Relations between the Divisors of the First n Numbers.

By J. W. L. Glaisher, Sc.D., F.R.S.

[*Read May 14th,* 1891.]

Introduction. § 1.

1. The present paper contains various generalizations of the recurring formula

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + 9\sigma(n-10) - \&c. = 0,^*$$

in which $\sigma(n)$ denotes the sum of the divisors of the number n, and $\sigma(0)$, when it occurs, is to have the value $\frac{1}{3}n$.

It will be seen that the theorems in their most general form relate to the actual divisors themselves, and not necessarily to their sums or other numbers obtained from them by any method of combination. We may, however, deduce various theorems of the same kind as the one quoted above by so combining them; and numerous examples of results obtained in this manner occur in the paper. These formulæ relate not only to the sums of the divisors, but also to the sums of their m^{th} powers, m being any uneven number.†

Notation. § 2.

2. Let $G_n\{\phi(d), \psi(d), \chi(d), \ldots\}$ denote the group of numbers

$$\begin{array}{l} \phi(d_1),\ \phi(d_2),\ \phi(d_3),\ \ldots,\ \phi(d_f), \\ \psi(d_1),\ \psi(d_2),\ \psi(d_3),\ \ldots,\ \psi(d_f), \\ \chi(d_1),\ \chi(d_2),\ \chi(d_3),\ \ldots,\ \chi(d_f), \\ \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \end{array}$$

---

* *Quarterly Journal of Mathematics*, Vol. xix., p. 220, and *Proc. Camb. Phil. Soc.*, Vol. v., p. 109.

† The general theorems in §§ 3, 11, 29, 46, 58 were obtained in December, 1887, and the first of the theorems relating to the actual divisors (§ 3), with the derived numerical theorem (§ 11), was brought before the Society at its meeting on January 12th, 1888 ("A Theorem connecting the Divisors of a certain Series of Numbers," Vol. xix., p. 143); but no paper was prepared for publication at that time. I did not return to the subject till December, 1890; since then I have worked out the details and made considerable additions, forming the present paper. A few short notes containing investigations arising out of the subject, and which have been published elsewhere, are referred to in the notes to §§ 21, 43, 53.

where $d_1, d_2, d_3, \ldots, d_r$ are all the divisors of the number n (which, it will be observed, occurs as the suffix of the letter G). The numbers 1 and n are to be included among the divisors; and it is supposed that

$$-G_n\{\phi(d), \psi(d), \chi(d), \ldots\},$$

is the same as $\quad G_n\{-\phi(d), \ -\psi(d), \ -\chi(d), \ldots\}$.

General Theorem relating to the Actual Divisors of the Numbers $n, n-1, n-3,$ *&c.* §§ 3–8.

3. Using the above notation, the general theorem may be expressed as follows:—The numbers given by the formula

$$G_n(d) - G_{n-1}(d, d\pm 1) + G_{n-3}(d, d\pm 1, d\pm 2)$$
$$- G_{n-6}(d, d\pm 1, d\pm 2, d\pm 3) + \&c.$$

all cancel each other, if n is not a triangular number, and reduce to

$$(-1)^{g-1} \text{ (one 1, two 2's, three 3's, ... } g\ g\text{'s)},$$

if n is the g^{th} triangular number $\frac{1}{2}g(g+1)$.

4. For example, putting $n = 9$, which is not a triangular number, the theorem asserts that the numbers given by the formula

$$G_9(d) - G_8(d, d\pm 1) + G_6(d, d\pm 1, d\pm 2) - G_3(d, d\pm 1, d\pm 2, d\pm 3)$$

all cancel each other.

Writing the numbers $d+1$, $d+2$, &c., in lines above the divisors d, and the numbers $d-1$, $d-2$, &c., in lines below, so that the central line consists of the divisors of the numbers 9, 8, 6, 3, *i.e.*, of the numbers 1, 3, 9; 1, 2, 4, 8; 1, 2, 3, 6; 1, 3, the formula becomes

$$\{1, 3, 9\} - \begin{Bmatrix} 2, 3, 5, 9 \\ 1, 2, 4, 8 \\ 0, 1, 3, 7 \end{Bmatrix} + \begin{Bmatrix} 3, 4, 5, 8 \\ 2, 3, 4, 7 \\ 1, 2, 3, 6 \\ 0, 1, 2, 5 \\ -1, 0, 1, 4 \end{Bmatrix} - \begin{Bmatrix} 4, 6 \\ 3, 5 \\ 2, 4 \\ 1, 3 \\ 0, 2 \\ -1, 1 \\ -2, 0 \end{Bmatrix};$$

or, changing the signs of the numbers occurring in the groups to which the negative sign is prefixed,

$$\begin{array}{rrrr}
 & & & -4,\ -6, \\
 & & 3,\ 4,\ 5,\ 8, & -3,\ -5, \\
 & -2,\ -3,\ -5,\ -9, & 2,\ 3,\ 4,\ 7, & -2,\ -4, \\
1,\ 3,\ 9, & -1,\ -2,\ -4,\ -8, & 1,\ 2,\ 3,\ 6, & -1,\ -3, \\
 & 0,\ -1,\ -3,\ -7, & 0,\ 1,\ 2,\ 5, & 0,\ -2, \\
 & & -1,\ 0,\ 1,\ 4, & 1,\ -1, \\
 & & & 2,\ \ 0, \\
 & & & \ldots
\end{array}$$

and it is easy to verify that these numbers exactly cancel each other, *i.e.*, there are five 1's and five -1's, four 2's and four -2's, four 3's and four -3's, three 4's and three -4's, two 5's and two -5's, one 6 and one -6, one 7 and one -7, one 8 and one -8, one 9 and one -9. The zeros are retained for the sake of regularity in writing the numbers, but no account is to be taken of them.

