Notes on a Ternary Cubic. By H. W. LLOYD TANNER, M.A.

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In the general ternary cubic

$$(a, b, c; f, g, h; i, j, k; l)(x, y, z)^3$$
$$= ax^3 + by^3 + cz^3 + 6lxyz$$
$$+ 3fy^2z + 3gz^2x + 3hx^2y + 3iyz^2 + 3jzx^2 + 3kxy^2,$$

we assume $a = 1$, $h = 0$, and that the form is the product of three factors linear in x, y, z. Denoting this by $F(x, y, z)$, we have

$$F(x, y, z) = x^3 + by^3 + cz^3 + 6lxyz$$
$$+ 3fy^2z + 3gz^2x + 3iyz^2 + 3jzx^2 + 3kxy^2$$
$$= (x + \theta y + \phi z)(x + \theta_1 y + \phi_1 z)(x + \theta_2 y + \phi_2 z),$$

where the θ, ϕ are to be determined. For this purpose write $y = -1$, $z = 0$, and it is seen that the three θ are roots of the equation

$$F(\theta, -1, 0) = (1, 0, k, b)(\theta, -1)^3 = 0.$$

From a comparison of the coefficients of x^2z, xyz, and y^2z in the two expressions for $F(x, y, z)$, we find

$$\phi = (j\theta^2 - 2l\theta + f)/(\theta^2 + k) = A\theta^2 + B\theta + C,$$

where $\quad A = 2kf - 2bl - 2k^2j\ (\div), \quad B = bf + 4k^2l - kbj\ (\div),$

$$C = 4k^2f - 4kbl + b^2j\ (\div),$$

the common denominator being $= b^2 + 4k^3 = \Delta$.

The θ, ϕ represent any one of the pairs θ, ϕ; θ_1, ϕ_1; θ_2, ϕ_2.

In this way the first factor of $F(x, y, z)$ becomes

$$x + y\theta + z\theta^2 = x + Cz + (y + Bz)\theta + Az \,.\, \theta^2$$
$$= x' + y'\theta + z'\theta^2, \text{ say},$$

which is in the standard form of a complex integer, if x', y', z' are integral. And then

$$F(x, y, z) = N(x' + y'\theta + z'\theta^2).$$

When the norm is developed and the symmetric functions of θ, θ_1, θ_2 are replaced by their values in terms of k, b, we get

$$N(x'+y'\theta+z'\theta^2) = (1, b, b^2;\ 0, 3k^2, 0;\ kb, -2k, k;\ -\tfrac{1}{2}b)(x', y', z')^3$$
$$= \Phi(x', y', z').$$

It is this form Φ which is considered in the present communication. Obviously it has the important property that the coordinates of a complex property $x'+y'\theta+z'\theta^2$ of any real integer m give a representation of m by the form Φ.

In general the forms F, Φ are not equivalent, since the transformations are not integral.

When $k = 0$, the discriminant of $(1, 0; k, b)$, namely, b^2+4k^3, is positive, and two of the factors of F (or Φ) are imaginary. This case has been discussed in the *Proceedings** by Professor Mathews, who gives references to several memoirs. The case in which all the factors are real will be here considered, and it will be assumed that b, k are integers, such that b^2+4k^3 is negative, and that $(1, 0, k, b)(\theta, -1)^3$ is irreducible.

Units and Automorphs.

Let u, v, w be a representation of 1 by the form Φ, so that

$$\Phi(u, v, w) = 1 = N(u+v\theta+w\theta^2),$$

and let $x+y\theta+z\theta^2$, $\xi+\eta\theta+\zeta\theta^2$ be two integers, such that

$$x+y\theta+z\theta^2 = (u+v\theta+w\theta^2)(\xi+\eta\theta+\zeta\theta^2).$$

Hence
$$\begin{aligned}\Phi(x, y, z) &= N(x+y\theta+z\theta^2)\\ &= N(u+v\theta+w\theta^2)(\xi+\eta\theta+\zeta\theta^2)\\ &= N(\xi+\eta\theta+\zeta\theta^2)\\ &= \Phi(\xi, \eta, \zeta).\end{aligned}$$

