

moved gradually nearer to the source and arrived at the source when the ear was 16 inches from the wall.

A similar experiment with a note of wave-length 19·4 inches gave the image:—

on line parallel to the wall,	with ear distant 17, 39, 61 inches
at the source,	with ear distant 24, 48, 58 inches
40° from the wall,	with ear distant 32, 52 inches.

From these experiments it is clear that in noting the direction of a fog-horn at sea the observer should be well away from any reflecting surfaces of any kind. (In one experiment an umbrella held to one side of the head at a distance of 2 feet displaced the sound-image 20°.) I find that it adds to correctness in fixing the direction to have a flat board slung on the shoulders vertical and parallel to the axis of the ears. This increases the intensity in front and shuts off sound from the rear. I think it would also be better to have two short blasts of 3 seconds each, every half minute, at sea, rather than a long blast every minute.

Also fog-horns should be placed well above any reflecting surfaces, but it might add to their carrying power if a large disk or sounding-board was placed horizontally directly over them.

XXXII. *On certain Bessel Integrals and the Coefficients of Mutual Induction of Coaxial Coils.* By T. H. HAVELOCK, M.A., D.Sc.; Fellow of St. John's College, Cambridge; Lecturer in Applied Mathematics, Armstrong College, Newcastle-on-Tyne*.

THE calculation of coefficients of mutual induction has been discussed by several writers from the time of Maxwell to the present, more accurate expressions being required as experimental methods have become more refined. The expressions are generally in one of two forms: they are either given in elliptic integrals, in which case numerical calculations are tedious, or given by a certain number of terms of a series.

The present paper brings forward another method of expressing the coefficients, namely in terms of integrals involving Bessel functions; series are obtained from these integrals, their general terms found, and their convergence tested. In certain cases series are obtained which seem to be simpler and better adapted for numerical calculation than

* Communicated by the Author.

This series is convergent for k less than unity; it converges rapidly, and in most cases we shall find the first three terms sufficient.

Consider now the integral

$$I = \int_0^\infty e^{-p\mu} J_1^2(\mu) d\mu. \quad . \quad . \quad . \quad . \quad (5)$$

If p is large we can easily find a suitable series for I in ascending inverse powers of p ; we substitute for the Bessel function its equivalent series in ascending powers of μ and then integrate each term separately. Then, since we have

$$J_1^2(\mu) = \sum_{s=0}^{\infty} (-1)^s \frac{(2s+2)!}{s!(s+2)! \{(s+1)!\}^2} \left(\frac{\mu}{2}\right)^{2s+2},$$

and

$$\int_0^\infty e^{-p\mu} \mu^{2s} d\mu = \frac{(2s)!}{p^{2s+1}},$$

we obtain the series

$$\int_0^\infty e^{-p\mu} J_1^2(\mu) d\mu = \sum_{s=0}^{\infty} (-1)^s \frac{\{(2s+2)!\}^2}{2^{2s+2} s! (s+2)! \{(s+1)!\}^2} \frac{1}{p^{2s+3}}; \quad (6)$$

$$\int_0^\infty e^{-p\mu} J_1^2(\mu) \frac{d\mu}{\mu} = \sum_{s=0}^{\infty} (-1)^s \frac{(2s+1)!(2s+2)!}{2^{2s+2} s! (s+2)! \{(s+1)!\}^2} \frac{1}{p^{2s+2}}; \quad (7)$$

$$\begin{aligned} \int_0^\infty e^{-p\mu} J_1^2(\mu) \frac{d\mu}{\mu^2} &= \sum_{s=0}^{\infty} (-1)^s \frac{(2s)!(2s+2)!}{2^{2s+2} s! (s+2)! \{(s+1)!\}^2} \frac{1}{p^{2s+1}} \\ &= \frac{1}{4p} - \frac{1}{8p^3} + \frac{5}{32p^5} - \frac{35}{128p^7} + \dots \quad (8) \end{aligned}$$

The series are convergent for $p > 2$.