As a second example, putting $n = 10$, which is the fourth triangular number, so that $g = 4$, the theorem asserts that the numbers given by the formula

$$G_{10}(d) - G_9(d,\ d\pm 1) + G_7(d,\ d\pm 1,\ d\pm 2) - G_4(d,\ d\pm 1,\ d\pm 2,\ d\pm 3)$$

all cancel each other, excepting only

$$-1,\ -2,\ -2,\ -3,\ -3,\ -3,\ -4,\ -4,\ -4,\ -4.$$

The divisors of 10, 9, 7, 4 are 1, 2, 5, 10; 1, 3, 9; 1, 7; 1, 2, 4 respectively, so that the numbers given by the formula are

$$\{1,\ 2,\ 5,\ 10\} - \left\{\begin{array}{rrr} 2, & 4, & 10 \\ 1, & 3, & 9 \\ 0, & 2, & 8 \end{array}\right\} + \left\{\begin{array}{rr} 3, & 9 \\ 2, & 8 \\ 1, & 7 \\ 0, & 6 \\ -1, & 5 \end{array}\right\} - \left\{\begin{array}{rrr} 4, & 5, & 7 \\ 3, & 4, & 6 \\ 2, & 3, & 5 \\ 1, & 2, & 4 \\ 0, & 1, & 3 \\ -1, & 0, & 2 \\ -2, & -1, & 1 \end{array}\right\};$$

that is

$$\begin{array}{rrrrrrrrrrrr}
 & & & & & & & & & -4, & -5, & -7, \\
 & & & & & & & 3, & 9, & -3, & -4, & -6, \\
 & & & & -2, & -4, & -10, & 2, & 8, & -2, & -3, & -5, \\
1, & 2, & 5, & 10, & -1, & -3, & -9, & 1, & 7, & -1, & -2, & -4, \\
 & & & & 0, & -2, & -8, & 0, & 6, & 0, & -1, & -3, \\
 & & & & & & & -1, & 5, & 1, & 0, & -2, \\
 & & & & & & & & & 2, & 1, & -1,
\end{array}$$

all of which cancel each other, excepting only one -1, two -2's, three -3's, and four -4's.

5. It is evident that we may express the theorem also in the form:—If $\alpha, \beta, \ldots,$ be the divisors of n; $\alpha_1, \beta_1, \ldots,$ those of $n-1$; $\alpha_3, \beta_3, \ldots,$ those of $n-3$; $\alpha_6, \beta_6, \ldots$ of $n-6$, and so on, then the numbers

$$\{\alpha, \beta, \ldots\} - \left\{\begin{matrix} \alpha_1+1, & \beta_1+1, & \ldots \\ \alpha_1, & \beta_1, & \ldots \\ \alpha_1-1, & \beta_1-1, & \ldots \end{matrix}\right\} + \left\{\begin{matrix} \alpha_3+2, & \beta_3+2, & \ldots \\ \alpha_3+1, & \beta_3+1, & \ldots \\ \alpha_3, & \beta_3, & \ldots \\ \alpha_3-1, & \beta_3-1, & \ldots \\ \alpha_3-2, & \beta_3-2, & \ldots \end{matrix}\right\} - \&c.$$

cancel each other in all cases if we make a certain convention with respect to the group having the central line $\alpha_0, \beta_0, \ldots,$ *i.e.*, if we suppose that this group, which can only occur when $n = \frac{1}{2}g(g+1)$, and which would be written

$$\left\{\begin{matrix} \alpha_0+g, & \beta_0+g, & \ldots \\ \ldots & \ldots & \ldots \\ \alpha_0+1, & \beta_0+1, & \ldots \\ \alpha_0, & \beta_0, & \ldots \\ \alpha_0-1, & \beta_0-1, & \ldots \\ \ldots & \ldots & \ldots \\ \alpha_0-g, & \beta_0-g, & \ldots \end{matrix}\right\},$$

is to be conventionally replaced by the group of numbers

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ \ldots,\ g,\ g,\ g\ \ldots\ (g \text{ times}).$$

6. To obtain the system of numbers given by the formula, we first write down the divisors of n, $n-1$, $n-3$, $n-6$, &c. (forming the

central line in the above scheme). Commencing with the divisors of $n-1$, we next write down the numbers derived from them by increasing and diminishing each divisor by 1 (forming the lines next above and next below the central line). Commencing with the divisors of $n-3$, we next write down the numbers derived from the divisors in the central line by increasing and diminishing them by 2 (forming the next lines above and below); and so on. We then change the signs of the divisors of $n-1$, $n-6$, $n-15$, &c., and also of all the numbers derived from them. The theorem asserts that in the system of numbers so formed (if we adopt the above convention with respect to the group depending upon the divisors of $n-n$, when it occurs) every number appears an equal number of times with the positive and with the negative sign, so that all the numbers in the system cancel each other.

7. Retaining the G-notation, the theorem may be conveniently stated in the form :—The system of numbers given by the formula

$$G_n(d) - G_{n-1}\begin{Bmatrix} d+1 \\ d \\ d-1 \end{Bmatrix} + G_{n-1-2}\begin{Bmatrix} d+2 \\ d+1 \\ d \\ d-1 \\ d-2 \end{Bmatrix} - G_{n-1-2-3}\begin{Bmatrix} d+3 \\ d+2 \\ d+1 \\ d \\ d-1 \\ d-2 \\ d-3 \end{Bmatrix} + \text{\&c.},$$

all cancel each other, subject to the convention that, if

$$n = 1+2+3+\ldots+g,$$

then the last term

$$G_{n-1-2-\ldots-g}\begin{Bmatrix} d+g \\ \ldots \\ d+1 \\ d \\ d-1 \\ \ldots \\ d-g \end{Bmatrix}$$

is to be replaced by the group of numbers

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ \ldots\ g,\ g,\ g,\ \ldots\ (g \text{ times}).$$