Also
$$\begin{aligned}&x+y\theta+z\theta^2\\ &= (u+v\theta+w\theta^2)(\xi+\eta\theta+\zeta\theta^2)\\ &= u\xi+(v\xi+u\eta)\,\theta+(w\xi+v\eta+u\zeta)\,\theta^2+(w\eta+v\zeta)\,\theta^3+w\zeta\theta^4\end{aligned}$$

Reducing this by means of the equation

$$\theta^3+3k\theta-b = 0,$$

* Vol. XXI., pp. 280–287.

we obtain

$$x+y\theta+z\theta^2$$
$$= u\xi+bw\eta+bv\zeta+\left[v\xi+u\eta+3k\,(w\eta+v\zeta)+bw\zeta\right]\theta$$
$$+(w\xi+v\eta+u\zeta-3kw\zeta)\,\theta^2,$$

and hence, on account of the irreducibility of the equation for θ,

$$\begin{aligned} x &= u\xi+bw\eta+bv\zeta, \\ y &= v\xi+(u-3kw)\,\eta+(bw-3kv)\,\zeta, \\ z &= w\xi+\quad v\eta \quad +(u-3kw)\,\zeta. \end{aligned}$$

These equations may be written

$$x,\ y,\ z = v\,(\xi,\ \eta,\ \zeta),$$

where

$$v = \left(\begin{array}{ccc} u, & bw, & bv \\ v, & u-3kw, & bw-3kv \\ w, & v, & u-3kw \end{array}\right),$$

and, since

$$\Phi\,(x,\ y,\ z) = \Phi\,(\xi,\ \eta,\ \zeta),$$

it follows that v is an automorph of Φ, and, as is seen from its genesis, a proper automorph.

There is plainly a correspondence between the proper automorphs, v, of the form Φ, and the complex units $u+v\theta+w\theta^2$ ($=U$ say), the coordinates of the unit being the terms of the first column of the corresponding automorph. And it is easily proved that the product v_1v_2 of two proper automorphs corresponds to the product U_1U_2 of the two corresponding units.

For, if $(x,\ y,\ z) = v_1\,(x',\ y',\ z')$ and $(x',\ y',\ z') = v_2\,(x'',\ y'',\ z'')$,

then, for the corresponding units,

$$(x+y\theta+z\theta^2) = U_1\,(x'+y'\theta+z'\theta^2), \quad (x'+y'\theta+z'\theta^2) = U_2\,(x''+y''\theta+z''\theta^2).$$

But, from the matrical equations,

$$(x,\ y,\ z) = v_1\,(x',\ y',\ z') = v_1v_2\,(x'',\ y'',\ z''),$$

and from the unit equations,

$$x+y\theta+z\theta^2 = U_1U_2\,(x''+y''\theta+z''\theta^2),$$

which shows that v_1v_2 and U_1U_2 correspond; and further proves that the sequence of factors in a product of proper automorphs is indifferent, assuming—what will presently appear—that to every unit corresponds only one proper automorph. The results extend to any number of factors, which need not all be different.

The coordinates of a complex unit give the elements of the first column of the corresponding proper automorph; and this first column determines the whole of the automorph. For the second column is formed from the first, and the third from the second by the matrix

$$\begin{pmatrix} . & . & b \\ 1 & . & -3k \\ . & 1 & . \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

acting on the three elements of the preceding column. Thus the scheme v includes circulants as a particular case. And, again, recalling the properties of circulants, the development of a determinant formed by this rule from the first column x, y, z is the form $\Phi(x, y, z)$.

There are some curious relations (which may be useful) arising from the nature of the matrix β. For instance,

$$\Phi \,.\, \beta (x, y, z) = b \,.\, \Phi (x, y, z),$$

so that, if (x, y, z) is a representation of m, then we have a representation also of $b^r.m$, viz., $\beta^r(x, y, z)$. The inverse matrix β^{-1} may also be used to derive a representation of m from a known representation (bx, y, z) of mb, in which the first element is a multiple of b. We have, in fact,

$$\Phi \,.\, \beta^{-1} (bx, y, z) = \Phi (3kx+y, z, x),$$

and $$\Phi \,.\, \beta^{-1} (bx, y, z) = \frac{1}{b}\, \Phi (bx, y, z).$$

Fundamental Improper Automorphs.