Further we have *

$$J_1(\mu) J_1(\lambda\mu) = \lambda \sum_{s=0}^{\infty} (-1)^s \frac{F(-s-1, -s, 2, \lambda^2)}{s!(s+1)!} \left(\frac{\mu}{2}\right)^{2s+2}, \quad (9)$$

* Nielsen, *Cylinderfunctionen*, p. 20.

where $\lambda < 1$, and F is the hypergeometric series given by

$$F(a, b, c, x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{2! c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3! c(c+1)(c+2)}x^3 + \dots$$

Then, using the same method, we obtain series for more general integrals suitable for large values of p . We obtain thus

$$\int_0^\infty e^{-p\mu} J_1(\mu) J_1(\lambda\mu) d\mu = \lambda \sum_{s=0}^\infty (-1)^s \frac{(2s+2)! F(-s-1, -s, 2, \lambda^2)}{s!(s+1)! 2^{2s+2}} \frac{1}{p^{2s+3}}; \quad (10)$$

$$\int_0^\infty e^{-p\mu} J_1(\mu) J_1(\lambda\mu) \frac{d\mu}{\mu} = \lambda \sum_{s=0}^\infty (-1)^s \frac{(2s+1)! F(-s-1, -s, 2, \lambda^2)}{s!(s+1)! 2^{2s+2}} \frac{1}{p^{2s+2}}; \quad (11)$$

$$\begin{aligned} \int_0^\infty e^{-p\mu} J_1(\mu) J_1(\lambda\mu) \frac{d\mu}{\mu^2} &= \lambda \sum_{s=0}^\infty (-1)^s \frac{(2s)! F(-s-1, -s, 2, \lambda^2)}{s!(s+1)! 2^{2s+2}} \frac{1}{p^{2s+1}} \\ &= \lambda \left[\frac{1}{4} \frac{1}{p} - \frac{1}{16} (1 + \lambda^2) \frac{1}{p^3} + \frac{1}{32} (1 + 3\lambda^2 + \lambda^4) \frac{1}{p^5} \right. \\ &\quad \left. - \frac{5}{256} (1 + 6\lambda^2 + 6\lambda^4 + \lambda^6) \frac{1}{p^7} + \dots \right]. \quad (12) \end{aligned}$$

To obtain series suitable for small values of p we have

$$\begin{aligned} y &= \int_0^\infty e^{-p\mu} J_1^2(\mu) d\mu \\ &= -\frac{1}{\pi} \int_0^\pi \cos \alpha d\alpha \int_0^\infty e^{-p\mu} J_0(2\mu \cos \frac{1}{2}\alpha) d\mu \\ &= -\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta d\theta}{\sqrt{p^2 + 4 \cos^2 \theta}}. \end{aligned}$$

The summation in this integral is now divided into two parts, one between the limits 0 and $\frac{\pi}{2} - \epsilon$ and the other between $\frac{\pi}{2} - \epsilon$ and ϵ ; ϵ may be taken indefinitely small

ultimately, but at present it is regarded as indefinitely larger than p .

Then we have

$$y = -\frac{2}{\pi} \int_0^{\frac{\pi}{2} - \epsilon} \frac{\cos 2\theta d\theta}{\sqrt{p^2 + 4 \cos 2\theta}} + \frac{2}{\pi} \int_0^{\epsilon} \frac{\cos 2\theta d\theta}{\sqrt{p^2 + 4 \sin^2 \theta}}$$

$$= -\frac{1}{\pi} y_1 + \frac{2}{\pi} y_2. \quad \dots \quad (13)$$

In y_1 we write, as far as terms in p^4 ,

$$\frac{2 \cos 2\theta}{\sqrt{p^2 + 4 \cos^2 \theta}} = \frac{\cos 2\theta}{\cos \theta} \left[1 - \frac{1}{8} \frac{p^2}{\cos^2 \theta} + \frac{3}{128} \frac{p^4}{\cos^4 \theta} \right].$$

Then we integrate the terms separately, substitute the limits 0 and $\frac{\pi}{2} - \epsilon$, and assuming ϵ small we expand as far as necessary; we obtain thus

$$y_1 = 2 + \log \frac{1}{2} \epsilon - \frac{1}{8} p^2 \left(\frac{1}{12} - \frac{1}{2\epsilon^2} - \frac{3}{2} \log \frac{1}{2} \epsilon \right)$$

$$+ \frac{3}{128} p^4 \left(-\frac{133}{1440} + \frac{7}{12\epsilon^2} - \frac{1}{4\epsilon^4} - \frac{5}{8} \log \frac{1}{2} \epsilon \right). \quad \dots \quad (14)$$

