Confocal Paraboloids. *By* A. G. Greenhill.

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The geometrical and analytical theory of confocal central quadrics has received considerable attention from its important applications to problems in Hydrodynamics, Electricity, Magnetism, and Attractions; but except for § 154, Chapter x., Vol. I., of Maxwell's *Electricity*, the corresponding theorems and applications of confocal paraboloids have not received special treatment;* and it is the object of this article to develop this mathematical treatment from an independent standpoint.

It will be found analytically interesting and instructive to carry this out, as the elliptic functions required in the general case of confocal central quadrics degenerate in the special case of confocal paraboloids into the ordinary circular and hyperbolic functions; and consequently the problems discussed do not require a knowledge of anything more than the properties of the functions employed in elementary mathematics.

1. Taking the ordinary system of three rectangular axes Ox, Oy, Oz in space, and two points S, S' on the axis of x, each at a distance a from the origin O, then the two foci S and S', and the two coordinate planes zOx, xOy are sufficient to define a system of confocal paraboloids.

Any point A being taken in the axis of x as the vertex of a paraboloid, the two principal sections of the surface made by the coordinate planes zOx and xOy will be the parabolas in these planes, having a common vertex at A and foci S and S' respectively; these parabolas may conveniently be called the *principal* or *directing parabolas* of the paraboloid.

If A is taken anywhere between the foci S and S', the paraboloid will be *hyperbolic*; but if A is taken anywhere beyond S or S' on either side, the paraboloid will be *elliptic.*

* [I have just received *Parabolische Koordinaten*, von Dr. Karl Baer, Frankfurt a/O, 1888. A. G. G., 19th April, 1888.]

2. Suppose a vertex A_1 taken beyond S on the positive side of the axis of x at a distance from O, which we shall denote by $a \cosh \alpha$; then

$$SA_1 = a(\cosh \alpha - 1) = 2a \sinh^2 \tfrac{1}{2}\alpha,$$

$$S'A_1 = a(\cosh \alpha + 1) = 2a \cosh^2 \tfrac{1}{2}\alpha;$$

and the equations of the directing parabolas in the coordinate planes of zOx and xOy, with common vertex at A_1 and foci S and S', are

$$y = 0, \quad z^2 = 8a \sinh^2 \tfrac{1}{2}\alpha\,(a \cosh \alpha - x),$$

$$z = 0, \quad y^2 = 8a \cosh^2 \tfrac{1}{2}\alpha\,(a \cosh \alpha - x);$$

and therefore the equation of the corresponding elliptic paraboloid is

$$\frac{y^2}{\cosh^2 \frac{1}{2}\alpha} + \frac{z^2}{\sinh^2 \frac{1}{2}\alpha} = 8a(a \cosh \alpha - x) \quad \ldots\ldots\ldots\ldots\ldots (1).$$

For, putting $y = 0$ and $z = 0$ alternately in equation (1), the equations of the corresponding directing parabolas are obtained.

3. Next suppose a vertex A_2 taken between S and S', at a distance from the origin O, which we shall denote by $a \cos \beta$; then

$$A_2S = a(1 - \cos \beta) = 2a \sin^2 \tfrac{1}{2}\beta,$$

$$S'A_2 = a(1 + \cos \beta) = 2a \cos^2 \tfrac{1}{2}\beta.$$

The equations of the directing parabolas being now

$$y = 0, \quad z^2 = 8a \sin^2 \tfrac{1}{2}\beta\,(x - a \cos \beta),$$

$$z = 0, \quad y^2 = 8a \cos^2 \tfrac{1}{2}\beta\,(a \cos \beta - x);$$

the equation of the corresponding hyperbolic paraboloid will be

$$\frac{y^2}{\cos^2 \frac{1}{2}\beta} - \frac{z^2}{\sin^2 \frac{1}{2}\beta} = 8a(a \cos \beta - x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These hyperbolic paraboloids will have generating lines, parallel to the asymptotic planes

$$\frac{y^2}{\cos^2 \frac{1}{2}\beta} - \frac{z^2}{\sin^2 \frac{1}{2}\beta} = 0.$$

4. Lastly, suppose a vertex A_3 taken beyond S', at a distance $a \cosh \gamma$ from the origin O; then

$$A_3S = a(\cosh \gamma + 1) = 2a \cosh^2 \tfrac{1}{2}\gamma,$$

$$A_3S' = a(\cosh \gamma - 1) = 2a \sinh^2 \tfrac{1}{2}\gamma;$$

and the equations of the directing parabolas being

$$y = 0, \quad z^2 = 8a \cosh^2 \tfrac{1}{2}\gamma\,(a \cosh \gamma + x),$$

$$z = 0, \quad y^2 = 8a \sinh^2 \tfrac{1}{2}\gamma\,(a \cosh \gamma + x),$$

the equation of the corresponding elliptic paraboloid will be

$$\frac{y^2}{\sinh^2 \frac{1}{2}\gamma} + \frac{z^2}{\cosh^2 \frac{1}{2}\gamma} = 8a\,(a \cosh \gamma + x) \;\ldots\ldots\ldots\ldots\ldots (3).$$