If θ, θ_1, θ_2 be the three roots of the equation

$$\theta^3+3k\theta-b = 0,$$

we have $$\theta_1 = 2kq+p\theta+q\theta^2,$$

$$\theta_2 = -2k(p+z)+kq\theta-(p+1)\,\theta^2,$$

where $$q = \pm k\sqrt{-3/\Delta}, \quad 2p+1 = bq/k.^*$$

Assigning one of these values to q, the other value of q and the

* If $k = 0$, $q = 0$, and $p^2+p+1 = 0$.

corresponding value of p are $-q$, $-p-1$. Therefore

$$\theta_2 = -2kq-(p+1)\,\theta-q\theta^2,$$

$$\theta_2^2 = \quad 2k\,(p-1)-kq\theta+p\theta^2.$$

These relations are merely a particular form of the well-known homographic relation between the roots of a cubic, but it is more easy to obtain them by writing coefficients for θ^0, θ, θ^2, which are determined by the results of the next paragraph.

The selection of one value for q determines the sequence of θ, θ_1, θ_2, but it in no wise identifies θ, which may be any one of the three roots.

From the above formulæ,

$$\begin{aligned} x+y\theta_1+z\theta_1^2 &= x+(2kq+p\theta+q\theta^2)\,y+\left[-2k\,(p+z)+kq\theta-(p+1)\,\theta^2\right]z \\ &= x+2kqy-2k\,(p+z)\,z+(py+kqz)\,\theta+\left[qy-(p+1)\,z\right]\theta^2 \\ &= \xi+\eta\theta+\zeta\theta^2, \end{aligned}$$

if

$$\xi,\ \eta,\ \zeta = \left[\begin{array}{ccc} 1, & 2kq, & -2k\,(p+2) \\ \cdot & p, & kq \\ \cdot & q, & -p-1 \end{array}\right](x,\ y,\ z).$$

Similarly, if

$$x+y\theta_2+z\theta_2^2 = \xi'+\eta'\theta+\zeta'\theta^2,$$

$$\xi',\ \eta',\ \zeta' = \left[\begin{array}{ccc} 1, & -2kq, & 2k\,(p-1) \\ \cdot & -p-1, & -kq \\ \cdot & -q, & p \end{array}\right](x,\ y,\ z).$$

Now the equation $\quad x+y\theta_1+z\theta_1^2 = \xi+\eta\theta+\zeta\theta^2$

is tantamount to $\quad x+y\theta+z\theta^2 = x+\eta\theta_2+\zeta\theta_2^2.$

But we have $\quad \xi'+\eta'\theta+\zeta'\theta^2 = x+y\theta_2+z\theta_2^2.$

Hence the first matrix of this article, which changes (x, y, z) to (ξ, η, ζ), is the inverse of the second, which changes (x, y, z) to (ξ', η', ζ'). In a similar way it is found that each matrix is the square of the other, and that each is a cube root of unity. Accordingly they will be represented by the symbols γ, γ^2. The results just found enable us to write down the formulæ of the last article.

The matrices γ, γ^2 are automorphs of Φ, for, since

$$x+y\theta_1+z\theta_1^2 = \xi+\eta\theta+\zeta\theta^2,$$

their norms are equal, that is to say,

$$\Phi(x, y, z) = \Phi(\xi, \eta, \zeta) = \Phi\gamma(x, y, z).$$

Since they change the θ in the complex factor $x+y\theta_1+z\theta_1^2$, they are improper automorphs.

