

Finally, I give a table showing the number of types of group for all orders of the form  $p^3q$  less than 400.

Order.	Factors of Order.	Number of Types.
24	$2^3 \cdot 3$	15
40	$2^3 \cdot 5$	14
54	$3^3 \cdot 2$	15
56	$2^3 \cdot 7$	13
88	$2^3 \cdot 11$	12
104	$2^3 \cdot 13$	14
135	$3^3 \cdot 5$	5
136	$2^3 \cdot 17$	15
152	$2^3 \cdot 19$	12
184	$2^3 \cdot 23$	12
189	$3^3 \cdot 7$	12
232	$2^3 \cdot 29$	14
248	$2^3 \cdot 31$	12
250	$5^3 \cdot 2$	15
296	$2^3 \cdot 37$	14
297	$3^3 \cdot 11$	5
328	$2^3 \cdot 41$	15
344	$2^3 \cdot 43$	12
351	$3^3 \cdot 13$	13
375	$5^3 \cdot 3$	7
376	$2^3 \cdot 47$	12

*On the Complete System of Multilinear Differential Covariants of a single Pfaffian Expression, and of a set of Pfaffian Expressions.* By J. BRILL, M.A. Received January 31st, 1899. Read February 9th, 1899. Received in revised form April 5th, 1899.

1. An account of the bilinear covariant of a Pfaffian expression is to be found in Forsyth's *Theory of Differential Equations*, Part I., ch. xi. This covariant involves the first set of Pfaffians belonging to the given expression, and is derived from the said expression by

means of a differential operation. As Forsyth points out, a repetition of this method of derivation upon the covariant itself merely produces an expression which vanishes identically. We can, however, by making use, alternately, of algebraical and differential methods of derivation, produce a series of covariants of the given expression which involve the various orders of derived functions associated with it. The theory of the derivation of these successive covariants is dependent upon a method of successive derivation of the Pfaffians and their allied functions, to which I have called attention in a paper recently published in the *Quarterly Journal of Mathematics*\*.

We shall suppose our given Pfaffian expression to be

$$X_1 dx_1 + X_2 dx_2 + \dots + X_n dx_n.$$

In connexion with it we shall have the first set of Pfaffians

$$[12] = \frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}, \quad [13] = \frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3}, \quad \&c.,$$

connected by a set of relations of the form

$$\frac{\partial}{\partial x_1} [23] - \frac{\partial}{\partial x_2} [13] + \frac{\partial}{\partial x_3} [12] = 0.$$

From the first set of Pfaffians we form the first set of allied functions according to the formulæ

$$[0123] = X_1 [23] - X_2 [13] + X_3 [12], \quad \&c.$$

The second set of Pfaffians may be derived from these by means of differential operations; for we have

$$\begin{aligned} \frac{\partial}{\partial x_1} [0234] - \frac{\partial}{\partial x_2} [0134] + \frac{\partial}{\partial x_3} [0124] - \frac{\partial}{\partial x_4} [0123] \\ = 2 \{ [12] [34] - [13] [24] + [14] [23] \} = 2 [1234]. \end{aligned}$$

The members of the second set of Pfaffians are connected by a set of relations of the type

$$\frac{\partial}{\partial x_1} [2345] - \frac{\partial}{\partial x_2} [1345] + \frac{\partial}{\partial x_3} [1245] - \frac{\partial}{\partial x_4} [1235] + \frac{\partial}{\partial x_5} [1234] = 0,$$

as may readily be proved by taking account of their method of derivation from the first set of allied functions.

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\* "Suggestions towards the Formation of a General Theory of Systems of Pfaffian Equations," Pt. I., Vol. xxx., pp. 221-242.

For the second set of allied functions we have

$$[012345] = X_1[2345] - X_2[1345] + X_3[1245] - X_4[1235] + X_5[1234],$$

&c.

