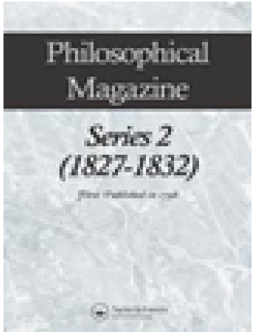

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XVI. General solution of the problem: to represent the parts of a given surface on another given surface, so that the smallest parts of the representation shall be similar to the corresponding Parts of the Surface represented. Answer to the Prize Question proposed by the Royal Society of Sciences at Copenhagen

C.F. Gauss

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This mineral was analysed some years ago by Vauquelin and Dolomieu; but the numbers which they have mentioned, owing to the insufficient mode of analysis employed at that time, are not entitled to any confidence.—(*Journal des Mines* ix. 778.)

XVI. *General Solution of the Problem: to represent the Parts of a given Surface on another given Surface, so that the smallest Parts of the Representation shall be similar to the corresponding Parts of the Surface represented. By* C. F. Gauss. *Answer to the Prize Question proposed by the Royal Society of Sciences at Copenhagen*.*

Ab his via sternitur ad majora.

THE author of this paper believes that he must consider the repeated selection by the Royal Society of the question which forms the subjects of it, as a proof of the importance which the Royal Society attaches to it; and has thereby been induced to submit a solution found by him some considerable time since, as the lateness of the time at which he was informed of the prize question would otherwise have prevented him from sending an answer. He regrets that the latter circumstance has obliged him to limit his inquiry to the essential part only, besides hinting some obvious applications to the projection of maps and the higher branches of geodetics. Had it not been for the near approach of the term fixed by the Society, he would have followed up several inquiries, and have detailed numerous applications of the subject to geodetical operations; all which he must now reserve to himself for another moment and another place.

December 1822.

1. The nature of a curve surface is determined by an equation between the coordinates belonging to every point of the same x, y, z. In consequence of this equation, every one of these three variable quantities may be considered as a function of the two others. It is still more general to introduce two new variable quantities t, u, and to represent each of the quantities x, y, z as a function of t and u, by which at least generally speaking, determinate values of t and u always belong to every determinate point of the surface, and *vice versâ*.

2. Let X, Y, Z, T, U have the same signification for a second surface, which x, y, z, t, u had in reference to the first.

3. To *represent* the former surface on the second means to establish

* From Prof. Schumacher's *Astronomische Abhandlungen*, No. 3.

establish a law by which a determinate point of the second surface is to correspond to every point of the first. This will have been effected if T and U have been made equal to two functions of t and u. These functions will cease to be arbitrary as soon as they are required to satisfy certain conditions. As X, Y, Z next become likewise functions of t and u, these functions must, therefore, besides satisfying the conditions required by the nature of the second surface, also fulfill those of the representation.

The problem of the Royal Society of Sciences prescribes that the representation shall be similar to the object represented in the smallest parts. It is, therefore, first required to find an analytical expression for this condition. Let us suppose that the following equations are the result of the differentiation of the functions of t and u expressing the values of x, y, z, X, Y, Z.

$$
\begin{aligned}
dx &= a\,dt + a'du \\
dy &= b\,dt + b'du \\
dz &= c\,dt + c'du \\
dX &= A\,dt + A'du \\
dY &= B\,dt + B'du \\
dZ &= C\,dt + C'du
\end{aligned}
$$

The condition prescribed requires first that all infinitely small lines proceeding from one point of the first surface and situate in it, shall be proportionate to the corresponding lines on the second surface; and next, that the former shall form between them the same angles as the latter.

Such a linear element on the first surface has this expression

$$\sqrt{\left((a^2+b^2+c^2)dt^2+2(aa'+bb'+cc')\,dt\,du+(a'^2+b'^2+c'^2)du^2\right)}$$

and the corresponding one on the second surface is

$$\sqrt{\left((A^2+B^2+C^2)\,dt^2+2\,(AA'+BB'+CC')\,dt\,du+(A'^2+B'^2+C'^2)du^2\right)}.$$

If both are to be in a certain ratio independent of dt and du, the three quantities

$$a^2+b^2+c^2,\ aa'+bb'+cc',\ a'^2+b'^2+c'^2$$

must evidently be respectively proportional to the three quantities $A^2+B^2+C^2,\ AA'+BB'+CC',\ A'^2+B'^2+C'^2$.