It is sometimes convenient to write the automorphs γ, γ^2 in another way, which displays the irrational elements (if any). Writing $\Delta = -3r^2$, the symbol r is real but in general irrational. In the numerical example (p. 195) it is rational, and this is probably the case in most of the cubics that occur in connexion with cyclotomy. All the elements of γ, γ^2 can now be expressed in terms of k, b, r, and the result is

$$\gamma, \gamma^2 = \tfrac{1}{2}\begin{pmatrix} 2, & \cdot & -6k \\ \cdot & -1, & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \pm \frac{1}{2r}\begin{pmatrix} \cdot & 2k^2, & -kb \\ \cdot & b, & 2k^2 \\ \cdot & 2k, & -b \end{pmatrix}.$$

Associated Automorphs.

Let v be a proper automorph whose first column consists of the coordinates of the complex unit $u+v\theta+w\theta^2$, and let $x+y\theta+z\theta^2$ be any complex integer. Then, as on p. 186, the coordinates of the product $(u+v\theta+w\theta^2)(x+y\theta+z\theta^2)$ are $v(x, y, z)$.

Again (p. 191), if

$$\gamma(x, y, z) = (\xi, \eta, \zeta),$$

we have

$$\xi+\eta\theta+\zeta\theta^2 = x+y\theta_1+z\theta_1^2.$$

Hence the complex integer in θ whose coordinates are $\gamma(x, y, z)$ may be written $x+y\theta_1+z\theta_1^2$.

Hence it follows that $v\gamma(x, y, z)$ are the coordinates of the complex number $(u+v\theta+w\theta^2)(x+y\theta_1+z\theta_1^2)$; while $\gamma u(x, y, z)$ are the coordinates of $(u+v\theta_1+w\theta_1^2)(x+y\theta_1+z\theta_1^2)$. In the first case, γ increases the subscript of θ, and the unit multiplication is then effected; in the second case, the action of v corresponds to multiplying $x+y\theta+z\theta^2$ by the complex unit, and the subsequent action of γ increases by 1 the subscript of every θ.

The automorphs most nearly related to v may be shown in a diagram :—

$$\begin{array}{ccc} v & v\gamma & v\gamma^2 \\ \gamma v\gamma^2 & \gamma v & \gamma v\gamma \\ \gamma^2 v\gamma & \gamma^2 v\gamma^2 & \gamma^2 v \end{array}$$

and the preceding explanations will make it easy to see that the first column contains the proper automorphs, the second contains improper automorphs of the γ-kind, and the third improper automorphs of the γ^2-kind. The automorphs of the first, second, and third rows correspond to the complex units $u+v\theta+w\theta^2$, $u+v\theta_1+w\theta_1^2$, and $u+v\theta_2+w\theta_2^2$, respectively. It follows that no two of the nine automorphs can be equivalent.

The automorphs v, γ satisfy the identity

$$u\gamma v\gamma v\gamma = u\gamma^2 v\gamma^2 v\gamma^2 = 1,$$

which differs from the similar identity for the elliptic modular function group only by containing a periodic automorph of the third instead of the second order. The proof of the identity comes from the observation that $v\gamma v\gamma v\gamma\,(x, y, z)$ are the coördinates of

$$(u+v\theta+w\theta^2)(u+v\theta_1+w\theta_1^2)(u+v\theta_2+w\theta_2^2)(x+y\theta+z\theta^2),$$

that is, of $$x+y\theta+z\theta^2,$$

so that $$v\gamma v\gamma v\gamma\,(x, y, z) = (x, y, z)\,;$$

and so for $v\gamma^2 v\gamma^2 v\gamma^2$.