As a type of the method of derivation of the third set of Pfaffians from these, we have

$$\begin{aligned} & \frac{\partial}{\partial x_1} [023456] - \frac{\partial}{\partial x_2} [013456] + \frac{\partial}{\partial x_3} [012456] - \frac{\partial}{\partial x_4} [012356] \\ & \quad + \frac{\partial}{\partial x_5} [012346] - \frac{\partial}{\partial x_6} [012345] \\ = & [12][3456] - [13][2456] + [14][2356] - [15][2346] + [16][2345] \\ & \quad + [23][1456] - [24][1356] + [25][1346] - [26][1345] \\ & \quad + [34][1256] - [35][1246] + [36][1245] + [45][1236] \\ & \quad - [46][1235] + [56][1234] \\ = & 3 \{ [12][34][56] - [12][35][46] + [12][36][45] - [13][24][56] \\ & \quad + [13][25][46] - [13][26][45] + [14][23][56] \\ & \quad - [14][25][36] + [14][26][35] - [15][23][46] \\ & \quad + [15][24][36] - [15][26][34] + [16][23][45] \\ & \quad - [16][24][35] + [16][25][34] \} \\ = & 3[123456]. \end{aligned}$$

These methods of successive derivation are clearly general, a fact which could be rigorously demonstrated with the aid of the method of mathematical induction.

The whole set of derived functions connected with our given Pfaffian expression culminates in a single function, which is either a Pfaffian or an allied function according as the number of independent variables is even or odd.

2. Within the continuum symbolized by our system of independent variables, there will be  $n$  independent displacements of the point  $(x_1, x_2, \dots, x_n)$ . The elements of any other displacement will be expressible linearly in terms of the elements of these. We will suppose the operating symbols  $d_1, d_2, \dots, d_n$  to relate to such a set of displacements, and will write  $d_{23\dots(r+1)}(x_1, x_2, \dots, x_r)$  as an

abbreviation for the differential determinant

$$\begin{vmatrix} d_2 x_1, & d_2 x_2, & \dots, & d_2 x_n \\ d_3 x_1, & d_3 x_2, & \dots, & d_3 x_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ d_{s+1} x_1, & d_{s+1} x_2, & \dots, & d_{s+1} x_n \end{vmatrix},$$

making use of a similar notation to denote all such determinants as can be formed.

We will use the symbol  $U$  as an abbreviation for our Pfaffian expression, affixing to it a subscript identical with that affixed to the  $d$ 's involved in it, *i.e.*, we will write

$$U_r \equiv X_1 d_r x_1 + X_2 d_r x_2 + \dots + X_n d_r x_n.$$

Now we will suppose our expression to be transformed with the aid of a point transformation defined by the equations

$$x'_1 = f_1(x_1, x_2, \dots, x_n), \quad x'_2 = f_2(x_1, x_2, \dots, x_n), \quad \&c.$$

The transformed expression and its derivatives will be indicated by adding dashes to the symbols that denote the corresponding untransformed quantities.

$$\text{Then, since} \quad U_1 = U'_1, \quad U_2 = U'_2,$$

$$\text{we have} \quad d_1 U_2 - d_2 U_1 = d_1 U'_2 - d_2 U'_1.$$

This establishes the existence of the first covariant

$$[12]d_{12}(x_1, x_2) + [13]d_{13}(x_1, x_3) + \dots + [23]d_{23}(x_2, x_3) + \&c.,^*$$

$$\text{since we have} \quad d_1 d_2 \equiv d_2 d_1.$$

We will denote this covariant by the symbol  $U_{12}$ . We now have

$$U_1 = U'_1, \quad U_2 = U'_2, \quad U_3 = U'_3,$$

$$U_{12} = U'_{12}, \quad U_{13} = U'_{13}, \quad U_{23} = U'_{23},$$

and therefore

$$U_1 U_{23} - U_2 U_{13} + U_3 U_{12} = U'_1 U'_{23} - U'_2 U'_{13} + U'_3 U'_{12}.$$

This gives rise to the second covariant, which we will denote by the symbol  $U_{123}$ . Written in full it will be of the form

$$[0123]d_{123}(x_1, x_2, x_3) + [0124]d_{123}(x_1, x_2, x_4) + \dots \\ \dots + [0234]d_{123}(x_2, x_3, x_4) + \&c.,$$

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\* The number of terms in this expression is

$$\frac{1}{2}n(n-1).$$

It is to be noted that, throughout this paper, the figures within the square brackets are retained in their numerical order. This expression is easily reconciled with that given by Forsyth, by remembering that

$$[2'] = -[12], \quad [31] = -[13], \quad \&c.$$

and will therefore involve the first set of allied functions in a similar manner to that in which the first set of Pfaffians is involved in  $U_{11}$ .