If we suppose that the values t, u and $t+\delta t,\ u+\delta u$ correspond to the extreme points of a second element on the first surface, the cosine of the angle formed between the two elements on that surface, will be

$$\frac{(a\,dt+a'du)(a\delta t+a'\delta u)+(b\,dt+b'du)(b\delta t+b'\delta u)+(c\,dt+c'du)(c\delta t+c'\delta u)}{\sqrt{\left[(adt+a'du)^2+(bdt+b'du)^2+(cdt+c'du)^2\right]\cdot\left[(a\delta t+a'\delta u)^2+(b\delta t+b'\delta u)^2+(c\delta t+c'\delta u)^2\right]}}$$

and we shall obtain an exactly similar expression for the cosine of the corresponding angle on the second surface by changing a, b, c, a', b', c' into A, B, C, A′, B′, C′. The two expressions become clearly equal if the above-mentioned proportionality takes place, and the second condition is already comprehended in the first, as a little reflection will easily show.

The analytical expression of the condition of our problem is, therefore, this:

$$\frac{A^2+B^2+C^2}{a^2+b^2+c^2} = \frac{AA'+B.B'+C.C'}{aa'+bb'+cc'} = \frac{A'A'+B'B'+C'C'}{a'^2+b'^2+c'^2}.$$

Let the value of these equal quantities, which must be a finite function of t and u, be $= m^2$. The quantity m is therefore the index of the ratio in which linear quantities on the first surface are increased or diminished in their representation on the second surface (according as m is greater or smaller than 1). This ratio will, generally speaking, be different in different places: in the particular case in which m is constant, there will be a perfect similarity also in the finite parts; and if m is besides $= 1$, there will be a perfect equality, and the one surface may be developed on the other. Putting for brevity

$$(a^2+b^2+c^2)\,dt^2+2(aa'+bb'+cc')\,dt.du+(a'^2+b'^2+c'^2)du^2=\omega$$

we remark that the differential equation $\omega=0$ admits of two integrations. Representing the trinomial ω as the product of two factors linear with respect to dt and du, either of the two may be $= 0$, which will give two different integrations. One of the integrations will be derived from the equation : $0 =$

$$(a^2+b^2+c^2)\,dt+\{aa'+bb'+cc'+i\sqrt{[(a^2+b^2+c^2)(a'^2+b'^2+c'^2)-(aa'+bb'+cc'^2)]}\}du$$

(where i is written for brevity instead of $\sqrt{-1}$, for it will be easily seen that the irrational part of the expression must become imaginary), the other integration will be the result of a similar equation, which will be obtained by putting $-i$ in place of i in the former. If the integral of the first equation be this

$$p+iq = \text{const.}$$

where p and q denote real functions of t and u, the other integral will be

$$p-iq = \text{const.}$$

It follows from this, that $(dp+idq)(dp-idq)$ or dp^2+dq^2 must be a factor of ω, or that

$$\omega = n\,(dp^2+dq^2)$$

where n is a finite function of t and u.

Let us now denote by Ω the trinomial into which dX^2+

dY^2+

$dY^2 + dZ^2$ will be converted by substituting for dX, dY, and dZ their values expressed by T, U, dT and dU; and let us assume that in a similar manner, as before, the two integrals of the equation $\Omega = 0$ are as follow:

$$P + iQ = \text{const.} \qquad ; \ P - iQ = \text{const. and}$$

$\Omega = N.(dP^2 + dQ^2)$ where P, Q, N denote real functions of T and U.

These integrations may evidently be effected (without taking into consideration the general difficulties of integrating) before the solution of our principal problem.

