POYNTING'S THEOREM AND THE EQUATIONS OF ELECTROMAGNETIC ACTION.

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THE mathematical theory of the electromagnetic field is usually developed in terms of electric and magnetic intensities in space, and the equations are almost unintelligible to the electrical engineer, who is in the habit of expressing everything in terms of voltage and current. Everyone who understands electromagnetic theory, however, knows that *voltage* is a generalized parameter which completely represents the electric field distribution, and that *current* is a generalized parameter which completely represents the magnetic field distribution when the form of the circuit is given. Therefore it is possible, for any given shape of circuit, so to transform [1] the electromagnetic equations that voltage and current may replace electric and magnetic field intensities. When this is done the unfamiliar electromagnetic equations reduce to the extremely simple and universally familiar voltage and current equations of the electrical engineer. It is the object of this paper to reduce the usual expression of Poynting's Theorem and the usual equations of electromagnetic wave motion to familiar expressions involving voltage and current. This reduction is rather difficult if carried out strictly and in detail, and therefore a general statement of the nature of the transformations is given instead of the algebraic transformations themselves.

POYNTING'S THEOREM.

The rate at which energy streams past a point on a transmission line is equal to EI, where E is the voltage across the line at the point and I is the line current at the point (outgoing current in one wire and returning current in the other wire).

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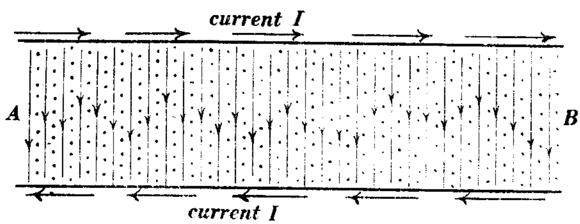
[1] It is always easiest to think of an algebraic transformation as a new type of space measurement, as explained below.

To say that this is Poynting's Theorem is more or less ridiculous, because this relation was known before Poynting's Theorem was ever thought of, but it is Poynting's Theorem all the same, as can be shown most easily in the case of a "transmission line" which consists of two broad, flat parallel metal ribbons. In this case the electric field is uniform, as represented by the fine vertical lines in Fig. 1, and the magnetic field is uniform (the dots in Fig. 1 represent an end view of the lines of force of the magnetic field). Let E be the voltage between the ribbons, then E/d is the intensity of the electric field, where d is the distance of the ribbons apart. Let I be the current, then $4\pi\ I/w$ is the intensity of the magnetic field.[2] The total energy stream is EI ergs per second from left to right, or EI/wd

FIG. 1.



ergs per second per square centimetre. Substituting electric field intensity $f = E/d$ and magnetic field intensity $H = 4\pi\ I/w$, the expression for the energy stream in ergs per second per square centimetre becomes $\frac{1}{4\pi} \cdot f\ H$, which is the usual form in which Poynting's Theorem is expressed.

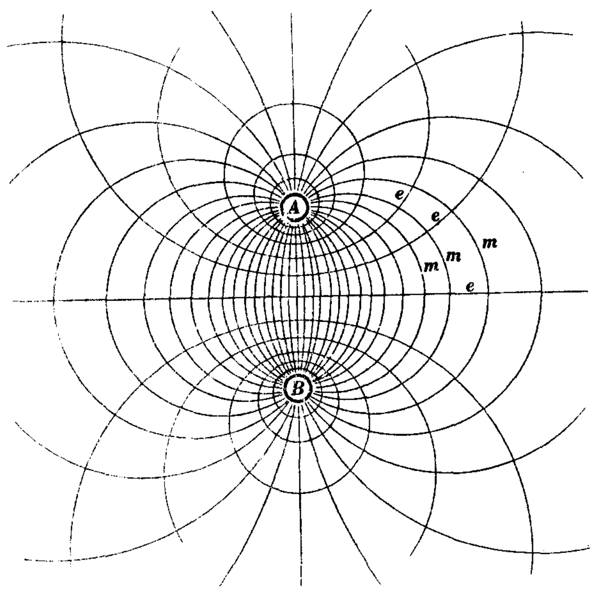
The following is a general statement of the transformation which would be necessary to reduce the usual expression for Poynting's Theorem to the familiar form involving voltage and current for the case of a transmission line consisting of two parallel cylindrical wires.

The two sets of curved lines $e\ e\ e$ and $m\ m\ m$, Fig. 2, represent electric equipotential surfaces and magnetic equipotential surfaces in the region surrounding two parallel cylindrical wires with a certain voltage between the wires and a certain current

[2] C.g.s. electromagnetic units are used throughout.

flowing in them in opposite directions. Imagine measurements in space to be so altered that the distance between equipotential surfaces is everywhere considered unity. Then the electric potential at any point is proportional to the distance (so measured) of the point from the surface of the wire A, and the magnetic potential at any point is proportional to the distance (so measured) of the point from, say, the plane through A and B. But when potentials are proportional to measured distances,

FIG. 2.

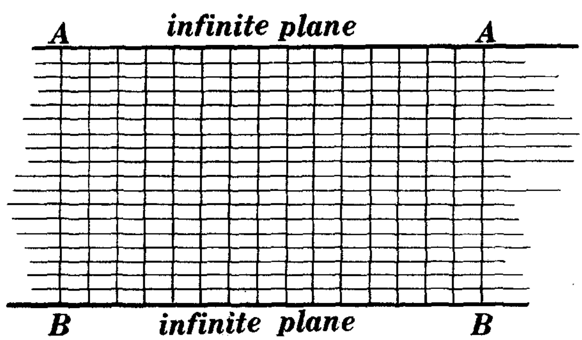


potential gradients are uniform; and therefore Fig. 2 represents in ordinary space what would be a uniform field distribution like Fig. 3 in space as above measured.