In the second integral y_2 , p is small compared with θ throughout the range; we substitute for $\cos 2\theta$ and $\sin \theta$ their expansions in powers of θ and expand by the binomial theorem. We obtain

$$y_2 = \int_0^{\epsilon} \left[1 - 2\theta^2 + \frac{2}{3} \theta^4 + \frac{2}{3} \frac{\theta^4}{p^2 + 4\theta^2} - \frac{64}{15} \frac{\theta^6}{p^2 + 4\theta^2} + \frac{2}{3} \frac{\theta^8}{(p^2 + 4\theta^2)^2} \right] \frac{d\theta}{\sqrt{p^2 + 4\theta^2}},$$

including all parts which will give terms containing p^4 on integration. Integrating the parts separately and expanding as far as p^4/ϵ^4 we find

$$= \frac{1}{2} \left(1 + \frac{3}{16} p^2 - \frac{15}{1024} p^4 \right) \left(\log \frac{4\epsilon}{p} + \frac{p^2}{16\epsilon^2} - \frac{3p^4}{512\epsilon^4} \right) - \frac{\epsilon^4}{144} \left(1 - \frac{3p^2}{8\epsilon^2} + \frac{15p^4}{128\epsilon^4} \right)$$

$$- \left(\frac{3}{8} \epsilon^2 + \frac{5}{64} \epsilon^4 - \frac{15}{512} p^2 \epsilon^2 \right) \left(1 + \frac{p^2}{8\epsilon^2} - \frac{p^4}{128\epsilon^4} \right)$$

$$- \left(\frac{1}{12} \epsilon^2 - \frac{31}{240} \epsilon^4 \right) \left(1 - \frac{p^2}{8\epsilon^2} + \frac{3p^4}{128\epsilon^4} \right). \quad \dots \quad (15)$$

Substituting these values of y_1 and y_2 in (13) we find that the terms in $\log \epsilon$ and inverse powers of ϵ cancel; then making ϵ indefinitely small the terms involving positive powers of ϵ vanish and we obtain as far as terms in p^4

$$y = \int_0^\infty e^{-p\mu} J_1^2(\mu) d\mu = \frac{1}{\pi} \left\{ \left(1 + \frac{3}{16} p^2 - \frac{15}{1024} p^4 \right) \log_e \frac{8}{p} - 2 - \frac{1}{16} p^2 + \frac{31}{2048} p^4 \right\}. \quad (16)$$

Further, if we integrate (11) with respect to p and put the constant of integration equal to $\frac{1}{2}$ on account of (1), we obtain

$$\int_0^\infty e^{-p\mu} J_1^2(\mu) \frac{d\mu}{\mu} = -\frac{1}{\pi} \left\{ \left(p + \frac{1}{16} p^3 - \frac{3}{1024} p^5 \right) \log_e \frac{8}{p} - \frac{\pi}{2} - p + \frac{5}{2048} p^5 \right\}. \quad (17)$$

Integrating again with respect to p and taking account of (2) we find

$$\begin{aligned} \int_0^\infty e^{-p\mu} J_1^2(\mu) \frac{d\mu}{\mu^2} &= \frac{1}{\pi} \left\{ \frac{1}{2} p^2 \left(1 + \frac{1}{32} p^2 - \frac{1}{1024} p^4 \right) \log_e \frac{8}{p} \right. \\ &\quad \left. + \frac{4}{3} - \frac{\pi}{2} p - \frac{1}{4} p^2 + \frac{1}{256} p^4 + \frac{1}{3072} p^6 \right\} \\ &= \frac{1}{\pi} \left\{ \frac{4}{3} - \frac{\pi}{2} p + \frac{1}{2} p^2 \left(\log \frac{8}{p} - \frac{1}{2} \right) \right. \\ &\quad \left. + \frac{1}{64} p^4 \left(\log \frac{8}{p} + \frac{1}{4} \right) - \frac{1}{2048} p^6 \left(\log \frac{8}{p} - \frac{2}{3} \right) \right\}. \quad (18) \end{aligned}$$