8. The theorem seems to be a very curious one, relating as it does to the mutual destruction of certain numbers depending upon the divisors of numbers separated from each other by fixed intervals. It assigns all the divisors of n when we know the divisors of the numbers which are inferior to n and separated from it by the intervals 1, 3, 6, &c. It thus effects the complete resolution of any number n into its factors, or in other words, lays bare the structure of the number n as regards its real divisors, when the structure of the numbers $n-1$, $n-3$, $n-6$, &c., is known. The recurring formula in § 1 suffices to determine whether n is prime or not, when we know the sums of the divisors of $n-1$, $n-3$, $n-6$, &c., for in that case $\sigma(n)$ is equal to $n+1$; but the general theorem goes further, and gives all the divisors of n by means of those of $n-1$, $n-3$, &c.

Numerical Theorems relating to the Sums of Powers of the Divisors of n, $n-1$, $n-3$, &c. §§ 9–13.

9. Since the actual numbers cancel each other in the general theorem, we may replace them by any function of themselves, so long as the function is the same for all and changes sign with the argument. We may, further, combine all the functions in each group by addition, and thus derive from the general theorem a numerical equation connecting together functions of the divisors of the numbers n, $n-1$, $n-3$, &c.

Let $$\Sigma_n \{\phi(d),\ \psi(d),\ \chi(d),\ \dots\}$$

denote the sum

$$\begin{aligned} &\phi(d_1)+\phi(d_2)+\dots+\phi(d_f)\\ &+\psi(d_1)+\psi(d_2)+\dots+\psi(d_f)\\ &+\chi(d_1)+\chi(d_2)+\dots+\chi(d_f)+\dots, \end{aligned}$$

where $d_1, d_2, \dots d_f$ are, as in § 2, all the divisors of n.

By replacing $G_r(d, d\pm 1, \dots)$ by $\Sigma_r\{\phi(d), \phi(d\pm 1), \dots\}$, ϕ being any uneven function, we find that

$$\begin{aligned} \Sigma_n\phi(d) - \Sigma_{n-1}\{\phi(d)+\phi(d\pm1)\} + \Sigma_{n-1-2}\{\phi(d)+\phi(d\pm1)+\phi(d\pm2)\}\\ -\Sigma_{n-1-2-3}\{\phi(d)+\phi(d\pm1)+\phi(d\pm2)+\phi(d\pm3)\}+\&c. \end{aligned}$$

$= 0$, if n is not a triangular number, and $\qquad = (-1)^{g-1}\{\phi(1)+2\phi(2)+3\phi(3)+\ldots+g\phi(g)\}$,

if $\qquad n = 1+2+3+\ldots+g = \tfrac{1}{2}g(g+1)$.

10. As a particular case, putting

$$\phi(d) = d,$$

the equation becomes

$$\Sigma_n d - \Sigma_{n-1}(3d) + \Sigma_{n-3}(5d) - \Sigma_{n-6}(7d) + \Sigma_{n-10}(9d) - \&c.$$
$$= 0 \text{ or } (-1)^{g-1}(1^2+2^2+3^2+\ldots+g^2),$$

according as n is not a triangular number or is

$$= 1+2+3+\ldots+g.$$

Putting $\qquad \phi(d) = d^3,$

it becomes

$$\Sigma_n(d^3) - \Sigma_{n-1}(3d^3+6d) + \Sigma_{n-3}\{5d^3+6(1^2+2^2)\,d\}$$
$$-\Sigma_{n-6}\{7d^3+6(1^2+2^2+3^2)d\} + \Sigma_{n-10}\{9d^3+6(1^2+2^2+3^2+4^2)\,d\} - \&c.$$
$$= 0, \text{ or } (-1)^{g-1}(1^4+2^4+3^4+\ldots+g^4),$$

according as n is not a triangular number or is

$$= 1+2+3+\ldots+g.$$

Putting $\phi(d) = d^5$,

$$\Sigma_n(d^5) - \Sigma_{n-1}(3d^5+20d^3+10d) + \Sigma_{n-3}\{5d^5+20(1^2+2^2)d^3+10(1^4+2^4)d\}$$
$$-\Sigma_{n-6}\{7d^5+20(1^2+2^2+3^2)\,d^3+10(1^4+2^4+3^4)\,d\} + \&c.$$
$$= 0, \text{ or } (-1)^{g-1}(1^6+2^6+3^6+\ldots+g^6),$$

according as n is not a triangular number or is

$$= 1+2+3+\ldots+g;$$

and so on.

11. Denoting by $\sigma_r(n)$ the sum of the r^{th} powers of the divisors of n, we may write these equations in the following form, the additional term in square brackets coming into existence only in the case when n is a triangular number $\tfrac{1}{2}g(g+1)$.

