"Sur la Résolution des Equations du 2me, 3me, et 4me degré par la fonction $\frac{n}{r}$(X)," par Dr. Axel S. Guldberg (20 Dec. 1872).

(i.) "Zur Theorie des Integralitäts-faktors." (ii.) Veralgemeinerung und neue Verwerthung der Jacobischen Multiplicator-Theorie," von Sophus Lie.

The above from Det Kongelige Norske Universitat i Christiania.

Numerical Values of the First Twelve Powers of π, of their Reciprocals, and of certain other related quantities. By J. W. L. Glaisher, M.A., F.R.S.

[*Read January* 11*th*, 1877.]

1. I have so often wanted the first few powers of π to more figures than can be obtained by the use of seven-figure logarithms, that, some months since, I determined to have them calculated once for all to a sufficient extent to meet all cases that were likely to arise. My intention originally was merely to obtain the first twelve powers of π to twenty figures; but, when these were calculated, it seemed desirable to deduce from them also the values of their reciprocals.

2. The two following tables contain the values of π, π^2, π^3, ... π^{12} and of π^{-1}, π^{-2}, π^{-3} ... π^{-12} to twenty-two or more figures.

TABLE I.

n	π^n					
1		3·14159	26535	89793	23846	264
2		9·86960	44010	89358	61883	449
3		31·00627	66802	99820	17547	63
4		97·40909	10340	02437	23644	0
5		306·01968	47852	81453	2627	
6		961·38919	35753	04437	0302	
7		3020·29322	77767	92067	514	
8		9488·53101	60705	74007	129	
9		29809·09933	34462	11666	51	
10		93648·04747	60830	20973	72	
11	2	94204·01797	38905	97105	7	
12	9	24269·18152	33741	86222	6	

TABLE II.

n	π^{-n}					
1	0·31830	98861	83790	67153	7768	
2	0·10132	11836	42337	77144	3879	
3	0·03225	15344	33199	48918	4422	
4	0·01026	59822	54684	33518	9153	
5	0·00326	77636	43053	38547	2628	
6	0·00104	01614	73295	85229	6090	
7	0·00033	10936	80177	56676	43260	
8	0·00010	53903	91653	49366	63317	
9	0·00003	35468	03572	08869	12874	0
10	0·00001	06782	79226	86153	36620	4
11	0·00000	33990	01845	34103	10277	22
12	0·00000	10819	35890	52899	80492	70

The value of π^{-1} is well-known: it is given by Paucker (Grunert's "Archiv," t. i., p. 10) to 140 figures. All the other values in the tables were obtained by calculation, as explained in the next section. The values of π^2 and π^{-2} have been published before to more figures than my calculation extends, viz., in Maynard's list of constants* π^2 is given to 31 figures, and Paucker, in the paper just cited, has given the value of π^{-2} to 50 figures. I may mention also that in the same paper Paucker has given $\sqrt{\pi}$ to about 140 figures, and $\pi^{-\frac{1}{2}}$, $\pi^{\frac{1}{3}}$, $\pi^{\frac{1}{4}}$ as well as some other quantities involving π, to about 50 figures.

3. Table I. was calculated as follows:—The value of π to 24 figures was multiplied by itself, 24 figures of the result being retained: this value of π^2 was then multiplied by the value of π previously used, and 24 figures of the result retained, giving π^3: this was multiplied by π and so on up to π^{12}; there were thus eleven multiplications of 24 figures by 24 figures to 24 figures. The whole calculation was performed in duplicate, and the two calculations were brought into agreement.

Maynard's 31-figure value of π was then squared, and 31 figures of the square retained. This 31-figure value of π^4 was squared, and 31 figures of π^8 retained; π^8 was then multiplied twice by π^2 giving π^{10} and π^{12}. None of these values differed from those previously found by so much as a unit in the twenty-third figure, and accordingly the results given in the table (generally to 22 figures) should be correct to the last

* This list will be found in Templeton's "Millwright and Engineer's Pocket Companion" (1871), pp. 169—180.

figure. A small correction was applied to the values of π^2 and π^3, which was equivalent to the use of 25 figures of π.

Table II. was constructed in the first instance by dividing unity by each of the 24-figure values of $\pi^2, \pi^3, \ldots \pi^{12}$ that were obtained by the duplicate 24-figure calculation. The divisions were performed to 24 significant figures. A second calculation was then made by dividing the value of π^{-2} by π, giving π^{-3}, which divided by π gave π^{-4} and so on to π^{-12}.

After these two calculations had been made completely accordant by the correction of a few slight errors, a third calculation was made (by another computer) as follows: the value of π^{-2} was divided by π^2, giving π^{-4}; this was divided by π^2, giving π^{-6}; and so on, to π^{-12}. For the verification of the uneven powers π^{-3} was divided by π^2 repeatedly till π^{-11} was reached. All the three calculations agreed to at least 23 figures.

The results in both Table I. and Table II. are, I believe, correct to the last figure (indeed in most cases it would have been quite safe to have given one figure more), but it is, of course, possible that any one of them may be in error by a unit in this figure. The last figure has been contracted, *i.e.*, it has been increased by unity, when the succeeding figures exceeded 50...

