1</sup>

"This 'ugab is most probably the same as the bagpipe, which is of course a very primitive and widely spread instrument, familiar to us as the national instrument of the Scotch, and best known in continental Europe as the *pifferari* of Italy."

As a matter of fact, however, it is not possible to say what manner of musical instrument is referred to in the Old Testament

<sup>1</sup> C. H. Cornill, *loc. cit.*, p. 251.