In the same way, if $p^2 + (1-\lambda)^2$ is small compared with λ , and if $\lambda > 1$, we obtain from (11) the more general expansion

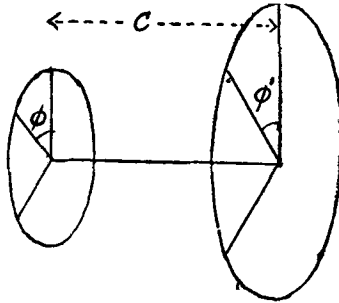
$$\begin{aligned} \pi \lambda^{\frac{1}{2}} \int_0^\infty e^{-p\mu} J_1(\mu) J_1(\lambda\mu) d\mu &= \left[1 + \frac{3}{16} \frac{p^2 + (1-\lambda)^2}{\lambda} - \frac{15}{1024} \left\{ \frac{p^2 + (1-\lambda)^2}{\lambda} \right\}^2 \right] \log 8 \sqrt{\frac{\lambda}{p^2 + (1-\lambda)^2}} \\ &\quad - 2 - \frac{1}{16} \frac{p^2 + (1-\lambda)^2}{\lambda} + \frac{31}{2048} \left\{ \frac{p^2 + (1-\lambda)^2}{\lambda} \right\}^2 \quad \dots \quad (19) \end{aligned}$$

Finally, integrating (19) with respect to p and taking account of (3) we can obtain a similar series for the integral

$$\int_0^\infty e^{-p\mu} J_1(\mu) J_1(\lambda\mu) \frac{d\mu}{\mu}. \quad \dots \quad (20)$$

§ 2. *Mutual Induction of Single-layer Coil and Coaxial Circle.*

If we have two coaxial circles of radii a and b , with a distance c between their planes, we have their coefficient of



mutual induction given by

$$\begin{aligned} M &= \iint \frac{\cos \epsilon}{r} ds ds' = \int_0^{2\pi} \int_0^{2\pi} \frac{ab \cos (\phi - \phi') d\phi d\phi'}{\sqrt{c^2 + a^2 + b^2 - 2ab \cos (\phi - \phi')}} \\ &= \int_0^{2\pi} \int_0^{2\pi} ab \cos (\phi - \phi') d\phi d\phi' \int_0^\infty e^{-\lambda c} J_0(\lambda \sqrt{a^2 + b^2 - 2ab \cos (\phi - \phi')}) d\lambda \\ &= 4\pi^2 ab \int_0^\infty e^{-\lambda c} J_1(\lambda a) J_1(\lambda b) d\lambda. \quad \dots \dots \dots (21) \end{aligned}$$

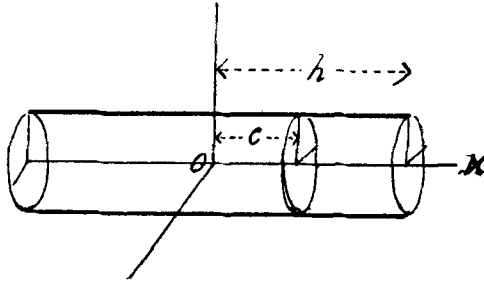
From the integrals (10) and (19) in the previous section series for M could be obtained suitable for c large or small compared with a or b . Further, if we have a single-layer solenoid of length $2h$, radius b , and n turns of wire per unit length together with a concentric coaxial circle of radius a , less than b , we obtain their coefficient of mutual induction by integrating (21) with respect to c between the limits $-h$ and $+h$. Hence we obtain

$$M = 8\pi^2 abn \int_0^\infty (1 - e^{-\lambda h}) J_1(\lambda a) J_1(\lambda b) \frac{d\lambda}{\lambda}. \quad \dots (22)$$

Using then the integrals given in (11) and (20) we have series suitable both for long and for short coils. However, in the latter case the difference between the radii of the coil and the circle must be small compared with one of them, and unless this holds series already in use probably give a better approximation than those obtained from (22)*. Finally, if

* Cf. E. B. Rosa, Bulletin of the Bureau of Standards, vol. iii. p. 209, 1907.

we have a solenoid of length $2h$ and radius a , and a circle of the same radius at a distance c from the central section of