Now, if for T, U such functions of t and u are substituted as will fulfill the condition of our principal problem, Ω will be changed into $m^2\omega$, and we shall have

$$\frac{(dP + i\,dQ).(dP - i\,dQ)}{(dp + i\,dq)\cdot(dp - i\,dq)} = \frac{m^2 n}{N}.$$

But it will be easily seen that the numerator in the first part of this equation cannot be divisible by the denominator, except if either

$dP + idQ$ is divisible by $dp + idq$, and $dP - idQ$ by $dp - idq$,

or,

$dP + idQ$ is divisible by $dp - idq$, and $dP - idQ$ by $dp + idq$.

In the former case $dP + idQ$ will therefore vanish if $dp + idq = 0$, or $P + iQ$ will be constant if $p + iq$ is supposed to be constant; that is to say, $P + iQ$ will be a function of $p + iq$ only, and in the same manner $P - iQ$ will be a function of $p - iq$. In the latter case $P + iQ$ will be a function of $p - iq$, and $P - iQ$ a function of $p + iq$. It is easy to perceive that the reverse of these positions likewise holds good, or that if for $P + iQ$ and $P - iQ$ functions of $p + iq$ or $p - iq$ (either respectively or inversely) are assumed the divisibility of Ω by ω, and consequently the above required proportionality will take place.

It will easily be conceived that if, for example, we suppose

$$P + iQ = f(p + iq),\ P - iQ = f'(pe - iq)$$

the nature of the function f' is already given by that of f. For if among the constant quantities which it involves, there are none but real quantities, the function f' must be identical with f; in order that real values of P and Q may correspond to real values of p and q: in the contrary case, f' will only be distinguished from f by having in the imaginary quantities which f involves $-i$ instead of $+i$.

We have next, $P = \tfrac{1}{2}f(p + iq) + \tfrac{1}{2}f'(p - iq)$

$$iQ = \tfrac{1}{2}f(p + iq) - \tfrac{1}{2}f'(p - iq),$$

or, which is the same, as the function f is assumed quite ar-

 bitrarily

bitrarily (involving at pleasure constant imaginary quantities), P is equal to the real, and iQ (or $-i$Q in the second solution) equal to the imaginary part of $f(p+iq)$, and by elimination T and U will be represented as functions of t and u. Thus the proposed problem is solved quite generally and completely.

6. If we represent any determinate function of $p+iq$ by $p'+iq'$ (where p' and q' are real functions of p and q), it will be easily seen that likewise the equations

$$p'+iq' = \text{const. and } p'-iq' = \text{const.}$$

will represent the integrals of the differential equation $\omega = 0$; indeed, these equations will respectively agree with the above

$$p+iq = \text{const. and } p-iq = \text{const.}$$

In like manner the integrals of the differential equation $\Omega = 0$, viz. $\quad P'+iQ' = \text{const. and } P'-iQ' = \text{const.}$

will agree with the above,

$$P+iQ = \text{const. and } P-iQ = \text{const.}$$

if $P'+iQ'$ represents any determinate function of $P+iQ$ (while P$'$ and Q$'$ are real functions of P and Q). Hence it is clear that in the general solution of our problem which we have given in the preceding article, p' and q' may be substituted for p and q, and P$'$ and Q$'$ for P and Q. Although this change does not add to the generality of the solution, yet in practice one form may be more applicable to one, and another to another purpose.

7. If the functions arising from the differentiation of the arbitrary function f and f' are denoted respectively by ϕ and ϕ', so that $d.f'v = \phi v . dv$ and $d.f'v = \phi' v . dv$. we shall have in conformity with our general solution

$$\frac{dP+idQ}{dp+idq} = \phi(p+iq), \quad \frac{dP-idQ}{dp-idq} = \phi'(p-iq),$$

therefore, $\quad \dfrac{m^2 n}{N} = \phi(p+iq) . \phi(p-iq).$

The scale of linear dimensions is determined by

$$m = \sqrt{\left\{\frac{dp^2+dq^2}{\omega} . \frac{\Omega}{dP^2+dQ^2} . \phi(p+iq) . \phi'(p-iq)\right\}}.$$

8. We shall now illustrate our general solution by some examples by which the manner of applying it, as well as the nature of some circumstances which may come into consideration, may be best explained.