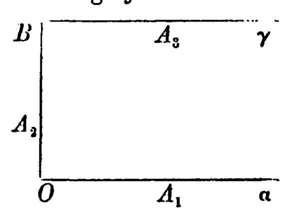
These equations (1), (2), and (3) represent a system of orthogonal confocal paraboloids in their simplest canonical form; and the parameters α, β, γ are the equivalents of Lamé's thermometric parameters for confocal ellipsoids and hyperboloids.

5. Solving these equations (1), (2), (3) for x, y, z in terms of α, β, γ, we find

$$\left.\begin{aligned} x &= a\,(\cosh \alpha + \cos \beta - \cosh \gamma) \\ y &= 4a \cosh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \sinh \tfrac{1}{2}\gamma \\ z &= 4a \sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

so that to agree with the corresponding expressions given by Maxwell, *Electricity and Magnetism*, Vol. I., p. 190, we must invert the positive direction of the axis of x, and interchange y and z.

The whole series of surfaces and of values of x, y, z is obtained by making α range from ∞ to 0, β from 0 to π, and γ from 0 to ∞.



Since $\cos \beta = \cosh i\beta$,

$$\cosh \gamma = -\cosh\,(i\pi + \gamma),$$

we may take a period parallelogram of infinite length, open at one end and bounded by the lines

$$y = 0, \quad x = 0, \quad \text{and} \quad y = \pi;$$

and then the vector of a point moving round the perimeter of the period parallelogram will give the series of confocal paraboloids; the vector being α anywhere on $A_1 O$, $i\beta$ on OA_2B, and $i\pi + \gamma$ on BA_3; so that now, writing α', β', γ' for α, $i\beta$ and $i\pi + \gamma$, we shall obtain the symmetrical expressions for x, y, z in terms of α', β', γ',

$$\left.\begin{aligned} x &= a\,(\cosh \alpha' + \cosh \beta' + \cosh \gamma') \\ y &= -4ia \cosh \tfrac{1}{2}\alpha' \cosh \tfrac{1}{2}\beta' \cosh \tfrac{1}{2}\gamma' \\ z &= -4a \sinh \tfrac{1}{2}\alpha' \sinh \tfrac{1}{2}\beta' \sinh \tfrac{1}{2}\gamma' \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots (5),$$

so that

$$y + iz = -4ia\,\{\exp \tfrac{1}{2}(\alpha' + \beta' + \gamma') + \exp \tfrac{1}{2}(\alpha' - \beta' - \gamma') + \exp \tfrac{1}{2}(-\alpha' + \beta' - \gamma') + \exp \tfrac{1}{2}(-\alpha' - \beta' + \gamma')\}.$$

6. The generating lines of the paraboloids are real only on the hyperbolic paraboloids given by equation (2); and their equations are of the form

$$\left.\begin{aligned}\frac{y}{\cos\frac{1}{2}\beta} \pm \frac{z}{\sin\frac{1}{2}\beta} &= 4a\lambda \\ \frac{y}{\cos\frac{1}{2}\beta} \mp \frac{z}{\sin\frac{1}{2}\beta} &= \frac{2(a\cos\beta - x)}{\lambda}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

so that the projections of the generating lines on the plane yOz are two sets of parallel lines inclined at an angle β.

This is well seen in the cardboard model of this surface made by Brill, of Darmstadt, which exhibits the series of different forms of confocal hyperbolic paraboloids made by the deformation of the model and its generating lines, when the angle β between the two sets of parallel planes of cardboard is altered; the focal parabolas being obtained in the two positions in which the model is flattened out.

With the values of x, y, z given in (4),

$$\lambda = \cosh\tfrac{1}{2}\alpha \sinh\tfrac{1}{2}\gamma \pm \sinh\tfrac{1}{2}\alpha \cosh\tfrac{1}{2}\gamma = \sinh\tfrac{1}{2}(\gamma \pm \alpha);$$

so that, keeping β constant, then, along a generating line of the corresponding hyperbolic paraboloid, $\gamma \pm \alpha$ is constant.