The identity shows that the cube of every improper automorph is 1; this is visible on writing out the cube, say $\gamma v\gamma\,\gamma v\gamma\,\gamma v\gamma$. It proves that the product of two proper automorphs is independent of the sequence of the factors. This comes by multiplying the equation

$$\gamma u\gamma v\gamma = \gamma^3 v\gamma^2 v\gamma \quad (= v^{-1})$$

by the multipliers indicated below, on the left, and then changing the association of the factors as is allowable.

$$\begin{array}{ll} 1 \;\ldots & \gamma v\gamma^2 \,.\, \gamma^2 v\gamma = \gamma^2 v\gamma \,.\, \gamma v\gamma^2, \\ \gamma^2 \ldots \gamma & v \,.\, \gamma v\gamma^2 = \gamma v\gamma^2 \,.\, v, \\ \gamma \;\ldots \gamma^2 & \gamma^2 v\gamma^2 \,.\, v = v \,.\, \gamma^2 v\gamma^2. \end{array}$$

But any other pair of automorphs in the scheme are not commutative. If we denote by ω any one of the automorphs, then any pair of automorphs in the table (extended if need be) are consecutive automorphs (i.) in a horizontal line, as ω, $\omega\gamma$; or (ii.) in a dexter line, as ω, $\gamma\omega$; or (iii.) in a sinister line (ω, $\gamma^2\omega\gamma^2$); or (iv.) in a vertical line (ω, $\gamma^2\omega\gamma$). It will therefore suffice to prove that each of these four pairs gives different products when the sequence of the factors is altered.

Since $\gamma\omega$, $\omega\gamma$ are two automorphs in the scheme, they are different. Hence, if we multiply, as indicated on the left, and then reassociate the factors suitably, the inequation

$$\omega\gamma \neq \gamma\omega,$$

we obtain $\quad \omega \ldots \qquad \omega \, . \, \omega\gamma \neq \omega\gamma \, . \, \omega$ ....(i.),

$\ldots \omega \qquad \omega \, . \, \gamma\omega \neq \gamma\omega \, . \, \omega$ ....(ii.),

$\omega^{-1} \ldots \omega^{-1} \qquad \gamma\omega^{-1} \neq \omega^{-1}\gamma.$

But since

$$\omega^{-1} = \gamma^2\omega\gamma^2\omega\gamma^2,$$

the last equation gives $\omega \, . \, \gamma^2\omega\gamma^2 \neq \gamma^2\omega\gamma^2 \, . \, \omega$ ....(iii.),

For automorphs in the same column it is necessary to take the column separately. For the second column the inequation

$$\gamma^2 \upsilon \, \gamma^2\upsilon\gamma^2 \neq \upsilon^2$$

gives $\quad \gamma^2 \ldots \qquad \gamma\upsilon \, . \, \gamma^2\upsilon\gamma^2 \neq \gamma^2\upsilon\gamma^2 \, . \, \gamma\upsilon,$

$\gamma \ldots \gamma \qquad \upsilon\gamma \, . \, \gamma\upsilon \neq \gamma\upsilon \, . \, \upsilon\gamma,$

$\ldots \gamma^2 \qquad \gamma^2\upsilon\gamma^2 \, . \, \upsilon\gamma \neq \upsilon\gamma \, . \, \gamma^2\upsilon\gamma^2,$

and similar results follow for the third column by using the inequation

$$\gamma\upsilon\gamma\upsilon\gamma \neq \upsilon^2.$$

There is an interesting speculation concerning the three proper automorphs υ, $\gamma\upsilon\gamma^2$, $\gamma^2\upsilon\gamma$ (or, say, for shortness υ, ν, ω) which is connected with these results. We know that the powers of any proper automorph are commutative in a product; and the question is whether υ, ν, ω can be represented as powers of one of them. Now, if $\nu = \upsilon^h$, we have also $\omega = \nu^h$ and $\upsilon = \omega^h$, so that $h^3 = 1$. But h obviously cannot be 1, and therefore the only values for this exponent are the imaginary cube roots of 1, viz., γ, γ^2. If we agree to define the symbol υ^γ by the equation $\upsilon^\gamma = \nu$, then any product

$$\upsilon^a \nu^b \omega^c = \upsilon^{a+b\gamma+c\gamma^2},$$

and, supposing v to be a fundamental automorph, every proper automorph of the form, is a power of a fundamental automorph whose exponent is a cubic integer. There is thus a doubly infinite series. For the forms in which two factors are imaginary the exponent is a real integer and there is a singly infinite series of automorphs.