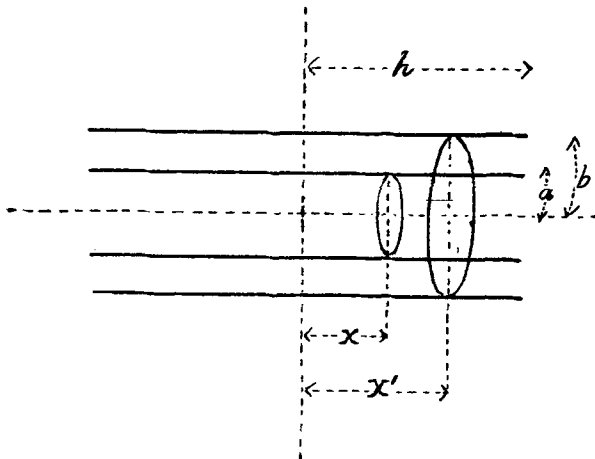
the coil, we obtain from (16)

$$\begin{aligned} M &= 4\pi^2 a^2 n \int_0^\infty d\lambda J_1^2(\lambda a) \left[\int_{-h}^c e^{-\lambda(c-x)} dx + \int_c^h e^{-\lambda(x-c)} dx \right] \\ &= 4\pi^2 a^2 n \int_0^\infty \left\{ 2 - e^{-\lambda(h+c)} - e^{-\lambda(h-c)} \right\} J_1^2(\lambda a) \frac{d\lambda}{\lambda}. \quad (23) \end{aligned}$$

$$= 4\pi^2 a^2 n \left[1 - \int_0^\infty \left\{ e^{-\frac{h+c}{a}\mu} - e^{-\frac{h-c}{a}\mu} \right\} J_1^2(\mu) \frac{d\mu}{\mu} \right]. \quad (24)$$

Substituting for these integrals from (7) and (17) according as $h+c$ and $h-c$ are large or small compared with a , we can obtain series giving the induction through any section of a solenoid, whether it is a long or a short coil.

§ 3. Two Coaxial Solenoids of Equal Length.



Suppose we have two single-layer coils of equal length $2h$ and of radii a and b , placed as in the figure; let n_1 and

n_2 be the number of turns of wire per unit length on the two cylinders. Then if we write $|x-x'|$ for the absolute value of $x-x'$, we have from (21)

$$\begin{aligned} M &= 4\pi^2 ab n_1 n_2 \int_0^\infty J_1(\lambda a) J_1(\lambda b) d\lambda \int_{-h}^h dx \int_{-h}^h dx' e^{-\lambda |x-x'|} \\ &= 16\pi^2 ab n_1 n_2 \int_0^\infty \left\{ \frac{h}{\lambda} - \frac{1}{2\lambda^2} + \frac{1}{2\lambda^2} e^{-2h\lambda} \right\} J_1(\lambda a) J_1(\lambda b) d\lambda. \end{aligned}$$

With $b > a$ and $h > b$, we use the series (3), (4) and (12); thus we find for the coefficient of mutual induction of the two coils:

$$\begin{aligned} M &= 8\pi^2 ab n_1 n_2 \left[\frac{h}{b} - \frac{1}{2} + \frac{1}{16} \left(\frac{a}{b} \right)^2 + \frac{1}{128} \left(\frac{a}{b} \right)^4 + \frac{5}{2048} \left(\frac{a}{b} \right)^6 + \dots \right. \\ &\quad + \frac{1}{8} \frac{b}{h} - \frac{1}{128} \left(1 + \frac{a^2}{b^2} \right) \left(\frac{b}{h} \right)^3 + \frac{1}{1024} \left(1 + 3 \frac{a^2}{b^2} + \frac{a^4}{b^4} \right) \left(\frac{b}{h} \right)^5 \\ &\quad \left. - \frac{5}{2^{15}} \left(1 + 6 \frac{a^2}{b^2} + 6 \frac{a^4}{b^4} + \frac{a^6}{b^6} \right) \left(\frac{b}{h} \right)^7 + \dots \right] \quad (25) \end{aligned}$$