4. The results were verified by means of the equations

$$1+\pi+\pi^2 \ldots +\pi^{11} = \frac{\pi^{12}-1}{\pi-1},$$

$$1+\pi^{-1}+\pi^{-2} \ldots +\pi^{-11} = \frac{1-\pi^{-12}}{1-\pi^{-1}}.$$

The sum of the first eleven powers of π, increased by unity, was

4 31579·82447 03967 63120 124 ...,

while the value of $(\pi^{12}-1) \div (\pi-1)$, found by division, was

4 31579·82447 03967 63120 127 ...;

and the sum of the first eleven negative powers, increased by unity, was, to 25 decimal places,

1·46694 06197 86836 85681 05748,

which was exactly accordant to the same number of decimal places, with the value found by dividing $1-\pi^{-12}$ by $1-\pi^{-1}$. It must be noticed, however, that these verifications are not so perfect as at first sight they might seem to be, on account of the varying number of figures on the left of the decimal point in the one case and of ciphers on the right of the decimal point in the other. Thus although by the agreement of $1+\pi+\pi^2 \ldots +\pi^{11}$ and $(\pi^{12}-1) \div (\pi-1)$, π^{11} and π^{12} are verified directly to 22 figures, π^6 is only verified to 19, and π^3 to 17 figures. Similarly, π^{-12} and π^{-11} are only verified to 20 figures (25 places) although π^{-2} is verified to 25 figures.

5. M. Lefort* states that, in the formation of the "Tables du Cadastre," a table of the first twenty-six powers of $\frac{1}{2}\pi$ to 28 figures was constructed, and that it formed Table I. of Prony's Introduction. There can be little doubt that this table has not been published.

6. Most of the cases in which I have had occasion for the powers of π have arisen in the numerical verification of identities, in which the values of the terms had to be calculated to a certain number of decimal places (not figures). In calculating the value of a series

$$A_0 - A_1 x + A_2 x^2 - A_3 x^3 + \&c.$$

for a value of x greater than unity, the terms frequently increase very rapidly at first, and then only slowly diminish as the continual decrease of the coefficients overcomes the increase of the powers (the calculation of sin 3π from the series for sin x would be a case in point). Thus, in series where the integral portions of the terms cancel one another, a good many figures of the lower powers of x may be required in order to obtain the value of the series to only a moderate number of decimal places. In verifying certain elliptic function identities in which the terms on one side of the identity involve expressions such as $e^{-m\pi}$ or $e^{m\pi} \pm e^{-m\pi}$, it is frequently preferable to expand in powers of π, and to proceed by ordinary arithmetic, rather than to use ten-figure logarithms, even in cases where the latter afford a sufficient number of figures.

7. If we write $\quad S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.,$

it is well known that

$$S_{2n} = \frac{1}{2} \frac{(2\pi)^{2n}}{1.2.3 \ldots 2n} B_n,$$

B_n being the n^{th} Bernoullian number, so that from the values of π^2, π^4, ... π^{12} may be deduced the values of S_2, S_4, ... S_{12}; the formulæ, in fact, being

$$S_2 = \frac{1}{6}\pi^2, \qquad S_8 = \frac{1}{9450}\pi^8,$$

$$S_4 = \frac{1}{90}\pi^4, \qquad S_{10} = \frac{1}{93555}\pi^{10},$$

$$S_6 = \frac{1}{945}\pi^6, \qquad S_{12} = \frac{691}{638512875}\pi^{12}.$$

The values in Table III. were obtained in this manner.

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* "Annales de l'Observatoire de Paris," t. iv. (1858). p. 135.

8. Writing
$$s_n = \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \&c.,$$
$$\sigma_n = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \&c.;$$

then
$$s_n = S_n - \frac{S_n}{2^{n-1}},$$
$$\sigma_n = S_n - \frac{S_n}{2^n}.$$

Tables IV. and V. were deduced from Table III. by means of these formulæ. In all three tables the last figure has been contracted (see end of § 3); it is liable to an error of a unit.

9. A table of the values of S_n from $n=1$ to $n=35$ to 16 decimal places was calculated by Legendre ("Traité des Fonctions Elliptiques," t. ii., p. 432). This table was an extension of one previously given by Euler, and there can be no doubt that it was obtained by Legendre directly from the series $1^{-n}+2^{-n}+3^{-n}+\&c.$ The values of s_n and σ_n (for the same values of n and also to 16 places), deduced as above from Legendre's Table of S_n, form Table IV. of my paper "On the Constants that occur in certain Summations by Bernoulli's Series." *

Legendre's values of S_n (as also the values of s_n and σ_n) are confirmed by those in the present paper up to the last figure of the former in the cases of $n = 2, 4, 6, 8$; for $n=10$ there is a unit-difference, viz. in the 16-place values the last three figures were, for S_n, 180; for s_n, 715; for σ_n, 447; while the extended values give 18085 ..., 71565 ..., 44825 .... For $n=12$, S_n and σ_n agree; but the ending of the 16-place value of s_n was 581, the extended value giving 58190 .... This is due to the accumulation of the errors produced by the contraction to 16 places; for the figures in the extended value are obtained by subtracting ... 49857 ... from ... 08048 ...; but the subtraction of ... 499 from ... 080 gives ... 581.

* Proceedings of the London Mathematical Society, t. 4, p. 56 (1872).

TABLE III.

n	S_n				
2	1·64493	40668	48226	43647	2415
4	1·08232	32337	11138	19151	600
6	1·01734	30619	84449	13971	45
8	1·00407	73561	97944	33937	87
10	1·00099	45751	27818	08533	71
12	1·00024	60865	53308	04829	86

TABLE IV.

n	s_n				
2	0·82246	70334	24113	21823	6208
4	0·94703	28294	97245	91757	650
6	0·98555	10912	97435	10409	84
8	0·99623	30018	52647	89922	73
10	0·99903	95075	98271	56563	92
12	0·99975	76851	43858	19085	32

TABLE V.

n	σ_n				
2	1·23370	05501	36169	82735	4311
4	1·01467	80316	04192	05454	625
6	1·00144	70766	40942	12190	65
8	1·00015	51790	25296	11930	30
10	1·00001	70413	63044	82548	82
12	1·00000	18858	48583	11957	59