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THEORY OF FUNCTIONS.

A Treatise on the Theory of Functions. By James Harkness, M.A., Associate Professor of Mathematics, Bryn Mawr College, Pa., and Frank Morley, M.A., Professor of Pure Mathematics in Haverford College, Pa. (London and New York: Macmillan and Co., 1893.)

IF evidence were wanted of the recent progress of the study of pure mathematics on English and American soil, none better could be furnished than the appearance on the two sides of the Atlantic, within a short interval, of two important works on the theory of functions of a complex variable. But a few years ago this great modern branch of mathematics was so little known to English-speaking mathematicians that scarcely a trace of its influence could be traced in their writings, and the majority of our text-books were disfigured by incompleteness, and not seldom by positive error arising from ignorance of its principles. Now the English reader has at his disposal two extensive works dealing with the fundamental principles of the theory from all the more important points of view; and also a very useful aid in Cathcart's valuable translation of Harnack's "Elements of the Differential and Integral Calculus." Probably nothing could serve better as an exorcist of the spirit of formalism which has oppressed the English school of mathematicians so heavily, in spite of all the great things that its leaders have done for the science, than the study of the theory of functions. In no other mathematical discipline is the fundamental unity of logic kept so constantly before the student; nowhere else in mathematics is it so clearly made evident that the manifold array of symbols is the clothing, and not the soul of mathematical thought; and nowhere else can we perceive so fully that progress is to be looked for mainly in strengthening our hold upon elementary conceptions, in continual refinement of definition and continual increase of stringency in inference, together with the necessary complement of this, viz. a continual widening of our power of imagining logical possibilities.[1] A single illustration of these general remarks may be cited here, viz. the important part now played in mathematics by the classification of the possible singularities of a function. Although as yet this classification has hardly proceeded beyond the first stage of distinguishing between what Weierstrass has called essential and non-essential singularities, yet the exceeding fruitfulness of the idea is very manifest in every part, not only of the theory itself, but of its applications. In this connection we may remark that anyone who is sceptical as to the value of function-theory, should compare the treatment of the theory of elliptic functions as given in chapter vii. of the treatise now before us, with the older method of dealing with the same subject. He will there find the theorems which used to be for many of us a mere savagery of riotous mathematical formulæ, sitting now clothed in their right minds—the cultured dependents of a few leading ideas.

Our first impulse, after dipping here and there into the work of Messrs. Harkness and Morley, and recognising its substantial character, was to regret that so much learning and ability had been wasted in a field already covered by the admirable treatise of Forsyth. A more careful reading convinced us that this feeling was a mistake. The subject is wide enough to allow of two independent treatises; and the two works are independent so far as two mathematical works, each partly historical, dealing with the same subject, can be. Like Forsyth, Harkness and Morley are full of valuable references, not only to the great writers and the great memoirs on the subject, but also to the minor writers and to memoirs dealing with points of detail. So much is this the case, that we doubt whether in the matter of history and references the continental student has anything to equal, and certainly he has nothing to surpass, what the English student now possesses in Forsyth, combined with Harkness and Morley.

The more recent work does not, it is true, rival Forsyth in style and width of view. It is constructed more nearly on the model of a continental treatise, not reaching the airy elegance of a French work, but happily avoiding the intolerable prolixity and dulness of too many continental books, where a parade of generality not unfrequently engenders obscurity, or covers a poverty of fruitful ideas. It is inseparable from the nature of the subject that the unskilled reader should at times find passages that seem obscure. In such cases he will find it of great advantage to turn from Forsyth to Harkness and Morley, or from Harkness and Morley to Forsyth. The greater detail in some of the demonstrations in certain parts of the subject which characterises the treatise before us will often be a help to the reader who has run aground in Forsyth. A mere remark which constitutes a full demonstration to a mind properly prepared or naturally sufficiently nimble to receive it, often proves an enigma to another mind not so well "disposed," or, what is worse, is taken after the manner of the patient who, instead of taking his doctor's medicine, swallowed the prescription. If we might advise the beginner, we should say, first read Forsyth rapidly, possibly superficially with judicious omission, in order to get a good idea of the nature and aims of the theory; then proceed to work carefully through Harkness and Morley: and, finally, again read Forsyth carefully; so that the last impressions should be of the "poetry of the subject."

Chapter i. of Harkness and Morley's work is a very elegant and valuable geometric introduction to the subject, containing, besides the usual matter, a number of excellent graphical illustrations of the theory of invariants by means of Argand's diagram. Chapter ii. gives an account of the more recent refinements in the theory of functions of a real variable, in so far as such are necessary for the purpose in hand. In chapter iii. the theory of infinite series is dealt with in sufficient detail, and the reader is thus rapidly introduced to Weierstrass's theory of the analytic function, its continuation, its singular points and lacunary spaces. Chapter iv. deals specially with the algebraic function, its zeros, poles, and branch

[1] It is in this particular that the peculiar originality of Cauchy, Riemann, and Weierstrass, the three great leaders in the theory of functions, has been so conspicuous.

points, the expansions which represent it in the neighbourhood of ordinary and singular points, its cycles, &c. Chapter v., on integration, introduces the fundamental theorems of Cauchy, with their applications to the establishment of the theorems of Weierstrass and Mittag-Leffler regarding the general expressions for functions with assigned singularities. In chapters vi. and ix. we have the substance of Riemann's theory, both direct and inverse. The account of the inverse theory consists largely of an exposition of Schwarz's solution of Dirichlet's problem, on which depends the proof of the existence of "functions of position" on a given Riemannian surface. The applications of the theory are amply illustrated in chapter vii., which contains an admirable sketch, already alluded to, of the Weierstrassian theory of doubly periodic functions; and in chapters viii. and x. on double theta-functions and Abelian integrals.