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + 9\sigma(n-10) - \&c.$$
$$= \left[(-1)^{g-1}(1^2+2^2+3^2+\ldots+g^2)\right],$$

$$\sigma_3(n)-3\sigma_3(n-1)+5\sigma_3(n-3)-7\sigma_3(n-6)+9\sigma_3(n-10)-\&c.$$
$$= 6\left\{\sigma(n)-(1^2+2^2)\,\sigma(n-3)+(1^2+2^2+3^2)\,\sigma(n-6)-\&c.\right\}$$
$$+\left[(-1)^{g-1}(1^4+2^4+3^4+\ldots+g^4)\right],$$

$$\sigma_5(n)-3\sigma_5(n-1)+5\sigma_5(n-3)-7\sigma_5(n-6)+9\sigma_5(n-10)-\&c.$$
$$= 20\left\{\sigma_3(n)-(1^2+2^2)\,\sigma_3(n-3)+(1^2+2^2+3^2)\,\sigma_3(n-6)-\&c.\right\}$$
$$+10\left\{\sigma(n)-(1^4+2^4)\,\sigma(n-3)+(1^4+2^4+3^4)\,\sigma(n-6)-\&c.\right\}$$
$$+\left[(-1)^{g-1}(1^6+2^6+3^6+\ldots+g^6)\right],$$

$$\sigma_7(n)-3\sigma_7(n-1)+5\sigma_7(n-3)-7\sigma_7(n-6)+9\sigma_7(n-10)-\&c.$$
$$= 42\left\{\sigma_5(n-1)-(1^2+2^2)\,\sigma_5(n-3)+(1^2+2^2+3^2)\,\sigma_5(n-6)-\&c.\right\}$$
$$+70\left\{\sigma_3(n-1)-(1^4+2^4)\,\sigma_3(n-3)+(1^4+2^4+3^4)\,\sigma_3(n-6)-\&c.\right\}$$
$$+14\left\{\sigma(n-1)-(1^6+2^6)\,\sigma(n-3)+(1^6+2^6+3^6)\,\sigma(n-6)-\&c.\right\}$$
$$+\left[(-1)^{g-1}(1^8+2^8+3^8+\ldots+g^8)\right];$$

and, in general, m being any uneven number,

$$\sigma_m(n)-3\sigma_m(n-1)+5\sigma_m(n-3)-7\sigma_m(n-6)+9\sigma_m(n-10)-\&c.$$
$$= 2\,\frac{m(m-1)}{2!}\left\{\sigma_{m-2}(n-1)-(1^2+2^2)\,\sigma_{m-2}(n-3)\right.$$
$$\left.+(1^2+2^2+3^2)\,\sigma_{m-2}(n-6)-\&c.\right\}$$
$$+2\,\frac{m(m-1)(m-2)(m-3)}{4!}\left\{\sigma_{m-4}(n-1)-(1^4+2^4)\,\sigma_{m-4}(n-3)\right.$$
$$\left.+(1^4+2^4+3^4)\,\sigma_{m-4}(n-6)-\&c.\right\}$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$+2m\left\{\sigma(n-1)-(1^{m-1}+2^{m-1})\,\sigma(n-3)\right.$$
$$\left.+(1^{m-1}+2^{m-1}+3^{m-1})\,\sigma(n-6)-\&c.\right\}$$
$$+\left[(-1)^{g-1}(1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1})\right].$$

12. This general theorem expresses the sum of the m^{th} powers (m being uneven) of the divisors of n in terms of the sums of the m^{th} and lower powers of the divisors of $n-1$, $n-3$, $n-6$, &c. The form of the right hand member of the equation is curious; all the coefficients, and even the additional term which occurs when n is equal to $\frac{1}{2}g(g+1)$, being sums of even powers of the natural

numbers. The nature of the theorem is best seen by taking a particular case. Putting therefore $n=9$ (which is not a triangular number, so that the additional term does not occur), the theorem gives

$$1^m+3^m+9^m-3(1^m+2^m+4^m+8^m)+5(1^m+2^m+3^m+6^m)-7(1^m+3^m)$$

$$= 2\frac{m(m-1)}{2!}\{1^{m-2}+2^{m-2}+4^{m-2}+8^{m-2}$$
$$-(1^2+2^2)(1^{m-2}+2^{m-2}+3^{m-2}+6^{m-2})$$
$$+(1^2+2^2+3^2)(1^{m-2}+3^{m-2})\}$$

$$+2\frac{m(m-1)(m-2)(m-3)}{4!}\{1^{m-4}+2^{m-4}+4^{m-4}+8^{m-4}$$
$$-(1^4+2^4)(1^{m-4}+2^{m-4}+3^{m-4}+6^{m-4})$$
$$+(1^4+2^4+3^4)(1^{m-4}+3^{m-4})\}$$

... ...

$$+2m\{1+2+4+8-(1^{m-1}+2^{m-1})(1+2+3+6)$$
$$+(1^{m-1}+2^{m-1}+3^{m-1})(1+3)\}.$$

Thus the theorem connects together the sums of uneven powers of the divisors of numbers separated from each other by the intervals 1, 2, 3, ..., and the sums of even powers of the natural numbers.

13. It will be observed that in the general theorem in § 11, we may dispense with the additional term if we assign to $\sigma_m(0)$ the conventional meaning

$$\frac{1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1}}{2g+1},$$

and put $\sigma_{m-2}(0)$, $\sigma_{m-4}(0)$, ... $\sigma(0)$ equal to zero. The coefficient of $\sigma_m(0)$ is $2g+1$, and we are therefore to replace $(2g+1)\sigma_m(0)$ by $1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1}$, the value of g being derivable from the coefficient of $\sigma_m(0)$ without the necessity of having recourse to the formula $n=\frac{1}{2}g(g+1)$.*

Another convention by which the additional term in the general theorem may be conveniently represented is given in §§ 20–22.

---

* The number g is also equal to the number of terms in the series on the left-hand side of the equation, the term $(2g+1)\sigma_m(0)$ not being counted.

The particular cases of $m = 1$, 3, *and* 5. §§ 14–17.