7. Employing Maxwell's notation, in Chapter x., *Electricity and Magnetism*, let us denote by ds_1, ds_2, ds_3 the elements of the normal to the surfaces α, β, γ; then

$$\left(\frac{ds_1}{d\alpha}\right)^2 = \left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2,$$

$$\begin{aligned}\frac{1}{a^2}\left(\frac{ds_1}{d\alpha}\right)^2 &= \sinh^2\alpha + 4\sinh^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\beta\sinh^2\tfrac{1}{2}\gamma + 4\cosh^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\beta\cosh^2\tfrac{1}{2}\gamma \\ &= \sinh^2\alpha + \tfrac{1}{2}(\cosh\alpha - 1)(1+\cos\beta)(\cosh\gamma - 1) \\ &\qquad + \tfrac{1}{2}(\cosh\alpha + 1)(1-\cos\beta)(\cosh\gamma + 1) \\ &= \sinh^2\alpha + 1 - \cos\beta\cosh\gamma + \cosh\gamma\cosh\alpha - \cosh\alpha\cos\beta \\ &= (\cosh\alpha - \cos\beta)(\cosh\alpha + \cosh\gamma);\end{aligned}$$

and, similarly,

$$\frac{1}{a^2}\left(\frac{ds_2}{d\beta}\right)^2 = (\cosh\alpha - \cos\beta)(\cos\beta + \cosh\gamma),$$

$$\frac{1}{a^2}\left(\frac{ds_3}{d\gamma}\right)^2 = (\cosh\alpha + \cosh\gamma)(\cos\beta + \cosh\gamma).$$

Denoting by l_1, m_1, n_1 the direction-cosines of the normal to the surface α, then

$$l_1 = \frac{dx}{ds_1} = \frac{dx}{d\alpha} \bigg/ \frac{ds_1}{d\alpha} = \frac{\sinh \alpha}{\surd\{(\cosh \alpha - \cos\beta)(\cosh\alpha + \cosh\gamma)\}},$$

$$m_1 = \frac{2 \sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \sinh \tfrac{1}{2}\gamma}{\surd\{(\cosh \alpha - \cos\beta)(\cosh \alpha + \cosh \gamma)\}},$$

$$n_1 = \frac{2 \cosh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma}{\surd\{(\cosh \alpha - \cos\beta)(\cosh \alpha + \cosh \gamma)\}}.$$

Writing D_1^2 for $\cos\beta + \cosh\gamma$, D_2^2 for $\cosh\alpha + \cosh\gamma$, D_3^2 for $\cosh\alpha - \cos\beta$, by analogy with Maxwell's notation, then l_2, m_2, n_2 and l_3, m_3, n_3 denoting the direction-cosines of the normals to the surfaces β and γ, a similar investigation proves that

$$l_2 = -\frac{\sin\beta}{D_3 D_1}, \qquad m_2 = -\frac{2\cosh\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta \sinh\tfrac{1}{2}\gamma}{D_3 D_1},$$

$$n_2 = \frac{2\sinh\tfrac{1}{2}\alpha \cos\tfrac{1}{2}\beta \cosh\tfrac{1}{2}\gamma}{D_3 D_1};$$

$$l_3 = -\frac{\sinh\gamma}{D_1 D_2}, \qquad m_3 = \frac{2\cosh\tfrac{1}{2}\alpha \cos\tfrac{1}{2}\beta \cosh\tfrac{1}{2}\gamma}{D_1 D_2},$$

$$n_3 = \frac{2\sinh\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta \sinh\tfrac{1}{2}\gamma}{D_1 D_2}.$$

Thus

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$$

$$l_3 l_1 + m_3 m_1 + n_3 n_1 = 0,$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0;$$

verifying that the surfaces α, β, γ form a triply orthogonal system.

8. Laplace's equation with α, β, γ for variables now becomes (Maxwell, § 148)

$$\nabla^2 V = D_1^2 \frac{d^2 V}{d\alpha^2} + D_2^2 \frac{d^2 V}{d\beta^2} + D \frac{d^2 V}{d\gamma^2} = 0,$$

or

$$\nabla^2 V = (\cos\beta + \cosh\gamma)\frac{d^2 V}{d\alpha^2} + (\cosh\gamma + \cosh\alpha)\frac{d^2 V}{d\beta^2}$$

$$+ (\cosh\alpha - \cos\beta)\frac{d^2 V}{d\gamma^2} = 0,$$

Supposing $V = AB\Gamma$, where A is a function of a only, B of β, and Γ of γ, then Laplace's equation is equivalent to

$$\frac{1}{A}\frac{d^2A}{da^2} = g\cosh a + h, \quad \frac{1}{B}\frac{d^2B}{d\beta^2} = -g\cos\beta - h,$$

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{d\gamma^2} = -g\cosh\gamma + h,$$

three degenerate forms of Lamé's equation.