Let the two surfaces be in the first place planes, in which case we may put

$$x = t, \quad y = u, \quad z = 0$$
$$X = T, \quad Y = U, \quad Z = 0.$$

The

The differential equation $\omega = dt^2 + du^2 = 0$ gives these two integrals

$$t + iu = \text{const.},\ t - iu = \text{const.}$$

and in like manner the two integrals of the equation $\Omega = dT^2 + dU^2 = 0$, are the following $T + iU = \text{const.}, T - iU = \text{const.}$ The two general solutions of the problem are accordingly

$$\text{I. } T + iU = f(t + iu),\ T - iU = f'(t - iu)$$
$$\text{II. } T + iU = f(t - iu),\ T - iU = f'(t + iu).$$

This result may be thus expressed: f signifying an arbitrary function, the real part of $f(x + iy)$ is to be taken for x, and the imaginary part divided by i for y or for $-y$.

If the functional characteristics ϕ, ϕ' are taken in the same signification which they have in article 7, and if we put

$$\phi(x + iy) = \xi + i\eta,\ \phi'(x - iy) = \xi - i\eta$$

where ξ and η will be clearly real functions of x and y, we have by the first solution

$$dX + idY = (\xi + i\eta)(dx + idy)$$
$$dX - idY = (\xi - i\eta)(dx - idy)$$

and consequently, $dX = \xi dx - \eta dy$
$dY = \eta dx + \xi dy$

If we now put $\xi = \sigma . \cos j$, $\eta = \sigma . \sin j$
$dx = ds . \cos g$, $dy = ds . \sin g$
$dX = dS . \cos G$, $dY = dS . \sin G$,

so that ds is a linear element in the first plane, g its inclination to the line of abscissae, dS the corresponding linear element in the second plane, and G its inclination to the line of abscissae, the above equations give

$$dS . \cos G = \sigma . ds . \cos(g + j)$$
$$dS . \sin G = \sigma ds \ \sin(g + j),$$ and consequently, if we consider σ as positive, as we may do

$$dS = \sigma . ds \quad , G = g + j.$$

We see, therefore (in conformity to article 7), that σ is the index of the ratio of increase of the element ds in the representation ds, and is, as it ought to be, independent of g; and in the same way the angle j being independent of g, proves that all linear elements of the first plane proceeding from one point are represented by elements in the second plane which form to each other, and, as we may add, in the same direction, the same angles.

If we now choose for f a linear function, so that $fv = A + Bv$ where the constant coefficients are of the form $A = a + b . i$,

B =

$B = c + e\,i$, we shall have $\varphi v = B = c + e\,i$, therefore $\sigma = \sqrt{(c^2 + e^2)}$, $j =$ arc tang $\frac{e}{c}$.

The ratio of increase or the scale is consequently constant throughout, and the whole representation similar to the surface represented. For every other function f, it may be easily proved that the scale cannot be constant, and that the similarity can only take place in the smallest part. If the places are given which are to correspond in the representation to a determinate number of given points of the first plane, we may easily determine by the common method of interpolation the simplest algebraical function f, which will fulfill those conditions. If we denote the values of $x + i\,y$ for the given points by a, b, c, &c. and the corresponding values of $X + i\,Y$ by A, B, C, &c. then it will be necessary to put

$$f v = \frac{(v-b)(v-c)\ldots}{(a-b)(a-c)\ldots} A + \frac{(v-a)(v-c)\ldots}{(b-a)(b-c)\ldots} B + \frac{(v-a)(v-b)\ldots}{(c-a)(c-b)\ldots} C + \&c.$$

which is an algebraical function of v of a degree one unity lower than the number of given points. For two points, where the function becomes linear, a perfect resemblance will consequently take place.

An useful application may be made of this in geodetics, for converting a map founded on moderately good measurements, which in its minute detail is good, but on the whole somewhat distorted, into a better one, if the correct position of a number of points is known.

Going through the second solution in the same manner, it will be found that the only difference is, that the similarity is a reversed one; that all elements form indeed with each other the same angles as in the original, but in a contrary direction, so that that which is to the right in the one, is to the left in the other. But this difference is not an essential one, and vanishes if the side of the plane which was first considered as the upper one is made the lower one. This latter remark may be always applied whenever one of the surfaces is a plane; and we shall confine ourselves in the following examples of this kind to the first solution.