The units $u+v\theta+w\theta^2$, $u+v\theta_1+w\theta_1^2$, $u+v\theta_2+w\theta_2^2$ are connected in the same way. In Eisenstein's great memoir on cubic forms much use is made of the "regulator," viz., the expression $\log A - \gamma \log B$; where A, B are two conjugate factors of the form, and $\log A$, $\log B$ mean the logarithm of the absolute values of A, B. As the use of a complex exponent has been fruitful, it seemed worth while to notice the extension to matrices.

There is one point to which reference may be made. In the numerical example following, the automorphs include fractional elements. It will be found, however, that in no product of the automorphs does the common denominator exceed 9, and this fact leads to Eisenstein's proof of the existence of units (and therefore automorphs) with integral elements. There are two reasons for using the automorphs with fractional elements (with the property that the common denominator in all powers and products has a superior limit). In the first place, the integral elements are liable to be inconveniently large, or the first automorph with integral elements is a power of the fractional automorph, and the exponent may be any integer less than 3^n, where n is the common divisor. In the second place, the number m of which a representation is sought generally contains extraneous factors, especially in cyclotomic problems (for example, $x^2-5y^2=4p$, and the form that follows) which are also the factors of the common denominators of the fractional elements of the automorphs. So far from being a hindrance, the presence of fractions with such denominators is a real help to the calculator.

Numerical Example.

In the determination of a 7-ic complex prime, a factor of $p\ (=14\mu+1)$, the following equation is to be solved:—

$$F(x, y, z) = x^3+7y^3-189z^3+189y^2z-63z^2x-6yz^2-21xy^2 = 216p.$$

The equation for θ is $\qquad \theta^3-21\theta-7=0,$

$$k=-7, \quad b=7, \quad b^2+4k^3, = \Delta, = -27\,.\,49, \quad r=21.$$

The equation for ϕ is $\qquad 3\phi = -28-\theta+2\theta^2,$

giving $\qquad 3x' = 3x-28z, \quad 3y' = 3y-z, \quad 3z' = 2z.$

These give integral values for x', y', z', only if z is a multiple of 3 (which is actually the case in the present problem). The transformed form is

$$\Phi(x, y, z) = x^3 + 7y^3 + 7^2z^3 + 21^2 . z^2x - 147yz^2 + 42zx^2 - 21xy^2 - 21xyz.$$

Automorphs

$$\gamma = \tfrac{1}{3}\begin{pmatrix} 3, & \bar{14}, & 56 \\ \cdot & \bar{2}, & \bar{7} \\ \cdot & 1, & \bar{1} \end{pmatrix}, \quad \gamma^2 = \tfrac{1}{3}\begin{pmatrix} 3, & 14, & 70 \\ \cdot & \bar{1}, & 7 \\ \cdot & \bar{1}, & \bar{2} \end{pmatrix}$$

$$(p, q) = \tfrac{1}{3}(-1, -1), \quad \text{or} \quad \tfrac{1}{3}(-2, 1),$$

$$\nu = \tfrac{1}{3}\begin{pmatrix} \bar{1}, & 0, & 7 \\ 1, & \bar{1}, & 21 \\ 0, & 1, & \bar{1} \end{pmatrix}, \quad \nu\gamma = \tfrac{1}{3}\begin{pmatrix} \bar{1}, & 7, & \bar{21} \\ 1, & 3, & 14 \\ \cdot & \bar{1}, & \bar{2} \end{pmatrix}, \quad \nu\gamma^2 = \tfrac{1}{3}\begin{pmatrix} \bar{1}, & \bar{7}, & \bar{28} \\ 1, & \bar{2}, & 7 \\ \cdot & \cdot & 3 \end{pmatrix},$$