Laplace's equation is also satisfied if V is a linear function of a, β, γ.

As an Electrostatic example, suppose two elliptic paraboloids, denoted by a_1 and a_2, are electrified to potentials V_1 and V_2 respectively; then the electric potential in the interspace will be

$$V = \frac{a - a_2}{a_1 - a_2} V_1 + \frac{a_1 - a}{a_1 - a_2} V_2;$$

and the electrification σ_1, at any point of the surface a_1, will be given by

$$4\pi\sigma_1 = -\frac{dV}{ds_1} = -\frac{V_1 - V_2}{a_1 - a_2}\frac{da}{ds_1} = -\frac{V_1 - V_2}{a_1 - a_2}\frac{D_2D_3}{a};$$

with a similar expression for the electrification σ_2 at any point of the surface a_2.

When two surfaces β_1 and β_2 or γ_1 and γ_2 are electrified to potentials V_1 and V_2, similar expressions to the above hold for the potential in the interspace, and for the electrification on either surface.

With regard to the geometrical interpretation of D_1, D_2, and D_3, we notice that, if A_1, A_2, A_3 denote the vertices of the paraboloids a, β, γ,

$$D_1^2 = a\,.\,A_2A_3, \quad D_2^2 = a\,.\,A_3A_1, \quad D_3^2 = a\,.\,A_1A_2.$$

With confocal central quadrics, D_2 and D_3 are the semi-axes of the central section of a which is conjugate to the diameter through the point of intersection of the surfaces a, β, γ, and these are parallel to ds_2 and ds_3.

But with paraboloids the centre has gone off to an infinite distance and a different geometrical interpretation must be devised.

9. As a Hydrodynamical application, consider the disturbance in the motion of infinite liquid flowing in the direction xO with uniform velocity U, due to the presence of a fixed obstacle in the shape of the elliptic paraboloid a_1; then we must seek to determine a velocity function ϕ satisfying Laplace's equation and also the conditions that

$$\frac{d\phi}{dx} = -U, \quad \frac{d\phi}{dy} = 0, \quad \frac{d\phi}{dz} = 0, \quad \text{when } \alpha = \infty,$$

and

$$\frac{d\phi}{ds_1} = 0, \quad \text{when} \quad \alpha = \alpha_1.$$

This can be effected by supposing the velocity function of the form

$$\phi = U(A\alpha - x),$$

obviously satisfying Laplace's equation as the equation of continuity, and then determining A so as to satisfy the boundary conditions.

Now

$$\frac{d\phi}{dx} = U\left(A\frac{d\alpha}{dx} - 1\right) = U\left(Al_1\frac{d\alpha}{ds_1} - 1\right)$$

$$= U\left(\frac{A}{a}\frac{\sinh\alpha}{D_2^2 D_3^2} - 1\right),$$

which when $\alpha = \infty$ becomes $-U$, since $\sinh\alpha / D_2^2 D_3^2$ is then ultimately zero; and similarly

$$\frac{d\phi}{dy} = 0, \quad \frac{d\phi}{dz} = 0, \quad \text{when} \quad \alpha = \infty.$$

Next, when $\alpha = \alpha_1$,

$$\frac{d\phi}{ds_1} = U\left(A\frac{d\alpha}{ds_1} - l_1\right) = 0,$$

so that

$$A = l_1\frac{ds_1}{d\alpha} = a\sinh\alpha_1;$$

and therefore the expression for the velocity function is

$$\phi = U(a\alpha\sinh\alpha_1 - x).$$

Similar investigations for a fixed obstacle in the shape of the hyperbolic paraboloid β_1, will show that the velocity function

$$\phi = U(a\beta\sin\beta_1 - x);$$

while the velocity function when the fixed obstacle is the elliptic paraboloid γ_1 is

$$\phi = U(x - a\gamma\sinh\gamma_1),$$

the liquid being now supposed originally flowing uniformly with velocity U in the direction Ox, in order to flow over the outer convex surface of the paraboloid γ_1.

A paraboloid receiving the stream of liquid on the concave side would stop the stream, instead of merely deflecting it, as the convex side does, and the preceding investigations are no longer applicable.

10. Next suppose the infinite stream of liquid originally flowing parallel to the axis of y with uniform velocity V, and disturbed by the presence of a fixed obstacle in the shape of the elliptic paraboloid α_1; to investigate the form of the velocity function ϕ.

We shall find that the required conditions can be satisfied by supposing ϕ to be composed of two terms, one term being

$$Vy = 4aV\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma,$$

and the other term being of the form

$$AV\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma.$$

For the first term Vy satisfies Laplace's equation, and so also does the second; so that now, putting

$$\phi = V(A\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma - y),$$

we must seek to determine A from the boundary conditions.