9. Let us now consider (as a second example) the representation of the surface of a perpendicular cone in a plane. As the equation of the former, we take

$$x^2 + y^2 - K^2 z^2 = 0$$

where we put $x = K\,t\,.\cos u$, $y = K\,t\,.\sin u$, $z = t$, and as before, $X = T$, $Y = U$, $Z = 0$.

The

The differential equation,

$\omega = (K^2+1)\, dt^2 + K^2 t^2 du^2 = 0$, gives the two integrals,

$$\log t \pm i\sqrt{\frac{K^2}{K^2+1}} \,.\, u = \text{const.}$$

We have, accordingly, the solution

$$X + iY = f\left(\log t + i\sqrt{\frac{K^2}{K^2+1}}\, u\right),$$

$$X - iY = f'\left(\log t + i\sqrt{\frac{K^2}{K^2+1}} \,.\, u\right);$$

that is to say, f denoting an arbitrary function, X is to be the real part of $f\left(\log t + i\sqrt{\frac{K^2}{K^2+1}}\, u\right)$, and Y the imaginary part, leaving out the factor i.

Let an exponential quantity be taken for f, or let $fv = h e^v$ where h is constant and e the base of the hyperbolical logarithms, and the most simple representation will be

$$X = h\,t \,.\, \cos\sqrt{\frac{K^2}{K^2+1}} \,.\, u \,.\quad Y = h\,t \,.\, \sin\sqrt{\frac{K^2}{K^2+1}} \,.\, u.$$

The application of the formulæ of article 7, gives in this case

$$n = (K^2+1)\, t^2 \quad N = 1$$

and $\phi\, v$ being $= \phi' v = h\, e^v$,

$$\phi\left(\log t + i\sqrt{\frac{K^2}{K^2+1}} \,.\, u\right) \,.\, \phi'\left(\log t - i\sqrt{\frac{K^2}{K^2+1}} \,.\, u\right) = h^2 t^2$$

consequently $m = \frac{h}{\sqrt{(K^2+1)}}$, and therefore constant. If now, besides, h is made $= \sqrt{(K^2+1)}$, the representation becomes a perfect development.

10. Let it next be required; to represent in a plane the surface of a sphere whose radius $= a$. We put here

$x = a \,.\, \cos t \,.\, \sin u, \quad y = a \,.\, \sin t \,.\, \sin u,$

$Z = a \,.\, \cos u$, by which we obtain

$\omega = a^2 \sin u^2\, dt^2 + a^2 du^2$. The differential equation $\omega = 0$ gives consequently

$$dt \mp i \,.\, \frac{du}{\sin u} = 0, \text{ and its integration}$$

$$t \pm i \log . \text{cotang} . \tfrac{1}{2} u = \text{const.}$$

If we denote therefore again by f an arbitrary function, X is to be put equal to the real, and iY to the imaginary part of $f(t + i \log \text{cotang} \tfrac{1}{2} u)$. We shall adduce some particular cases of this general solution. If we choose for f a linear function by putting $fv = kv$, we shall have $X = kt$ $Y = k \log \text{cotang} \tfrac{1}{2} u$.

This

This agrees evidently when applied to the earth with Mercator's projection, if we make t the geographical longitude, and $90°-a$ the latitude. For the scale of linear dimensions the formulæ of article 7 give $m = \frac{k}{a \sin u}$.

If we assume for f an imaginary exponential function, and in the first place the simplest of all, $fv = k\, e^{iv}$, we have $f(t + i \log \text{cotang} \frac{1}{2} u) = k\, e^{\log \text{tang} \frac{1}{2} u + it} =$ $k \text{ tang} \frac{1}{2} u (\cos t + i \sin t)$ and $X = k \text{ tang} \frac{1}{2} u . \cos t$, $Y = k \text{ tang} \frac{1}{2} u . \sin t$ which is, as will be easily seen, the stereographical projection.

