MATHEMATICAL NOTES.

143. [D. 2. a.] *A set of criteria for convergency or divergency of series of positive terms.*

It is common to regard I. and the first part of III. of the following more complete list as giving the fullest information on the subject which is afforded without the application of special methods, based on condensation, separation or rearrangement of terms, to particular series. There is, however, a natural reluctance to accept as final anything but a full reference to all possibilities as to the value of $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}$. There is besides a discontent with the inequality of allusion to convergency and divergency: as to convergency, I. gives an absolute test, free from any use of knowledge previously acquired, whereas III. gives only a test as to divergency which presupposes the possession of information about the divergency of $\Sigma\dfrac{1}{a_n}$. The most commonly used scale of included tests is obtained by taking for a_n in turn 1, n, $n\log n$, $n\log n\log(\log n)$, ... ; and III. enables us to say that certain series examined are divergent because $\Sigma\dfrac{1}{n}$, $\Sigma\dfrac{1}{n\log n}$, ... are divergent. But V. does not use these facts as already known. It is readily applied to prove them.

In the following, α, A, β, B mean assignable (and therefore finite) positive numerical constants, independent of n. $\Sigma u_n \equiv u_1+u_2+u_3+\dots$ is a series of positive terms, and a_1, a_2, a_3, ... a sequence of positive quantities (a_n dependent on n) assigned in any way at convenience.

I. *If from some value of n onwards $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}>\alpha>0$, then Σu_n is convergent.*

II. *If from some value of n onwards $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}>0$, whether tending to limit 0 or not, then Σu_n is convergent in case $\Sigma\dfrac{1}{a_n}$ is. In case, however, $\Sigma\dfrac{1}{a_n}$ is divergent, Σu_n is divergent provided $a_nu_n>\beta>0$.*

III. *If from some value of n onwards $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}<0$, then Σu_n is divergent in case $\Sigma\dfrac{1}{a_n}$ is. In case, however, $\Sigma\dfrac{1}{a_n}$ is convergent, Σu_n is convergent provided $a_nu_n<B<\infty$.*

IV. *If from some value of n onwards*

$$a_n\frac{u_n}{u_{n+1}}-a_{n+1}<-\alpha<0, \text{ and } a_nu_n<B<\infty,$$

then Σu_n is convergent.

V. *If from some value of n onwards*

$$a_n\frac{u_n}{u_{n+1}}-a_{n+1}<0, \text{ but } >-A>-\infty,$$

and if a_nu_n tends to infinity with n, then Σu_n is divergent.

Cases in which these criteria give no information are:

(1) when $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}$ has not always the same sign for large values of n;

(2) when $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}>0$ but has limit 0, and there is doubt as to the convergency or divergency of $\Sigma\dfrac{1}{a_n}$;

(3) when $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}>0$ but has limit 0, and a_nu_n has limit 0, and $\Sigma\dfrac{1}{a_n}$ is divergent;

(4) when $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}<0$ but has limit 0, and $a_nu_n<B<\infty$, and there is doubt as to the convergency or divergency of $\Sigma\dfrac{1}{a_n}$;

(5) when $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}$ tends to $-\infty$, and a_nu_n tends to infinity, and $\Sigma\dfrac{1}{a_n}$ is convergent or doubtful.

The overlapping of certain criteria gives the following information:

(i) If $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}>\alpha>0$, and $a_nu_n>\beta>0$, then $\Sigma\dfrac{1}{a_n}$ is convergent;

(ii) If $a_n\dfrac{u_n}{u_{n+1}}-a_{n+1}<-\alpha<0$, and $a_nu_n<B<\infty$, then $\Sigma\dfrac{1}{a_n}$ is convergent.

Of the five main criteria, II. and III. are only tests by comparison. They express in other terms, taking $\dfrac{1}{a_n}$ for c_n or d_n, that, if Σc_n is a convergent and Σd_n a divergent series of positive terms, Σu_n is convergent if from some n onwards the ratio $\dfrac{u_n}{c_n}$ never surpasses an assignable numerical magnitude, and on the other hand is divergent if $\dfrac{u_n}{d_n}$ never falls below an assignable magnitude greater than zero.

The establishment of the other tests, which have an absolute character, depends secondarily on these laws of comparison but primarily on the one fundamental fact as to convergency and divergency that $\Sigma(v_n-v_{n+1})$ is convergent or divergent according as the sequence $v_1, v_2, v_3, \ldots v_n, \ldots$ does or does not tend to a limit as n increases. If, for instance (to take IV. and V.), v_n-v_{n+1} is always negative from $n=r$ onwards, so is v_r-v_{n+1} by addition. No term of the sequence $v_{r+1}, v_{r+2}, \ldots$ is so small as v_r. Also the terms of the sequence increase as we go on in it, since every v_n-v_{n+1} is negative. If all are less than some B, there must be a definite quantity not greater than B which separates those quantities between v_r and B which v_n can be made to exceed, by increasing n sufficiently, from those which it cannot; and the sequence $v_1, v_2, v_3, \ldots$ tends to this limit. If no such B can be assigned it tends to no limit. Thus, taking a_nu_n for v_n, there is convergency of $\Sigma(a_nu_n-a_{n+1}u_{n+1})$ under the circumstances of IV. but not under those of V. Applying then the laws of comparison to the negative of this series, regarded as Σc_n or Σd_n, and the series Σu_{n+1}, the assertions of IV. and V. follow.

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144. [P. 3. b. a.] *To find the relation between two maps of the same contour on the stereographic projection.*

Let P' be any point of the curve σ lying on a sphere whose centre is O. Let α, β be the projections of σ from any two points A, B of the sphere on to the planes a, b through O parallel to the tangent planes at A and B. Let C be the projection of B from A on a. Rotate b about the intersection l of a and b through an angle AOB till it coincides with a. Then β is derived