As before, $\dfrac{d\phi}{dx} = 0,\quad \dfrac{d\phi}{dy} = -V,\quad \dfrac{d\phi}{dz} = 0,$ when $\alpha = \infty$;

also $$\frac{d\phi}{ds_1} = V\left(\tfrac{1}{2}A\cosh\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma\frac{d\alpha}{ds_1} - m_1\right) = 0,$$

when $\alpha = \alpha_1$, so that $A = 4a\tanh\tfrac{1}{2}\alpha_1$; and therefore

$$\begin{aligned}\phi &= V(4a\tanh\tfrac{1}{2}\alpha_1\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma - y)\\ &= 4aV\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma(\tanh\tfrac{1}{2}\alpha_1\sinh\tfrac{1}{2}\alpha - \cosh\tfrac{1}{2}\alpha)\\ &= 4aV\operatorname{sech}\tfrac{1}{2}\alpha_1\cosh\tfrac{1}{2}(\alpha_1-\alpha)\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma.\end{aligned}$$

If the stream was originally parallel to the axis of z, and the same fixed obstacle α_1 was introduced, then we should find, as before,

$$\begin{aligned}\phi &= W(4a\coth\tfrac{1}{2}\alpha_1\cosh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma - z)\\ &= 4aW\operatorname{cosech}\tfrac{1}{2}\alpha_1\cosh\tfrac{1}{2}(\alpha-\alpha_1)\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma.\end{aligned}$$

The corresponding expressions for the velocity function ϕ when the fixed obstacle is the hyperbolic paraboloid β_1, or the elliptic paraboloid γ_1, are now easily written down; for instance, we shall find, for the surface β_1,

$$\phi = -4aV\sec\tfrac{1}{2}\beta_1\cos\tfrac{1}{2}(\beta-\beta_1)\cosh\tfrac{1}{2}\alpha\sinh\tfrac{1}{2}\gamma,$$

or $$\phi = 4aW\operatorname{cosec}\tfrac{1}{2}\beta_1\cos\tfrac{1}{2}(\beta-\beta_1)\sinh\tfrac{1}{2}\alpha\cosh\tfrac{1}{2}\gamma,$$

according as the liquid was originally streaming parallel to the axis of y or z; while the corresponding expressions for the surface γ_1 are

$$\phi = 4aV\operatorname{cosech}\tfrac{1}{2}\gamma_1\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cosh\tfrac{1}{2}(\gamma-\gamma_1),$$

and $$\phi = -4aW\operatorname{sech}\tfrac{1}{2}\gamma_1\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}(\gamma-\gamma_1).$$

11. Suppose the interspace between a_1 and a_2 filled with liquid, and now suppose the surface a_1 to be moved with velocity U_1, and a_2 with velocity U_2, parallel to Ox; to determine the velocity function ϕ of the initial motion of the liquid filling the interspace.

This is obtained by putting

$$\phi = Aa + Bx,$$

thus satisfying Laplace's equation of continuity; and then A and B are determined from the conditions that

$$\frac{d\phi}{ds_1} = U_1 l_1, \quad \text{when} \quad a = a_1;$$

$$\frac{d\phi}{ds_2} = U_2 l_1, \quad \text{when} \quad a = a_2.$$

Now $$\frac{da}{ds_1} = l_1 \Big/ \frac{dx}{da} = l_1 / a \sinh a,$$

and $$\frac{dx}{ds_1} = l_1,$$

so that dividing out l_1,

$$U_1 = \frac{A}{a} \operatorname{cosech} a_1 + B,$$

$$U_2 = \frac{A}{a} \operatorname{cosech} a_2 + B,$$

whence A and B are determined. Then

$$\phi = a \frac{\left\{ \begin{array}{c} (U_1 - U_2)\, a \sinh a_1 \sinh a_2 \\ -(U_1 \sinh a_1 - U_2 \sinh a_2)(\cosh a + \cos\beta - \cosh\gamma) \end{array} \right\}}{\sinh a_1 - \sinh a_2}.$$

Supposing $a_1 > a_2$, then, for the infinite liquid outside the surface a_1, the motion due to the velocity U_1 of a_1 is given by the velocity function $$\phi = aU_1 a \sinh a_1;$$
while, for the liquid filling the interior of the surface a_2, the velocity function is simply $$\phi = U_2 x.$$