This gives an expression for M which is easy of calculation and rapidly convergent; moreover, from (4) and (12), additional terms in the two series within the brackets in (25) can be calculated if required from the general terms

$$\frac{\{1.3.5\dots(2r-3)\}^2 (2r-1)}{2^{2r+1} r! (r+1)!} \left(\frac{a}{b} \right)^{2r}$$

and

$$(-1)^s \frac{(2s)! F\left(-s-1, -s, 2, \frac{a^2}{b^2}\right)}{2^{4s+3} s! (s+1)!} \left(\frac{b}{h} \right)^{2s+1}.$$

We consider as a numerical illustration a case which has been used in comparing other similar series, namely:—

$$a=5 \text{ cm.}; \quad b=10 \text{ cm.}; \quad h=100 \text{ cm.}; \quad n_1=n_2=n.$$

Then we have

$$\frac{h}{b} = 10; \quad \frac{b}{h} = \frac{1}{10}; \quad \frac{a}{b} = \frac{1}{2},$$

and from (25)

$$M = 2000\pi^2 n^2 \left(10 - \frac{1}{2} + \frac{1}{2^6} + \frac{1}{2^{11}} + \frac{5}{2^{17}} + \dots + \frac{1}{80} - \frac{5}{2^9} \frac{1}{10^3} + \dots \right).$$

Using the terms shown, we find that this gives

$$M/\pi^2 n^2 = 19057.28. \quad . \quad . \quad . \quad (26)$$

Further, by calculating a few more terms, we easily see that the result in (26) is correct as far as the figures shown. Other series for this case are those of Maxwell* and Heaviside†, while a complete expression in elliptic functions has been given by Cohen‡. In the last case, although an exact theoretical expression is found, yet in practice the accuracy depends upon tables of elliptic functions and upon the result of long and complicated calculations. These three expressions have been compared numerically by Rosa and Cohen§ for the case used above; they give the following results for $M/\pi^2 n^2$:—

Maxwell's series	19057.25
Heaviside's series	19067.08
Cohen's elliptic-function formula...	19057.36

Comparing these with the result given in (26) we infer that the two latter formulæ do not give better results than Maxwell's, at least when the ratio of length to diameter is large; Cohen's formula is applicable to all values of this ratio, but it is not suitable for calculation. The series given in (25) appears somewhat simpler than Maxwell's; it is convergent for all coils with the length greater than the radius of the outer coil, and as one knows the general term of the series the result can be calculated to any required degree of accuracy; it can easily be verified that the series converges quite rapidly even for coils whose length is not much greater than their breadth.

* Maxwell, 'Electricity and Magnetism,' vol. ii. § 678.

† Heaviside, 'Electrical Papers,' vol. ii. p. 277.

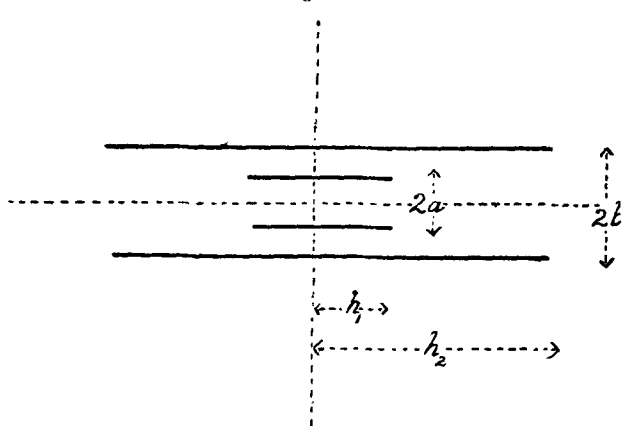
‡ Cohen, Bulletin of the Bureau of Standards, vol. iii. p. 295 (1907).

§ Rosa and Cohen, Bulletin of the Bureau of Standards, vol. iii. p. 316 (1907).

§ 4. *Short Coil inside a Long Coil.*

Another case is that of a coil of length $2h_1$ within concentric coaxial coil of length $2h_2$, as in the figure.

Fig. 4.