$$\gamma\nu\gamma^2 = \tfrac{1}{9}\begin{pmatrix} \bar{17}, & 7, & \bar{14} \\ \bar{2}, & 4, & \bar{35} \\ 1, & \bar{2}, & 4 \end{pmatrix}, \quad \gamma\nu = \tfrac{1}{9}\begin{pmatrix} \bar{17}, & 70, & \bar{329} \\ \bar{2}, & \bar{5}, & \bar{35} \\ 1, & \bar{2}, & 22 \end{pmatrix}, \quad \gamma\nu\gamma = \tfrac{1}{9}\begin{pmatrix} \bar{17}, & \bar{77}, & \bar{371} \\ \bar{2}, & 1, & \bar{14} \\ 1, & 4, & 16 \end{pmatrix},$$

$$\gamma^2\nu\gamma = \tfrac{1}{9}\begin{pmatrix} 11, & \bar{7}, & \bar{7} \\ \bar{1}, & \bar{10}, & \bar{28} \\ \bar{1}, & \bar{1}, & \bar{10} \end{pmatrix}, \quad \gamma^2\nu\gamma^2 = \tfrac{1}{9}\begin{pmatrix} 11, & \bar{49}, & 224 \\ \bar{1}, & 2, & 14 \\ \bar{1}, & 2, & \bar{13} \end{pmatrix}, \quad \gamma^2\nu = \tfrac{1}{9}\begin{pmatrix} 11, & 56, & 245 \\ \bar{1}, & 8, & \bar{28} \\ \bar{1}, & \bar{1}, & \bar{19} \end{pmatrix}.$$

Geometrical Theory.

A point P whose rectangular coordinates (x, y, z) satisfy the equation

$$\Phi(x, y, z) = 1$$

is upon a cubic surface; and every point on this surface with integral coordinates (x, y, z) corresponds to a unit $x + y\theta + z\theta^2$ and *vice versâ*. The surface has three real asymptotic planes

$$x + y\theta + z\theta^2 = 0, \quad x + y\theta_1 + z\theta_1^2 = 0, \quad x + y\theta_2 + z\theta_2^2 = 0,$$

where θ, θ_1, θ_2 are real irrational roots of

$$\theta^3 + 3k\theta - b = 0.$$

It is clear that, of the three units corresponding to any point P, all or only one must be positive. The surface consists of four sheets for which the three units have the signs $+++$; $+--$, $-+-$, $--+$, respectively. The first may be distinguished as the "positive" sheet. Of the eight compartments into which space is divided by the asymptote planes, that one which contains the axis of x also contains the positive sheet, which it meets at the "vertex" A, coordinates 1, 0, 0. This compartment is separated from each of the other occupied compartments by an edge. The tangent plane at the vertex (1, 0, 0) is given by the equation

$$x-2kz = 1.$$

Consider now the section of the surface by a plane

$$x-2kz = m.$$

This plane not being parallel to any of the asymptote planes, the section will always have three asymptotes that in general form a triangle. When m is greater than 1 the plane cuts the positive sheet, and the curve consists of an oval inside the triangle with three infinite branches (from the three negative sheets) in the outward angles of the triangle. The curve has three diameters, the medians of the asymptote triangle, which meet at the point where the axis of x pierces the plane. If the plane moves till $m = 1$, the oval in the curve shrinks up to an acnode, at the vertex A of the surface. These sections are two of the species added by Stirling to Newton's *Enumeratio*. When m is between 0 and 1, the triangle is empty and the infinite branches remain (Newton's 22nd species); when $m = 0$, the asymptotes are concurrent (Newton's 32nd species); and when m becomes negative, the triangle is empty and the infinite branches are separated from the triangle by its sides (Newton's 23rd species).

Consider now a point P, (x, y, z) on the unit surface, and its conjugate points whose coordinates are

$$\gamma(x, y, z) = x-3kz+(2ky-bz)/2r, \quad -\tfrac{1}{2}y+(by+2k^2z)/2r,$$
$$-\tfrac{1}{2}z+(2ky-bz)/2r,$$

$$\gamma^2(x, y, z) = \text{the same with } -r \text{ for } r.$$

It is then seen that $\qquad x-2kz$

has the same value for the three conjugate points P, γP, $\gamma^2 P$, so that these points lie upon a plane parallel to the tangent plane at the vertex, viz., the plane

$$x-2kz = \text{const.}$$

A second relation common to the three points can also be found by direct substitution or by determining the volume of tetrahedron $O, P, \gamma P, \gamma^2 P$, namely,

$$(x-2kz)(ky^2-byz-k^2z^2)/2r.$$

This volume is the same for each of the three points, and one of its factors is also constant. Hence the three points lie upon the cylinder whose axis is the axis of x,

$$-ky^2+byz+k^2z^2 = \text{const.},$$

which can be written $(2ky-bz)^2+3r^2z^2 = \text{const.}$,

and is seen to be elliptic.