If the surfaces a_1 and a_2 had been started with velocities V_1 and V_2 parallel to Oy, then we should have to put

$$\phi = (A \sinh \tfrac{1}{2}a + B \cosh \tfrac{1}{2}a) \cos \tfrac{1}{2}\beta \sinh \tfrac{1}{2}\gamma,$$

satisfying the equation of continuity; and then the boundary

conditions
$$\frac{d\phi}{ds_1} = V_1 m_1, \text{ when } \alpha = \alpha_1,$$
$$\frac{d\phi}{ds_1} = V_2 m_1, \text{ when } \alpha = \alpha_2,$$

lead to the equations

$$A \cosh \tfrac{1}{2}\alpha_1 + B \sinh \tfrac{1}{2}\alpha_1 = 4a\, V_1 \sinh \tfrac{1}{2}\alpha_1,$$
$$A \cosh \tfrac{1}{2}\alpha_2 + B \sinh \tfrac{1}{2}\alpha_2 = 4a\, V_2 \sinh \tfrac{1}{2}\alpha_2,$$

for the determination of A and B; so that

$$\phi = 4a \operatorname{cosech} \tfrac{1}{2}(\alpha_1 - \alpha_2)$$
$$\times \left\{ V_1 \sinh \tfrac{1}{2}\alpha_1 \sinh \tfrac{1}{2}(\alpha - \alpha_2) + V_2 \sinh \tfrac{1}{2}\alpha_2 \sinh \tfrac{1}{2}(\alpha_1 - \alpha) \right\} \cos \tfrac{1}{2}\beta \sinh \tfrac{1}{2}\gamma.$$

When the surfaces α_1 and α_2 have initial velocities W_1 and W_2 respectively parallel to Oz, then we must put the velocity function of the liquid in the interspace

$$\phi = (A \sinh \tfrac{1}{2}\alpha + B \cosh \tfrac{1}{2}\alpha) \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma;$$

and determine A and B from the boundary conditions

$$\frac{d\phi}{ds_1} = W_1 n_1, \quad \text{when } \alpha = \alpha_1,$$
$$\frac{d\phi}{ds_1} = W_2 n_1, \quad \text{when } \alpha = \alpha_2;$$

thus leading to the equations

$$A \cosh \tfrac{1}{2}\alpha_1 + B \sinh \tfrac{1}{2}\alpha_1 = 4a W_1 \cosh \tfrac{1}{2}\alpha_1,$$
$$A \cosh \tfrac{1}{2}\alpha_2 + B \sinh \tfrac{1}{2}\alpha_2 = 4a W_2 \cosh \tfrac{1}{2}\alpha_2,$$

for A and B; and finally giving

$$\phi = 4a \operatorname{cosech} \tfrac{1}{2}(\alpha_1 - \alpha_2)$$
$$\times \left\{ W_1 \cosh \tfrac{1}{2}\alpha_1 \cosh \tfrac{1}{2}(\alpha - \alpha_2) - W_2 \cosh \tfrac{1}{2}\alpha_2 \cosh \tfrac{1}{2}(\alpha_1 - \alpha) \right\} \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma.$$

Similar expressions can easily be written down for the motion of the liquid in the interspace between the two surfaces β_1 and β_2 or γ_1 and γ_2, due to arbitrary velocities V_1 and V_2 parallel to Oy, or W_1 and W_2 parallel to Oz, imparted to the surfaces.

12. As another example, suppose liquid filling the interspace of the surfaces β_1 and β_2 to be set in motion by communicating an angular

velocity p_1 to the surface β_1, and an angular velocity p_2 to the surface β_2, each about the axis of x; to determine ϕ, the velocity function of the initial motion of the liquid.

We must make ϕ satisfy the conditions

$$\nabla^2\phi = 0,$$

and $$\frac{d\phi}{ds_2} = \left\{ \begin{matrix} \text{normal component of the velocity of the} \\ \text{surface } \beta_1 \text{ due to the angular velocity } p_1 \end{matrix} \right\}$$

$$= -p_1 zm_2 + p_1 yn_2$$

$$= p_1 (yn_2 - zm_2), \text{ when } \beta = \beta_1$$

$$\frac{d\phi}{ds_2} = p_2 (yn_2 - zm_2), \text{ when } \beta = \beta_2,$$

$$= 2ap_2 \frac{\sinh\alpha \sinh\gamma}{D_3 D_1}.$$

The proper form to assume for the velocity function is

$$\phi = (A\cos\beta + B\sin\beta)\sinh\alpha\sinh\gamma,$$

and then, when $\beta = \beta_1$,

$$\frac{d\phi}{ds_2} = \frac{-A\sin\beta_1 + B\cos\beta_1}{aD_3 D_1}\sinh\alpha\sinh\gamma$$

$$= \frac{2ap_1}{D_3 D_1}\sinh\alpha\sinh\gamma,$$

so that the variable factors $\sinh\alpha\sinh\gamma$ and $D_3 D_1$ cancel, and then

$$-A\sin\beta_1 + B\cos\beta_1 = 2a^2 p_1,$$

and similarly $$-A\sin\beta_2 + B\cos\beta_2 = 2a^2 p_2,$$

whence A and B can be determined; and then

$$\phi = 2a^2 \operatorname{cosec}(\beta_1 - \beta_2)\{p_2\cos(\beta_1 - \beta) - p_1\cos(\beta - \beta_2)\}\sinh\alpha\sinh\gamma.$$