From this enumeration our mathematical readers will see that Messrs. Harkness and Morley have provided for them an ample and varied bill of fare; and we have no hesitation in saying that the feast is worthy of the bill. We would merely express, in conclusion, our desire to see this pair of authors soon abroad again in another of the many fields that still await the conscientious writer of English mathematical text-books. G. CH.

THE CONSTRUCTION OF DRUM ARMATURES AND COMMUTATORS.

Drum Armatures and Commutators. By F. M. Weymouth. (London: The *Electrician* Printing and Publishing Co., 1893.)

IN the preface to this book we are told that it is intended as "a useful guide or introduction to those who may ultimately wish to proceed with the mathematical treatment of the subjects," and further, that "the beginner will read these pages during the early period of his training, while he is studying his mathematics, and so may combine the two together at a later and more advanced stage." To such this work can be recommended, for the author has collected a good deal of information, which is well illustrated by woodcuts, showing how different makers have built up their armatures and commutators, thus giving the student a variety of experience in this direction.

In the first three chapters the drum armature is discussed from a general point of view. It is contrasted with that of Gramme, and the generation of electromotive force explained. The distinctive difference between "electromotive force" and "potential difference" might have been at this stage (p. 9) pointed out with greater clearness. For instance, in a direct current dynamo when working on open circuit the "electromotive force" of the machine and the "potential difference" at the brushes are the same in magnitude if no current flows through the armature. But when giving current to the external circuit between the brushes, a difference at once steps in, the "electromotive force" being greater than the "potential difference" by an amount represented by the current into the ohmic resistance of the armature.

In chapter iii. the winding of armatures for heavy currents is discussed generally; then follow some notes on balancing armatures properly.

With regard to an effect of current in the armature, it is stated on p. 30 that when "the field is bored concentric with the axis of the armature, Foucault currents arise principally, if not entirely, when the bars pass under the trailing horns of the pole-pieces, where the induction lines are particularly dense. By 'trailing' horn is meant the last horn of a pole-piece which the bars leave or recede from as they revolve." This statement is not sufficient. Take the case of a shunt-wound motor (of the ordinary type) when loaded and working with a negative lead. Here it is at the "trailing" horn that the induction per unit area of the polar-surface is *less* dense than at any other part of the surface.[1] It should also be impressed upon the beginner that it is the "loading" of generator or motor which brings about this disturbance.

Six chapters (iv.–ix.) on the details of drum armatures for heavy currents, specially with reference to the end-connections, follow. These have been carefully compiled, and it must be said that they give a good insight into the construction of drum armatures. In the first of this series of chapters the prevention of Foucault currents is dealt with. With regard to making the air-space longer near the horns in order to remove the *cause* of Foucault currents, a word could be added. In what is generally termed the "inverted horse-shoe" type of machine, so largely used at the present time, the pull upon the armature due to magnetism is in an upward direction, and with concentric fields outbalances the weight of the armature, thus causing a considerable pressure on the upper brasses of the bearings. When such a dynamo is direct-coupled to a steam-engine which works with a constant *downward* thrust, serious stresses are brought into play by these opposite forces acting at different points on the shaft. In such cases the widening of the air spaces near the two top horn pieces is usually resorted to, to relieve the pull on armature due to magnetism.

After describing the Edison "plate" end connection in detail, we come, in chapter vi., to the "evolute" end connection, which is described firstly in connection with bars cranked radially towards the shaft. Then follows a description of "evolutes," in which the cranked bar is dispensed with entirely. With regard to this latter, it is unfortunate that the author has not given details of the "Siemens" bar armature, which would have added to the value of the work. He of course recognises Von Hefner Alteneck as the inventor of evolute end-connections.

Eickemeyer's evolute wire-winding, Kapp's helical end-connection, and Swinburne's chord-winding are described in great detail. Chapter ix. treats of the Parson's helical outside end-winding, which is specially interesting on account of the enormous speed at which these armatures rotate. A description of Fritsche's winding is also given.

The subject of commutators claims chapters x.–xiii. In the introductory chapter (x.) "end play" in the bearings is mentioned as tending to more even wear of the surface of commutators. In this connection the author does not mention the "Halpin" gear, which has been introduced for the purpose of automatically moving the brushes longitudinally backward and forward on the

[1] See *Proceedings* of the Royal Society, vol. li. p. 49.