Let E be the voltage between the wires in Fig. 2, and let I be the current in the wires ($4 \pi I$ is the magnetic potential-drop along any curve encircling one of the wires). Also let d be the distance between the wires measured as above explained, and let w be the length of any one of the curves e e e in Fig. 2 (they are all of the same length when measured as above explained). Then E/d is the electric field intensity everywhere, and $4 \pi I/w$ is the magnetic field intensity everywhere. The total energy stream is EI, the entire sectional area of the energy stream is dw, and the ergs per second per unit sectional area is EI/dw, and this is equal to $\frac{1}{4\pi} \cdot f\, H$, exactly as in the above argument as applied to two broad, flat parallel ribbons.

The alteration of space measurement which makes Fig. 2 appear like Fig. 3 is the exact geometrical equivalent of the algebraic transformation which reduces the equations of the electric and magnetic fields around two parallel wires to the forms which express the electric and magnetic fields between two broad, flat metal ribbons. One who appreciates the significance of such

FIG. 3.



transformations as this, all electromagnetic field distributions look alike to him! everything is straight and square and uniform, it is no longer necessary to talk about anything but simplest cases, and the familiar equation of the electrical engineer, namely, ***voltage × current equals delivered power,*** **is all there** is left to Poynting's Theorem.

EQUATIONS OF ELECTROMAGNETIC ACTION.

The general equations of the electromagnetic field[8] are tremendously simplified by considering the case of a transmission line and using the generalized parameters E and I; and the

[8] The familiar equations of the electromagnetic field expressing time rates in terms of curl—for the simplest case, of course—are discussed on pages 60, 61, and 62 and in Chapter VI of Franklin's "Electric Waves."

integrals of these equations (the equations of the electromagnetic wave) assume a form, which for simplicity are equal to the formula for Ohm's Law.

The general equations are:

$$C \cdot \frac{dE}{dt} = \frac{dI}{dx} \qquad (1)$$

and

$$L \cdot \frac{dI}{dt} = -\frac{dE}{dx} \qquad (2)$$

where C is the capacity of the line per unit of length (the generalized[4] inductivity of the medium between the wires), and L is the inductance of the line per unit length (the generalized permeability of the medium between the wires). These two equations are easily derived by considering an element of the transmission line. The current which enters the element is greater than the current which flows out of the element by the amount $\frac{dI}{dx} \cdot \triangle x$, and this is equal to the rate at which charge is accumulating on the element of the line $\left(= C \cdot \triangle x \times \frac{dE}{dt}\right)$. The voltage across one end of the element exceeds the voltage across the other end by the amount $\frac{dE}{dx} \cdot \triangle x$, and this difference is the net voltage which is causing the current in the element to decrease $\left(= L \cdot \triangle x \times \frac{dI}{dt}\right)$.

The general integral of equations (1) and (2) involves undetermined functions of $(x \pm Vt)$, where $V = \sqrt{\frac{1}{LC}}$; but the simplest form of this integral may be established by imagining a current distribution over the line which *travels* at velocity V, and a voltage distribution which *travels* at velocity V, because the idea of travel is precisely equivalent to the use of the double variable $(x \pm Vt)$, as is well understood by everyone familiar with the equations of wave motion.

[4] It is assumed that the reader is familiar with the idea of generalized co-ordinates. If he is not, he cannot hope to understand much about voltage, current, capacity, and inductance.

Imagine a current distribution over a transmission line the value of the current at a given point being I. The magnetic flux between the wires of unit length of the line is LI. If the distribution is travelling at velocity V, then the flux LI moves unit distance forwards in $1/V$ second, the rate at which this flux sweeps across a line drawn from wire to wire is $LI \div 1/V$, and this is therefore the voltage E which must exist at the given point of the line so that

$$E = LIV \tag{3}$$

Imagine a voltage distribution over the transmission line, the value of the voltage at a given point being E. The charge on unit length of one of the wires is CE. If the voltage distribution is travelling at velocity V, then the charge CE moves forwards into the next unit length of the line in $1/V$ of a second, and therefore the current in the wire is equal to $CE \div 1/V$. Therefore the current which must exist at the given point of the line on account of the moving voltage distribution is

$$I = CEV \tag{4}$$

Now if the I in (4) which is due to the moving voltage distribution is the same I that produces the voltage distribution according to equation (3), and if the E which is produced by the moving I in equation (3) is the same E that produces the I according to equation (4), that is if E and I in (3) and (4) mutually sustain each other, as it were, then (3) and (4) are simultaneous equations, and by combining them we find

$$V = \sqrt{\frac{1}{LC}} \tag{5}$$

and

$$\tfrac{1}{2}LI^2 = \tfrac{1}{2}CE^2 \tag{6}$$

Now an unchanging distribution of E and I travelling along a line as here described constitutes a distortionless wave or a pure wave. Equation (5) gives the velocity of such a wave, and equation (6) shows that the electric energy in such a wave is always and everywhere equal to the magnetic energy.