Then if n_1 and n_2 are the turns per unit length, we have

$$\begin{aligned} M &= 4\pi^2 ab n_1 n_2 \int_0^\infty J_1(\lambda a) J_1(\lambda b) d\lambda \int_{-h_1}^{h_1} dx \int_{-h_2}^{h_2} e^{-\lambda |x-x'|} dx' \\ &= 8\pi^2 ab n_1 n_2 \left[\frac{a}{b} h_1 - b \int_0^\infty \left(e^{-\mu \frac{h_2-h_1}{b}} - e^{-\mu \frac{h_2+h_1}{b}} \right) J_1(\mu) J_1\left(\frac{a}{b} \mu\right) \frac{d\mu}{\mu^2} \right], \quad (27) \end{aligned}$$

Hence, if $h_2 - h_1$ is large compared with b , a suitable series can be obtained by using (12) in these two integrals. We find

$$\begin{aligned} M/8\pi^2 ab n_1 n_2 &= \frac{a}{b} h_1 - a \left[\frac{1}{4} \left(\frac{b}{h_2 - h_1} - \frac{b}{h_2 + h_1} \right) \right. \\ &\quad - \frac{1}{16} \left(1 + \frac{a^2}{b^2} \right) \left\{ \left(\frac{b}{h_2 - h_1} \right)^3 - \left(\frac{b}{h_2 + h_1} \right)^3 \right\} \\ &\quad \left. + \frac{1}{32} \left(1 + 3 \frac{a^2}{b^2} + \frac{a^4}{b^4} \right) \left\{ \left(\frac{b}{h_2 - h_1} \right)^5 - \left(\frac{b}{h_2 + h_1} \right)^5 \right\} - \dots \right]. \end{aligned}$$

The general term can be obtained from (12) if required :

for purposes of calculation the series can be written in the following way :

$$\begin{aligned} M/8\pi^2 a^2 n_1 n_2 = & h_1 - b^2 \left[\frac{1}{4}(u-v) - \frac{1}{16}d(u^3-v^3) \right. \\ & \left. + \frac{1}{32}(c^2+d^2)(u^5-v^5) - \frac{5}{256}d(3c^2+d^2)(u^7-v^7) + \dots \right], \quad (28) \end{aligned}$$

where

$$c=ab; \quad d=a^2+b^2; \quad u=\frac{1}{h_2-h_1}; \quad v=\frac{1}{h_2+h_1}.$$

We shall test this formula by the following numerical example which has been used to compare other series :

$$\begin{aligned} h_2 &= 15 \text{ cm.}; \quad h_1 = 2.5 \text{ cm.}; \quad b = 5 \text{ cm.}; \quad a = 4 \text{ cm.}; \\ n_1 &= 10, \text{ and } n_2 = 40 \text{ turns per cm.} \end{aligned}$$

Then we substitute in (28) the following values :

$$c=20; \quad d=41; \quad u=\frac{2}{25}; \quad v=\frac{2}{35}.$$

Calculating the terms shown, we find to the order indicated

$$M = .0012000 \text{ henry.}$$

From the form of the series we see that this is larger than the true value ; and in fact, by taking an extra term we find to the same order

$$M = .0011999 \text{ henry.}$$

Rosa and Cohen* have calculated the same example by three different series using a similar number of terms and give the results:—

M.	Series.
.001199896	Roiti.
.00119990	Searle and Airey.
.00119989	Russell.

§ 5. Short Coil outside a Long Coil.

With the same notation, suppose h_2 is small and h_1 large. It has been thought that in this case the formula for M is

* *Loc. cit.*

different from before and more complicated. But we have

$$\begin{aligned} \mathbf{M} &= 4\pi^2 abn_1n_2 \int_0^\infty J_1(\lambda a) J_1(\lambda b) d\lambda \int_{-h_1}^{h_1} dx \int_{-h_2}^{h_2} e^{-\lambda |x-x'|} dx' \\ &= 8\pi^2 abn_1n_2 \left[\frac{a}{b} h_2 - b \int_0^\infty \left(e^{-\mu \frac{h_1-h_2}{b}} - e^{-\mu \frac{h_1+h_2}{b}} \right) J_1(\mu) J_1\left(\frac{a}{b}\mu\right) \frac{d\mu}{\mu^2} \right]. \quad (29) \end{aligned}$$

Comparing this with (27) we see that h_1 and h_2 are merely interchanged; so that there is a similar series to (28) for this case also.

§ 6. Self-induction of a Cylindrical Coil.