14. In examining the first few particular cases of the general theorem in § 11, it seems worth while in the first place to take a numerical example of each theorem, both in the case when n is not, and when n is, a triangular number. Putting $n = 9$, which is not a triangular number, the theorems give

$$1+3+9-3\,(1+2+4+8)+5\,(1+2+3+6)-7\,(1+3)=0,$$

$$1^3+3^3+9^3-3\,(1^3+2^3+4^3+8^3)+5\,(1^3+2^3+3^3+6^3)-7\,(1^3+3^3)$$
$$= 6\,\{1+2+4+8-5\,(1+2+3+6)+14\,(1+3)\},$$

$$1^5+3^5+9^5-3\,(1^5+2^5+4^5+8^5)+5\,(1^5+2^5+3^5+6^5)-7\,(1^5+3^5)$$
$$= 20\,\{1^3+2^3+4^3+8^3-5\,(1^3+2^3+3^3+6^3)+14\,(1^3+3^3)\}$$
$$+10\,\{1+2+4+8-17\,(1+2+3+6)+98\,(1+3)\};$$

and, putting $n=10$, which $=1+2+3+4$, so that $y=4$, the theorems give

$$1+2+5+10-3\,(1+3+9)+5\,(1+7)-7\,(1+2+4)$$
$$=-(1^2+2^2+3^2+4^2),$$

$$1^3+2^3+5^3+10^3-3\,(1^3+3^3+9^3)+5\,(1^3+7^3)-7\,(1^3+2^3+4^3)$$
$$=6\,\{1+3+9-5\,(1+7)+14\,(1+2+4)\}$$
$$-(1^4+2^4+3^4+4^4),$$

$$1^5+2^5+5^5+10^5-3\,(1^5+3^5+9^5)+5\,(1^5+7^5)-7\,(1^5+2^5+4^5)$$
$$=20\,\{1^3+3^3+9^3-5\,(1^3+7^3)+14\,(1^3+2^3+4^3)\}$$
$$+10\,\{1+3+9-17\,(1+7)+98\,(1+2+4)\}$$
$$-(1^6+2^6+3^6+4^6).$$

The correctness of these equalities is easily verified.

15. In spite of the fact that some of the elegance and regularity of the formulæ is lost by summing the powers of the natural numbers which form the coefficients, it is still interesting to exhibit the series in a form in which the coefficients of the general terms are expressed algebraically.

Since $$1^2+2^2+3^2+\dots+i^2=\tfrac{1}{6}i\,(i+1)(2i+1),$$

the theorem in the case of $m=1$ may be written

$$\Sigma(-1)^i(2i+1)\,\sigma\{n-\tfrac{1}{2}i(i+1)\} = \left[(-1)^{g-1}\tfrac{1}{6}g(g+1)(2g+1)\right],$$

the summation extending from $i=0$ to $i=h$, where $\frac{1}{2}h(h+1)$ is the triangular number next inferior to n.

When $n=\frac{1}{2}g(g+1)$, we obtain the term $(-1)^g(2g+1)\,\sigma(0)$ by continuing the series one term further. We may therefore dispense with the additional term if we extend the summation from $i=0$ to $i=k$, where $\frac{1}{2}k(k+1)$ is the triangular number nearest to, and not exceeding, n (*i.e.*, so that $\frac{1}{2}k(k+1)$ is the triangular number next inferior to n, if n is not a triangular number, and is equal to n if n is a triangular number), and assign the conventional value $\frac{1}{3}n$ to $\sigma(0)$, when it occurs. This is the form in which the theorem was quoted in § 1.

16. Since

$$1^4+2^4+3^4+\ldots+i^4 = \tfrac{1}{30}i(i+1)(2i+1)(3i^2+3i-1),$$

we may write the theorem, in the case of $m=3$, in the form

$$\begin{aligned}&\Sigma(-1)^i(2i+1)\,\sigma_3\{n-\tfrac{1}{2}i(i+1)\}\\ &= \Sigma(-1)^{i-1}i(i+1)(2i+1)\,\sigma\{n-\tfrac{1}{2}i(i+1)\}\\ &\quad+\left[(-1)^{g-1}\tfrac{1}{30}g(g+1)(2g+1)(3g^2+3g-1)\right],\end{aligned}$$

the summation extending from $i=0$ to $i=h$, where $\frac{1}{2}h(h+1)$ is the triangular number next inferior to n.

The additional term, which only occurs when $n=\frac{1}{2}g(g+1)$, may be written in the form

$$(-1)^{g-1}\tfrac{1}{15}(2g+1)\,n\,(6n-1);$$

we may therefore dispense with it if we extend the summations so as to include the argument zero, and define $\sigma(0)$ and $\sigma_3(0)$ by any of the three following pairs of equations:—

$$\begin{aligned}&\text{(i.)}\ \sigma(0)=0, &&\sigma_3(0)=\tfrac{2}{5}n^2-\tfrac{1}{15}n,\\ &\text{(ii.)}\ \sigma(0)=\tfrac{1}{6}n-\tfrac{1}{30}, &&\sigma_3(0)=0,\\ &\text{(iii.)}\ \sigma(0)=\tfrac{1}{5}n, &&\sigma_3(0)=-\tfrac{1}{15}n.\end{aligned}$$

Adopting any one of these three pairs of simultaneous values of $\sigma(0)$ and $\sigma_3(0)$, we may write the theorem

$$\Sigma(-1)^i(2i+1)\,\sigma_3\{n-\tfrac{1}{2}i(i+1)\} + \Sigma(-1)^i i(i+1)(2i+1)\,\sigma\{n-\tfrac{1}{2}i(i+1)\}=0,$$

the summation now extending from $i = 0$ to $i = k$, where $\frac{1}{2}k(k+1)$ is the triangular number which is nearest to, and does not exceed, n.