The geometric construction of the two points γP, $\gamma^2 P$ conjugate with a given point P now becomes very simple. Let ABC be the asymptote triangle on the plane through P, parallel to the tangent at the vertex. Then P will be a point inside or outside the triangle according as it is on the positive sheet or one of the negative sheets. Through P draw lines parallel to the sides of the triangle, upon which mark points P', Q', R', respectively, so that the middle points of PP', QQ', RR' are the middle points of the segments intercepted by the triangle. A similar construction at P', or Q', or R', or all of them, will give two new points Q, R. Then P, Q, R are conjugate points, and so likewise are P', Q', R'.

The proof comes out thus. The six points are on the cubic curve because the intercepts between the cubic and two asymptotes on a line parallel to the third are equal. Hence, P being on the curve, P', Q', R', and therefore also Q, R, are on the curve.

Similarly for the conic, observing that the medians of the triangle ABC are diameters of the conic conjugate to the sides they bisect, it is seen that the six points are on the conic.

The medians AF, BG, CH divide the triangle into six parts with a common vertex at O', the centroid. The effect of γ is to transpose the asymptote planes cyclically. Hence A becomes B, B becomes C, C becomes A, while O' is unchanged. Hence the triangles $O'BF$, $O'CG$, $O'AH$ are conjugate, and any point in $O'BF$ has one conjugate in $O'CG$ and the other $O'AH$. But this is equivalent to the mode of distributing the six points into conjugate sets, as is seen at once from a figure.

As another application, consider a point U, coordinates (u, v, w); P, as before, the point (x, y, z), and examine the effect of υ, the proper automorph with u, v, w for its first column. This automorph determines a homogeneous strain in space; and, since it is proper, the three edges of the asymptote system remain unchanged. The point A (1, 0, 0) comes to U; the plane containing P, γP, $\gamma^2 P$, which was parallel to the tangent plane at A, when transformed, contains the points vP, $v\gamma P$, $v\gamma^2 P$, and is parallel to the tangent plane at U, the new position of A. In the same way, using the automorph γv, we find that the plane of the points γvP, $\gamma v\gamma P$, $\gamma v\gamma^2 P$ is parallel to the tangent plane at γU.

It may be noted that the fundamental units must be represented by points on the negative sheets, unless indeed there are no points with integral coordinates upon these sheets. For, if (u, v, w) is on the positive sheet, the three conjugate units are positive. Hence every power has three positive conjugates, and cannot lie on a negative sheet because units on a negative sheet have two of the conjugates negative.

Examples illustrating Lord Rayleigh's Theory of the Stability or Instability of certain Fluid Motions. By A. E. H. Love. Read January 9th, 1896.

In papers published in the *Proceedings*, Vols. XI. and XIX., Lord Rayleigh has discussed the oscillations possible in a stream of fluid flowing between two fixed planes, which arise from difference of spin (or molecular rotation) in different parts. He has especially attended to cases where the spin changes discontinuously at certain planes between the boundaries. A difficulty occurs in these solutions through the fact that the places where the stream-velocity is equal to the wave-velocity are singularities of the integrals of the differential equation on which the small varied motion depends. This difficulty Lord Rayleigh has sought to evade in a recent paper (*Proc.*, Nov., 1895). It was felt, however, that, in the case of continuously varying spin, the complete discussion of the problem for a particular law of velocity would be desirable. The present paper contains such a discussion as appears possible (without solving the differential equation) of a case where there are two separated singular places of the integral. The general conclusion seems to be that wave-