To obtain the third covariant we again have recourse to a differential operation. Its existence is demonstrated by the equation

$$d_1 U_{234} - d_2 U_{134} + d_3 U_{124} - d_4 U_{123} = d_1 U'_{234} - d_2 U'_{134} + d_3 U'_{124} - d_4 U'_{123}.$$

We will denote this covariant by the symbol  $U_{1234}$ , writing

$$d_1 U_{234} - d_2 U_{134} + d_3 U_{124} - d_4 U_{123} = 2U_{1234}.$$

We then have

$$U_{1234} = [1234] d_{1234}(x_1, x_2, x_3, x_4) + \dots + [pqrs] d_{1234}(x_p, x_q, x_r, x_s) + \&c.*$$

We will denote the fourth covariant by the symbol  $U_{12345}$ , defining it by the equation

$$U_1 U_{2345} - U_2 U_{1345} + U_3 U_{1245} - U_4 U_{1235} + U_5 U_{1234} = U_{12345}.$$

The fifth covariant will be defined by the equation

$$d_1 U_{23456} - d_2 U_{13456} + d_3 U_{12456} - d_4 U_{12356} + d_5 U_{12346} - d_6 U_{12345} = 3U_{123456}.$$

Proceeding in this manner we may obtain the whole set of covariants. Their number will be finite, and they will culminate in an expression involving a single differential determinant, viz.,

$$d_{123\dots n}(x_1, x_2, \dots, x_n).$$

The coefficient of this determinant will be the final derived function of the original Pfaffian expression.

From these covariants we may readily deduce a set of formulæ expressing the derived functions of the transformed expression in terms of the derived functions of the same order belonging to the original expression. These formulæ will involve the latter quantities linearly. They may be obtained with the aid of the known formula for transforming differential determinants.† As a special example we may instance the formula

$$[pq]' = [12] \frac{\partial(x_1, x_2)}{\partial(x'_p, x'_q)} + [13] \frac{\partial(x_1, x_3)}{\partial(x'_p, x'_q)} + \dots + [23] \frac{\partial(x_2, x_3)}{\partial(x'_p, x'_q)} + \&c.‡$$

\* It is readily verified that all differentials of the second order disappear from the result.

† See Donkin: "On a Class of Differential Equations, including those which occur in Dynamical Problems," *Phil. Trans.* 1854, p. 73.

‡ This result is easily obtained by applying the transformation to the left-hand side of the equation

$$U_{12} = U'_{12},$$

and then equating the coefficients of the differential determinants. It is to be noted that, although the number of independent displacements is finite, yet a suitable set of them may be chosen in an indefinite number of ways.

3. In the paper of mine to which I have referred in Article 1, I have developed for the case of several Pfaffian expressions a set of derived functions analogous to the Pfaffians and allied functions belonging to a single expression. I have shown that the vanishing of the various sets of these, taken in a reverse order, is a necessary consequence of a diminution in the number of the functions which constitute the integral system of the set of equations formed by equating the Pfaffian expressions severally to zero. I have not as yet proved the converse theorem.

These new expressions will be involved in the covariants of the system of Pfaffians in a similar manner to that in which the derived functions of a single Pfaffian expression are involved in its covariants. I shall here confine myself to briefly sketching out the method of derivation of these covariants.

Suppose, in the first place, that we have the two Pfaffian expressions

$$U^{(1)} \equiv X_{11}dx_1 + X_{12}dx_2 + \dots + X_{1n}dx_n,$$

$$U^{(2)} \equiv X_{21}dx_1 + X_{22}dx_2 + \dots + X_{2n}dx_n.$$

Each of these expressions will have a set of covariants of its own,

viz., 
$$U_{12}^{(1)}, U_{123}^{(1)}, U_{1234}^{(1)}, \dots,$$

and 
$$U_{12}^{(2)}, U_{123}^{(2)}, U_{1234}^{(2)}, \dots$$

In addition to these there will be a set of covariants belonging to the two expressions simultaneously. The first of these is obtained by means of a purely algebraical operation. We have, in fact,