If we put more generally $fv = k e^{i\lambda v}$, we have

$$X = k \text{ tang} \tfrac{1}{2} u^{\lambda} . \cos \lambda t \quad , \quad Y = k \text{ tang} \tfrac{1}{2} u^{\lambda} . \sin \lambda t.$$

For the scale of linear dimensions in the representation, we obtain here $n = a^2 \sin u^2$, $N = 1$, $\phi k v = i \lambda k e^{i\lambda v}$, and hence

$$m = \frac{\lambda k \text{ tang} \frac{1}{2} u}{a \sin u}.$$

It is evident that the representation of all points for which u is the same, will form a circle, and the representation of those points for which t is constant, a straight line, as also that the different circles corresponding to the different values of u are concentric. This affords a very useful projection for maps, if a part only of a sphere is to be represented. It will then be best to choose λ in such a manner as to make the scale the same for the extreme values of u which will make it smallest towards the middle. If we suppose the extreme values of u to be $u°$ and u', we must put

$$\lambda = \frac{\log \sin u' - \log \sin u°}{\log \text{tang} \frac{1}{2} u' - \log \text{tang} \frac{1}{2} u°}.$$

The sheets Nos. 19—26 of Prof. Harding's Celestial Maps, are drawn agreeably to this projection.

11. The general solution of the example given in the preceding article, may be exhibited in another form, which deserves to be mentioned on account of its neatness.

In conformity to what has been proved in article 6, we have [$\text{tang} \frac{1}{2} u (\cos t + i \sin t)$ being a function of $t + i \log \text{cotang} \frac{1}{2} u$ and $\text{tang} \frac{1}{2} u (\cos t + i \sin t) = \frac{\sin u . \cos t + i \sin u . \sin t}{1 + \cos u} = \frac{x + iy}{a + z}$] for the general solution likewise these formulæ:

$X + iY = f \frac{x+iy}{a+z}$, $X - iY = f' \frac{x-iy}{a+z}$ that is, X must be made equal to the real, and iY to the imaginary part of $f' \frac{x+iy}{a+z}$ f' denoting

f' denoting an arbitrary function. It will be easily seen that instead of $f\frac{x+iy}{a+z}$, any arbitrary function of $\frac{y+iz}{a+x}$, or of $\frac{z+ix}{a+y}$, may be taken.

[To be continued.]

XVII. *Analysis of two new Mineral Substances, consisting of Bi-seleniuret of Zinc and Sulphuret of Mercury, found at Culebras in Mexico. By Professor* Del Rio*.

EACH step of the traveller in this Republic discovers to him something new. Mr. Joseph Manuel Herrera, in an excursion to Culebras, near the mining district of El Doctor, found a mineral resembling cinnabar, accompanied by metallic quicksilver, in the limestone which overlies the red sandstone (*arenisca roxa*), and he gave me a few small specimens of this substance. Some considerable time afterwards Col. Robinson gave me an additional quantity, informing me at the same time that Dr. Magos had obtained two ounces and a half of quicksilver from sixteen ounces of the ore.

Under the blowpipe the red ore burns with a beautiful violet-coloured flame, accompanied by much smoke of a most offensive smell, resembling that of rotten cabbage: the residue is a grayish-white earthy matter.

Intimately admixed with the red mineral is another substance so strongly resembling light gray silver ore, that I acknowledge that I mistook it, at first, for this ore of silver. My only doubt on the subject arose from the consideration that gray silver ore and cinnabar are never found together. It differs, however, from gray silver in yielding a blacker powder when scraped, and which stains more than the powder of the latter. Under the blowpipe nearly the same phænomena are observed as when the red mineral is submitted to the same test. According to Mr. Chovell, the specific gravity of the gray substance is 5·56, after having been carefully cleared by washing from the calcareous spar of the matrix. That of the red substance, after having also been carefully separated from the spar, is 5·66; while the specific gravity of hepatic mercury exceeds 5·8.

The analysis of these minerals is very easy where great precision is not required. Nothing more is necessary than to put fifty grains of the ore in a small retort on the fire; mercury, selenium, and a small quantity of sulphur are imme-

* Communicated by A. F. Mornay, Esq.