For example, putting $n = 10$, and adopting the last of the three conventions, the theorem gives

$$\sigma_3(10) - 3\sigma_3(9) + 5\sigma_3(7) - 7\sigma_3(4) + 9\sigma_3(0)$$
$$= 6\sigma(9) - 30\sigma(7) + 84\sigma(4) - 180\sigma(0),$$

where $$\sigma(0) = 2, \quad \sigma_3(0) = -\tfrac{2}{3}.$$

17. Since

$$1^6 + 2^6 + 3^6 + \ldots + i^6 = \tfrac{1}{42} i(i+1)(2i+1)\{3i^4 + 6i^3 - 3i + 1\},$$

the theorem, in the case of $m = 5$, may be written

$$\Sigma(-1)^i(2i+1)\,\sigma_5\{n - \tfrac{1}{2}i(i+1)\}$$
$$= \tfrac{13}{3}\Sigma(-1)^{i-1}i(i+1)(2i+1)\,\sigma_3\{n - \tfrac{1}{2}i(i+1)\}$$
$$+ \tfrac{1}{3}\Sigma(-1)^{i-1}i(i+1)(2i+1)(3i^2+3i-1)\,\sigma\{n - \tfrac{1}{3}i(i+1)\}$$
$$+ \left[(-1)^{g-1}\tfrac{1}{42}g(g+1)(2g+1)(3g^4+6g^3-3g+1)\right],$$

the limits of summation being $i = 0$ and $i = h$, where h has the same meaning as in the preceding section.

The additional term may be written in the form

$$(-1)^{g-1}\tfrac{1}{21}(2g+1)\,n\,(12n^2 - 6n + 1);$$

we may therefore dispense with it if we extend the summations so as to include the argument zero, and define $\sigma(0)$, $\sigma_3(0)$, $\sigma_5(0)$ by any of the three sets of equations:

$$\text{(i.)}\ \sigma(0) = 0, \quad \sigma_3(0) = 0, \quad \sigma_5(0) = \tfrac{4}{7}n^3 - \tfrac{2}{7}n^2 + \tfrac{1}{21}n,$$

$$\text{(ii.)}\ \sigma(0) = 0, \quad \sigma_3(0) = \tfrac{3}{35}n^2 - \tfrac{3}{70}n + \tfrac{1}{140}, \quad \sigma_5(0) = 0,$$

$$\text{(iii.)}\ \sigma(0) = \tfrac{1}{7}n, \quad \sigma_3(0) = -\tfrac{1}{35}n, \quad \sigma_5(0) = \tfrac{1}{21}n.$$

Assigning to $\sigma(0)$, $\sigma_3(0)$, $\sigma_5(0)$ any one of these three sets of values, we may write the theorem in the form

$$\Sigma(-1)^i(2i+1)\,\sigma_5\{n - \tfrac{1}{2}i(i+1)\}$$
$$+ \tfrac{10}{3}\Sigma(-1)^i i(i+1)(2i+1)\,\sigma_3\{n - \tfrac{1}{2}i(i+1)\}$$
$$+ \tfrac{1}{3}\Sigma(-1)^i i(i+1)(2i+1)(3i^2+3i-1)\,\sigma\{n - \tfrac{1}{2}i(i+1)\} = 0.$$

For example, putting $n = 6$, and adopting the last of the three conventions, the theorem gives

$$\sigma_5(6) - 3\sigma_5(5) + 5\sigma_5(3) - 7\sigma_5(0) = 20\left\{\sigma_3(5) - 5\sigma_3(3) + 14\sigma_3(0)\right\}$$
$$+10\left\{\sigma(5) - 17\sigma(3) + 98\sigma(0)\right\},$$

where $\sigma(0) = \frac{6}{7}, \quad \sigma_3(0) = -\frac{6}{35}, \quad \sigma_5(0) = \frac{2}{7}.$

The same Theorems expressed in terms of c and t. §§ 18, 19.

18. The coefficients in the series may conveniently be expressed by means of c, the coefficient, and t, the triangular number, which occur in the general term of the simplest of the series, viz.,

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + 9\sigma(n-10) - \&c.$$

Let, therefore, t denote the i^{th} triangular number $\frac{1}{2}i(i+1)$, and let c denote the coefficient $2i+1$ of the term $\phi(n-t)$, so that the quantities c and t are connected by the relation

$$t = \tfrac{1}{8}(c^2 - 1).$$

Using these letters c and t, and extending the sign of summation to every triangular number which does not exceed n, zero being included, and also n itself, if n is not a triangular number, we may write the above series in the form

$$\Sigma(-1)^{\frac{1}{2}(c-1)} c\sigma(n-t);$$

or, if we employ Legendre's symbol $\left(\frac{-1}{c}\right)$ in place of its value $(-1)^{\frac{1}{2}(c-1)}$, in the form

$$\Sigma\left(\frac{-1}{c}\right) c\sigma(n-1).$$

Now it is known that

$$1^{2r} + 2^{2r} + 3^{2r} + \ldots + i^{2r} = ctF(t),$$

where $F(t)$ is a rational and integral function of t of the order $r-1$, so that the series

$$\phi(n-1) - (1^{2r} + 2^{2r})\,\phi(n-3) + (1^{2r} + 2^{2r} + 3^{2r})\,\phi(n-6) - \&c.$$

may always be expressed in the form

$$-\Sigma\left(\frac{-1}{c}\right) ct\,F(t)\,\phi(n-t),$$

the limits of summation being the same as above.

19. Expressed in this notation, the formulæ of §§ 15–17 assume the forms

$$\Sigma\left(\frac{-1}{c}\right) c\sigma\,(n-t) = 0,$$

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_3\,(n-t) + 2\Sigma\left(\frac{-1}{c}\right) ct\,\sigma\,(n-t) = 0,$$

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_5\,(n-t) + \tfrac{20}{3}\Sigma\left(\frac{-1}{c}\right) ct\,\sigma_3\,(n-t) + 4\Sigma\left(\frac{-1}{c}\right) ct^2\,\sigma\,(n-t)$$
$$-\tfrac{2}{3}\Sigma\left(\frac{-1}{c}\right) ct\,\sigma\,(n-t) = 0,$$

where, as in §§ 15–17, in the first formula, $\sigma\,(0) = \frac{1}{3}n$; in the second, $\sigma\,(0) = \frac{1}{5}n$, $\sigma_3\,(n) = -\frac{1}{15}n$; in the third, $\sigma\,(0) = \frac{1}{7}n$, $\sigma_3\,(0) = -\frac{1}{35}n$, $\sigma_5\,(0) = \frac{1}{21}n$.