If the interspace had been bounded by the surfaces α_1 and α_2, then we should have had

$$\phi = (A\cosh\alpha + B\sinh\alpha)\sin\beta\sinh\gamma,$$

and A and B determined by the equations

$$A\sinh\alpha_1 + B\cosh\alpha_1 = 2a^2 p_1,$$

$$A\sinh\alpha_2 + B\cosh\alpha_2 = 2a^2 p_2;$$

and then

$$\phi = 2a^2 \operatorname{cosech} (a_1 - a_2) \{p_1 \cosh (a - a_2) - p_2 \cosh (a_1 - a)\} \sin\beta \sinh\gamma.$$

If $p_1 = 0$, and $a_1 = \infty$, then

$$A + B = 0,$$

and
$$A = -B = -2a^2 p_2 e^{a_2},$$

so that
$$\phi = -2a^2 p_2 e^{-a+a_2} \sin\beta \sinh\gamma,$$

the velocity function due to the rotation of the surface a_2 about the axis of x with angular velocity p_1, in infinite liquid surrounding this elliptic paraboloid on the outside.

But, if $a_2 = 0$, then

$$\phi = 2a^2 \operatorname{cosech} a_1 \{ p_1 \cosh a - p_2 \cosh (a_1 - a) \} \sin\beta \sinh\gamma;$$

and, if $p_2 = 0$ also, then

$$\phi = 2a^2 p_1 \operatorname{cosech} a_1 \cosh a \sin\beta \sinh\gamma,$$

the velocity function of liquid inside the elliptic paraboloid a_1; but as pointed out by one of the referees of this paper, this state of motion implies that the focal parabola for which $a = 0$, and therefore $z = 0$,

$$y^2 = 8a(a - x),$$

must be looked upon as a fixed boundary.

When this boundary is removed, the motion of the liquid inside the elliptic paraboloid a_1, due to a rotation p_1 about Ox, will be given by a velocity function of the form

$$\phi = Ayz$$
$$= 2a^2 A \sinh a \sin\beta \sinh\gamma,$$

and then we shall find, as before,

$$A = p_1 \operatorname{sech} a_1;$$

so that
$$\phi = p_1 yz \operatorname{sech} a_1.$$

When the hyperbolic paraboloid β_1 is rotated about Ox with angular velocity p_1, then the motion of infinite liquid, on either side of the surface, is given by the velocity function

$$\phi = p_1 yz \sec\beta_1.$$

13. When the surfaces a_1 and a_2 are made to rotate with angular velocities q_1 and q_2 about the axis Oy, the velocity function of the initial motion in the interspace is more complicated, the boundary conditions being now

$$\frac{d\phi}{ds_1} = q_1 (zl_1 - xn_1), \text{ when } a = a_1,$$

or

$$\frac{d\phi}{da} = aq_1 (zl_1 - xn_1) D_2 D_3$$

$$= 2a^2 q_1 \{2 \sinh \tfrac{1}{2}a \sinh a - (\cosh a + \cos\beta - \cosh\gamma) \cosh \tfrac{1}{2}a\} \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma$$

$$= 2a^2 q_1 (\cosh a - \cos\beta + \cosh\gamma - 2) \cosh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma,$$

when $a = a_1$; and

$$\frac{d\phi}{ds_1} = q_2 (zl_1 - xn_1), \text{ when } a = a_2,$$

for all values of β and γ.

The form of the velocity function must be inferred by analogy from the corresponding expressions for confocal central quadrics.

Inside the surface a_2, the velocity function of the liquid motion would be of the form

$$\phi = Cxz$$

$$= 4Ca^2 (\cosh a + \cos\beta - \cosh\gamma) \sinh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma;$$

and, noticing that the terms a and $\cosh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma$ give the motion of the liquid due to translations parallel to Ox and Oy, we are led to infer that the required velocity function must be built up partly of terms of the form

$$(\cosh a + \cos\beta - \cosh\gamma) \cosh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma$$

and

$$a \sinh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma.$$

But, substituted in Laplace's equation of continuity of § 8, we find

$$\nabla^2 (\cosh a + \cos\beta - \cosh\gamma) \cosh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma$$
$$= -2 (\cos\beta + \cosh\gamma) \cosh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma,$$

$$\nabla^2 a \sinh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma = (\cos\beta + \cosh\gamma) \cosh \tfrac{1}{2}a \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma;$$

so that these two terms must be combined in the form

$$\{(\cosh a + \cos\beta - \cosh\gamma) \cosh \tfrac{1}{2}a + 2a \sinh \tfrac{1}{2}a\} \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma,$$

in order that the equation of continuity may be satisfied.