As a final example we consider the self-induction of a single-layer coil; then if we have

$2h$ = length of coil; a = radius;

$N = 2nh$ = total number of turns of wire;

we can easily deduce from the integrals in § 4 an expression for the self-induction of a coil in the form

$$\begin{aligned} L &= 4\pi^2 a^2 \frac{N^2}{h^2} \int_0^\infty \left\{ \frac{h}{\lambda} - \frac{e^{-\lambda h}}{\lambda^2} \sinh(\lambda h) \right\} J_1^2(\lambda a) d\lambda \\ &= 2\pi^2 \frac{a^2 N^2}{h} \left[1 - \frac{4}{3\pi} \frac{a}{h} + \frac{a}{h} \int_0^\infty e^{-\frac{2h}{a}\mu} J_1^2(\mu) \frac{d\mu}{\mu^2} \right]. \quad (30) \end{aligned}$$

For the integral in (30) we can now use one of the series (8) or (18).

If $h > a$, we have from (8) the series

$$\begin{aligned} L &= 2\pi^2 \frac{a^2 N^2}{h} \left\{ 1 - \frac{4}{3\pi} \frac{a}{h} + \frac{1}{8} \left(\frac{a}{h}\right)^2 - \frac{1}{2^6} \left(\frac{a}{h}\right)^4 \right. \\ &\quad \left. + \frac{5}{2^{10}} \left(\frac{a}{h}\right)^6 - \frac{35}{2^{14}} \left(\frac{a}{h}\right)^8 + \dots \right\}, \quad (31) \end{aligned}$$

where the general term is given by

$$\frac{(2s!) (2s+2)!}{s! (s+2)! \{ (s+1)! \}^2 2^{4s+3}} \left(\frac{a}{h}\right)^{2s+2}.$$

The first four terms of this series have been obtained by a different method by Russell*. We see that the series in general is simple and rapidly convergent for coils whose length is greater than their width.

* Russell, *Philosophical Magazine*, vol. xiii, p. 445 (1907).

For short coils we have also the alternative series given in (18). We find then

$$L = 2\pi a N^2 \left[2 \left\{ 1 + \frac{1}{8} \left(\frac{h}{a} \right)^2 - \frac{1}{64} \left(\frac{h}{a} \right)^4 \right\} \log_e \frac{4a}{h} - 1 \right. \\ \left. + \frac{1}{16} \left(\frac{h}{a} \right)^2 + \frac{1}{48} \left(\frac{h}{a} \right)^4 \right]. \quad \dots \quad (32)$$

By using an expression for L in terms of elliptic integrals and expanding, Coffin* has obtained a series for L with which (32) agrees; Coffin's series was evaluated up to terms in $(h/a)^4$. Instead of using one such complicated series, it seems that the two series (31) and (32) should cover between them all the cases that occur in practice.

XXXIII. *Effect of a Prism on Newton's Rings.*

By LORD RAYLEIGH, O.M., *Pres.R.S.*†

WHEN Newton's rings are regarded through a prism (or grating) several interesting features present themselves, and are described in the "Opticks." Not only are rings or arcs seen at unusual thicknesses, but a much enhanced number of them are visible, owing to approximate achromatism—at least on one side of the centre. The first part of the phenomenon was understood by Newton, and the explanation easily follows from the consideration of the case of a true wedge, viz. a plate bounded by plane and flat surfaces slightly inclined to one another. Without the prism, the systems of bands, each straight parallel and equidistant, corresponding to the various wave-lengths (λ) coincide at the black bar of zero order, formed where the thickness is zero at the line of intersection of the planes. Regarded through a prism of small angle whose refracting edge is parallel to the bands, the various systems no longer coincide at zero order, but by drawing back the prism, it will always be possible so to adjust the effective dispersive power as to bring the n th bars to coincidence for any two assigned colours, and therefore approximately for the entire spectrum.

"In this example the formation of visible rings at unusual thicknesses is easily understood; but it gives no explanation of the increased numbers observed by Newton. The width of the bands for any colour is proportional to λ , as well after the displacement by the prism as before. The manner of

* Coffin, Bulletin of Bureau of Standards, vol. ii. p. 113 (1906).

† Communicated by the Author.