It is evident that the general theorem in § 11 may be expressed as a linear relation connecting together series of the form

$$\Sigma\left(\frac{-1}{c}\right) ct^r\,\sigma_s\,(n-t),$$

where r is any number, and s any uneven number.

Method of representing the Additional Term in the general Theorem. §§ 20–23.

20. The most elegant of the different methods which have been made use of in the preceding sections for representing the additional term is the one in which the values of $\sigma\,(0)$, $\sigma_3\,(0)$, ... are simply proportional to n. It is easy to see that this method of representing the additional term is general. For, writing the general theorem (§ 11), when n is the triangular number $\frac{1}{2}g\,(g+1)$, in the form

$$\sigma_m\,(n) - 3\sigma_m\,(n-1) + 5\sigma_m\,(n-3) - 7\sigma_m\,(n-6) + 9\sigma_m\,(n-10) - \&c.$$
$$= \quad 2\,\frac{m^{(2)}}{2\,!}\,\{\sigma_{m-2}\,(n-1) - (1^2+2^2)\,\sigma_{m-2}\,(n-3)$$
$$+ (1^2+2^2+3^2)\,\sigma_{m-2}\,(n-6) - \&c.\}$$
$$+ 2\,\frac{m^{(4)}}{4\,!}\,\{\sigma_{m-4}\,(n-1) - (1^4+2^4)\,\sigma_{m-4}\,(n-3)$$
$$+ (1^4+2^4+3^4)\,\sigma_{m-4}\,(n-6) - \&c.\}$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$+ 2m\,\{\sigma\,(n-1) - (1^{m-1}+2^{m-1})\,\sigma\,(n-3)$$
$$+ (1^{m-1}+2^{m-1}+3^{m-1})\,\sigma\,(n-6) - \&c.\}$$
$$+ (-1)^{g-1}\,(1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1}),$$

where m is any uneven number, and $m^{(r)}$ denotes the factorial

$$m(m-1)\dots(m-r+1),$$

it is evident that, if $a_1, a_3, \dots, a_m$ are numerical quantities depending upon m, which are determined by the equation

$$\begin{aligned} 1^{m+1}+2^{m+1}+3^{m+1}+\dots+g^{m+1} & \\ = g(g+1)\Big\{ & m(1^{m-1}+2^{m-1}+3^{m-1}+\dots+g^{m-1})\,a_1 \\ & +\frac{m^{(3)}}{3!}(1^{m-3}+2^{m-3}+3^{m-3}+\dots+g^{m-3})\,a_3 \\ & \dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots \\ & +\frac{m^{(2)}}{2!}(1^2+2^2+3^2+\dots+g^2)\,a_{m-2} \\ & +\tfrac{1}{2}(2g+1)\,a_m, \end{aligned}$$

then the additional term disappears if we assign to $\sigma(0)$, $\sigma_3(0)$, ... the system of values

$$\sigma(0)=a_1 n,\quad \sigma_3(0)=a_3 n,\quad \sigma_5(0)=a_5 n,\ \dots,\quad \sigma_m(0)=a_m n.$$

21. By replacing the series

$$1^{m+1}+2^{m+1}+3^{m+1}+\dots+g^{m+1},\quad 1^{m-1}+2^{m-1}+3^{m-1}+\dots+g^{m-1},\ \&c.$$

by their summations in terms of Bernoullian numbers, and equating coefficients, we ultimately arrive at the following simple system of equations, which determine the values of $a_1, a_3, \dots, a_m$* :—

$$ma_1 = 1-\frac{2}{m+2},$$

* The details of this determination are given in a paper "On a Theorem relating to Sums of Powers of the Natural Numbers" (*Messenger of Mathematics*, Vol. xx., pp. 120-128). The determination was made the subject of a separate paper, as the investigation is somewhat lengthy, and it is only with the results that the present paper is concerned. It is shown that, if the series $1^n+2^n+3^n+\dots+g^n$ be denoted by S_n, then

$$S_{2n} = S_1\left\{A_2 S_{2n-2}+A_4 S_{2n-4}+\dots+A_{2n-2}S_2+A_{2n}(g+\tfrac{1}{2})\right\},$$

where $\tfrac{1}{2}A_2 = ma_1,\quad \tfrac{1}{2}A_4 = \frac{m^{(3)}}{3!}a_3,\ \dots,\quad \tfrac{1}{2}A_{2n-2} = \frac{m^{(2)}}{2!}a_{m-2},\quad \tfrac{1}{2}A_{2n} = a_m,$

m being $= 2n-1$, and $a_1, a_3, \dots, a_m$ having the values assigned to them in the text. In connexion with this investigation the theorems in § 11, which gave rise to it, are also given, but without proof, in a separate note (*Messenger*, Vol. xx., pp. 129-135).

$$\frac{m^{(3)}}{3!}a_3 - ma_1 \quad = -(m+1)\frac{B_1}{1},$$

$$\frac{m^{(5)}}{5!}a_5 - \frac{m^{(3)}}{3!}a_3 = \quad \frac{(m+1)^{(3)}}{3!}\frac{B_2}{2},$$

$$\frac{m^{(7)}}{7!}a_7 - \frac{m^{(5)}}{5!}a_5 = -\frac{(m+1)^{(5)}}{5!}\frac{B_3}{3},$$

&c., &c.,

$B_1, B_2, B_3, \ldots$ being the Bernoullian numbers.