To these terms may be added the terms

$$(\cosh\alpha+\cos\beta-\cosh\gamma)\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma$$

and
$$(P\cosh\tfrac{1}{2}\alpha+Q\sinh\tfrac{1}{2}\alpha)\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma,$$

obviously satisfying the equation of continuity; so that now in the general case we may put

$$\phi=\big\{(\cosh\alpha+\cos\beta-\cosh\gamma)(A\cosh\tfrac{1}{2}\alpha+B\sinh\tfrac{1}{2}\alpha)$$
$$+2A\alpha\sinh\tfrac{1}{2}\alpha+P\cosh\tfrac{1}{2}\alpha+Q\sinh\tfrac{1}{2}\alpha\big\}\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma,$$

and now we have sufficient disposable constants A, B, P, Q to satisfy the boundary conditions when $\alpha=\alpha_1$ and $\alpha=\alpha_2$.

Similar expressions can be constructed for the surfaces β_1 and β_2, or γ_1 and γ_2.

14. The velocity function

$$\phi=xyz$$
$$=2a^3(\cosh\alpha+\cos\beta-\cosh\gamma)\sinh\alpha\sin\beta\sinh\gamma$$

satisfies the equation of continuity, and gives the motion of the liquid inside a surface due to a torsional strain imparted to the surface about a principal axis.

The velocity function of the motion of the liquid in the interspace between two surfaces, due to arbitrary torsional strains of the surfaces, may then be constructed by analogy with the solution of the corresponding problem for confocal central quadrics, being built up of terms of the form

$$\alpha\sinh\alpha\sin\beta\sinh\gamma,$$

$$(\cosh\alpha+\cos\beta-\cosh\gamma)\,\frac{\cosh}{\sinh}\,\alpha\,\frac{\cos}{\sin}\,\beta\,\frac{\cosh}{\sinh}\,\gamma,$$

in addition to the terms employed in the previous solutions.

Similar investigations will enable us to determine the induced magnetism in sheets of soft iron, bounded by two confocal paraboloids of the same kind, due to a magnetic field of potential

$$Ax+By+Cz+Pyz+Qzx+Rxy+Sxyz.$$

January 12*th*, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. J. M. Dodds, M.A., Fellow of St. Peter's College, Cambridge; and Mr. G. G. Morrice, M.A., M.B., Trinity College, Cambridge, were elected members; and Mr. E. W. Hobson was admitted into the Society.

The following communications were made:—

The Theory of Distributions: Captain P. A. MacMahon, R.A.

On the Analogues of the Nine-Points Circle in Space of Three Dimensions: S. Roberts, F.R.S.

On a Theorem analogous to Gauss's in Continued Fractions, with applications to Elliptic Functions: L. J. Rogers, M.A.

A Theorem connecting the Divisors of a certain Series of Numbers: Dr. Glaisher, F.R.S.

On Reciprocal Theorems in Dynamics: Prof. H. Lamb, F.R.S.

The following presents were received:—

"Proceedings of the Royal Society," Nos. 259 and 260.

"Educational Times," for January, 1888.

"Nautical Almanac," for 1891.

"Annals of Mathematics," Vol. III., Number 5.

"Bulletin des Sciences Mathématiques," December 1887, and January 1888.

"Annales de l'École Polytechnique de Delft," Tome III., 3me Livr.

"Acta Mathematica," XI., 1.

"Annali di Matematica," Tome XV., Fasc. 3.

"Beiblätter zu den Annalen der Physik und Chemie," Band XI., Stück 11.

"Jahrbuch über die Fortschritte der Mathematik," Band XVII., H. 1.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo I., No. 5.

"Bollettino delle Pubblicazioni Italiane," Nos. 47 and 48.

Pamphlets by Maurice d'Ocagne: "Sur une Classe de Nombres remarquables"; "Sur la Relation entre les Rayons de Courbure de deux Courbes Polaires Réciproques"; "Les Coordonnées parallèles de Points"; "Sur les Courbes Algébriques de degré quelconque"; Quelques Propriétés du Triangle"; "Les Coordonnées Cycliques."

"American Journal of Mathematics," Vol. X., No. 2; Baltimore, January, 1888.