$$U_1^{(1)}U_2^{(2)} - U_2^{(1)}U_1^{(2)} = (U')_1^{(1)}(U')_2^{(2)} - (U')_2^{(1)}(U')_1^{(2)}.$$

We will write  $V_{12}$  for this first covariant, so that we have

$$U_1^{(1)}U_2^{(2)} - U_2^{(1)}U_1^{(2)} = V_{12}.$$

The second covariant  $V_{123}$  is derived from this by means of a differential operation, and is defined by the equation

$$d_1V_{23} - d_2V_{13} + d_3V_{12} = V_{123}.$$

From this two new covariants may be formed by means of algebraical operations. They are

$$U_1^{(1)}V_{23} - U_2^{(1)}V_{13} + U_3^{(1)}V_{12}$$

and 
$$U_1^{(2)}V_{23} - U_2^{(2)}V_{13} + U_3^{(2)}V_{12}.$$

For convenience we will denote these by the symbols

$$V_{123}^{(1)} \quad \text{and} \quad V_{123}^{(2)}.$$

From these, a new covariant  $V_{1234}$  may be formed by means of algebraical operations. It is defined by either of the expressions

$$U_1^{(1)} V_{234}^{(2)} - U_2^{(1)} V_{134}^{(2)} + U_3^{(1)} V_{124}^{(2)} - U_4^{(1)} V_{123}^{(2)}$$

and  $-\{U_1^{(2)} V_{234}^{(1)} - U_2^{(2)} V_{134}^{(1)} + U_3^{(2)} V_{124}^{(1)} - U_4^{(2)} V_{123}^{(1)}\}.$

It may also be written in the form

$$\{U_1^{(1)} U_2^{(2)} - U_2^{(1)} U_1^{(2)}\} V_{34} - \{U_1^{(1)} U_3^{(2)} - U_3^{(1)} U_1^{(2)}\} V_{24} + \&c.$$

This takes us as far as we can go with the aid of algebraical operations, as another step would produce an evanescent expression. We must, therefore, again have recourse to differential operations, and we deduce a new covariant  $V_{12345}$ , defined by the equation

$$d_1 V_{2345} - d_2 V_{1345} + d_3 V_{1245} - d_4 V_{1235} + d_5 V_{1234} = 2V_{12345}.$$

Treating this covariant in a similar manner to that in which we treated  $V_{12}$ , we may form two new covariants, and again a single one, from either of these. We shall then have to introduce differential operations again. Thus we see that the different methods of derivation will repeat themselves in cycles.

4. As a further instance we will take the case of the three Pfaffian expressions

$$U^{(1)} \equiv X_{11} dx_1 + X_{12} dx_2 + \dots + X_{1n} dx_n,$$

$$U^{(2)} \equiv X_{21} dx_1 + X_{22} dx_2 + \dots + X_{2n} dx_n,$$

$$U^{(3)} \equiv X_{31} dx_1 + X_{32} dx_2 + \dots + X_{3n} dx_n.$$

There will, of course, be the sets of invariants belonging to each expression separately, and to each pair combined. In addition, there will be a set belonging to all three combined. The first of this set,  $W_{123}$ , is defined by the equation

$$W_{123} = \begin{vmatrix} U_1^{(1)} & U_2^{(1)} & U_3^{(1)} \\ U_1^{(2)} & U_2^{(2)} & U_3^{(2)} \\ U_1^{(3)} & U_2^{(3)} & U_3^{(3)} \end{vmatrix}.$$

The next,  $W_{1234}$ , is derived from this by means of a differential operation, thus:

$$W_{1234} = d_1 W_{234} - d_2 W_{134} + d_3 W_{124} - d_4 W_{123}.$$

From this we may form three others,

$$W_{12345}^{(1)}, \quad W_{12345}^{(2)}, \quad W_{12345}^{(3)},$$

which are defined by means of the expressions

$$\begin{aligned} U_1^{(1)} W_{2345} - U_2^{(1)} W_{1345} + U_3^{(1)} W_{1245} - U_4^{(1)} W_{1235} + U_5^{(1)} W_{1234}, \\ U_1^{(2)} W_{2345} - U_2^{(2)} W_{1345} + U_3^{(2)} W_{1245} - U_4^{(2)} W_{1235} + U_5^{(2)} W_{1234}, \\ U_1^{(3)} W_{2345} - U_2^{(3)} W_{1345} + U_3^{(3)} W_{1245} - U_4^{(3)} W_{1235} + U_5^{(3)} W_{1234}. \end{aligned}$$

From each of these we may form two covariants, but there will in reality only be three new ones, which may be defined as follows:—

$$W_{123456}^{(23)} = \begin{vmatrix} U_1^{(2)} & U_2^{(2)} \\ U_1^{(3)} & U_2^{(3)} \end{vmatrix} W_{3456} - \begin{vmatrix} U_1^{(2)} & U_3^{(2)} \\ U_1^{(3)} & U_3^{(3)} \end{vmatrix} W_{2456} + \&c.$$

In the determinants contained in this expression we shall have for subscripts every combination of two out of the numbers 1, 2, 3, 4, 5, 6. For the subscript attached to the  $W$ , multiplying any determinant, we have those numbers out of the set 1, 2, 3, 4, 5, 6 which do not occur as subscripts within the corresponding determinant. With regard to the sign of each product, we have first to write down the subscripts of the symbols within the determinant in the order in which they occur, and then to write after them the numbers in the subscript of the corresponding  $W$ ; then, according as the number of displacements required to bring these numbers into their proper numerical order is even or odd, so will the required sign be plus or minus. In a similar manner, we may write down the expressions for

$$W_{123456}^{(13)} \quad \text{and} \quad W_{123456}^{(12)}.$$

Only one more covariant can be formed from these by algebraical methods. It may be derived from any one of the three, and the expression defining it is of the form

$$\Sigma \begin{vmatrix} U_1^{(1)} & U_2^{(1)} & U_3^{(1)} \\ U_1^{(2)} & U_2^{(2)} & U_3^{(2)} \\ U_1^{(3)} & U_2^{(3)} & U_3^{(3)} \end{vmatrix} W_{4567}.$$

The law for determining the signs of the various products under the  $\Sigma$  will be the same as in the case of the covariants immediately preceding.

We must now again have recourse to a differential method of derivation; and it is evident that, in this case, the different methods of derivation will repeat themselves in triple cycles.

5. The cases we have discussed are sufficient to illustrate the general method of derivation, the progress of which is perfectly

clear. It will be noted that the places at which differential operations occur are those which mark the passing from one group of cases into the next in the case of a set of equations obtained by equating our Pfaffian expressions severally to zero. Further, it will be found that the more general derived functions that I have introduced play a similar part, in reference to these latter covariants, to that which the derived functions of a single expression play in reference to its covariants. It will also be noticed that the transformation theory applies. One of the main difficulties in working out general proofs of propositions in this subject is the extraordinary complication of the notation. The chief desideratum is the invention of a notation to express the general type of derived functions as convenient as that introduced by Cayley to denote the derived functions of a single expression.

*Note on a Case of Divisibility of a Function of Two Variables by another Function.* By ARTHUR BERRY, Fellow of King's College, Cambridge. Received and read February 9th, 1899.

### § 1.

If  $f = 0$ ,  $\phi = 0$  are the equations, expressed in Cartesian coordinates, of two given algebraic curves, which have both simple and multiple points of intersection, and if  $\psi = 0$  is the equation of a third curve passing through all these points, and satisfying certain further conditions at the multiple points of intersection, then we have the identity

$$\psi \equiv Af + B\phi$$

where  $A$ ,  $B$  are polynomials in the coordinates  $x$ ,  $y$ . The above mentioned conditions were first stated, and the theorem rigorously proved, by Noether.\* A simpler proof of Noether's theorem was soon afterwards published by Halphen.† Dr. F. S. Macaulay has

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\* "Ueber einen Satz aus der Theorie der algebraischen Functionen," *Mathematische Annalen*, Vol. vi. (1873).

† "Sur une proposition d'Algèbre," *Bulletin de la Société Mathématique de France*, Vol. v. (1877); the article is reproduced in Benoist's French translation of Clebsch's *Vorlesungen über Geometrie*.