We thus find

$$a_1 = \frac{1}{m+2},$$

$$\frac{m^{(3)}}{3!}a_3 = \frac{m}{m+2} - (m+1)\frac{B_1}{1},$$

$$\frac{m^{(5)}}{5!}a_5 = \frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3!}\frac{B_2}{2},$$

$$\frac{m^{(7)}}{7!}a_7 = \frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3!}\frac{B_2}{2} - \frac{(m+1)^{(5)}}{5!}\frac{B_3}{3},$$

&c., &c.,

whence

$$\sigma(0) = \frac{1}{m+2}n,$$

$$\frac{m^{(3)}}{3!}\sigma_3(0) = \left\{\frac{m}{m+2} - (m+1)\frac{B_1}{1}\right\}n,$$

$$\frac{m^{(5)}}{5!}\sigma_5(0) = \left\{\frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3!}\frac{B_2}{2}\right\}n,$$

$$\frac{m^{(7)}}{7!}\sigma_7(0) = \left\{\frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3!}\frac{B_2}{2} - \frac{(m+1)^{(5)}}{5!}\frac{B_3}{3}\right\}n,$$

&c., &c.

Thus finally we obtain the formulæ:

$$\sigma(0) = \frac{1}{m+2}n,$$

$$\frac{\sigma_3(0)}{m+1} = \left\{\frac{3!\,m}{(m+2)^{(3)}} - \frac{3!}{m^{(3)}}\frac{B_1}{1}\right\}n,$$

$$\frac{\sigma_5(0)}{m+1} = \left\{ \frac{5!\,m}{(m+2)^{(7)}} - \frac{5!}{m^{(7)}}\frac{B_1}{1} + \frac{5^{(2)}}{(m-2)^{(3)}}\frac{B_2}{2} \right\} n,$$

$$\frac{\sigma_7(0)}{m+1} = \left\{ \frac{7!\,m}{(m+2)^{(9)}} - \frac{7!}{m^{(7)}}\frac{B_1}{1} + \frac{7^{(4)}}{(m-2)^{(5)}}\frac{B_2}{2} - \frac{7^{(2)}}{(m-4)^{(3)}}\frac{B_3}{3} \right\} n,$$

and, in general, r being any uneven number,

$$\frac{\sigma_r(0)}{m+1} = \left\{ \frac{r!\,m}{(m+2)^{(r+2)}} - \frac{r!}{m^{(r)}}\frac{B_1}{1} + \frac{r^{(r-3)}}{(m-2)^{(r-2)}}\frac{B_2}{2} - \frac{r^{(r-5)}}{(m-4)^{(r-4)}}\frac{B_3}{3} + \dots \right.$$

$$\left. \dots \pm \frac{r^{(2)}}{(m-r+3)^{(3)}}\frac{B_{\frac{1}{2}(r-1)}}{\frac{1}{2}(r-1)} \right\} n.$$

22. These equations give

$$\sigma\,(0) = \frac{1}{m+2}\,n,$$

$$\sigma_3(0) = -\frac{1}{(m+2)\,m}\,n,$$

$$\sigma_5(0) = \frac{3m+10}{3\,(m+2)\,m\,(m-2)}\,n,$$

$$\sigma_7(0) = -\frac{m^2+12m+56}{3\,(m+2)\,m\,(m-2)(m-4)}\,n,$$

&c., &c.

The values obtained by putting $m = 3$ and 5 in these expressions agree with the third system of results in §§ 16 and 17.

Putting $m = 7$, we find

$$\sigma\,(0) = \tfrac{1}{9}n, \quad \sigma_3(0) = -\tfrac{1}{63}n, \quad \sigma_5(0) = \tfrac{17}{945}n, \quad \sigma_7(0) = -\tfrac{1}{15}n.$$

If, therefore, we attribute these values to $\sigma(0)$, $\sigma_3(0)$, ..., $\sigma_7(0)$, we have, for all values of n,

$$\sigma_7(n) - 3\sigma_7(n-1) + 5\sigma_7(n-3) - 7\sigma_7(n-6) + 9\sigma_7(n-10) - \&c.$$

$$= 42\,\{\sigma_5(n-1) - (1^2+2^2)\,\sigma_5(n-3) + (1^2+2^2+3^2)\,\sigma_5(n-6) - \&c.\}$$

$$+ 70\,\{\sigma_3(n-1) - (1^4+2^4)\,\sigma_3(n-3) + (1^4+2^4+3^4)\,\sigma_3(n-6) - \&c.\}$$

$$+ 14\,\{\sigma\,(n-1) - (1^6+2^6)\,\sigma\,(n-3) + (1^6+2^6+3^6)\,\sigma\,(n-6) - \&c.\}.$$

23. With respect to the two methods of presenting the general theorem (§§ 11 and 20), *i.e.*, with or without the additional term, it is evidently an advantage to avoid the irregularity produced in the formula by the occasional presence of an extra term. Since the

quantities $\sigma(0)$, $\sigma_3(0)$, ... , $\sigma_m(0)$ only enter into the formula in the exceptional case when the additional term makes its appearance, it seems natural, following Euler,* to get rid of this term by assigning suitable values to these quantities. It is interesting to find that this can always be effected by means of a system of values, all of which are simply proportional to n. The form of the expressions for these values in terms of Bernoullian numbers (§ 21) is also not without interest. On the other hand, the simplicity and directness of the theorem, as stated in § 11, is impaired by these conventional meanings assigned to $\sigma(0)$, $\sigma_3(0)$, ... , $\sigma_m(0)$. Also, the additional term itself has a special interest of its own, being similar in form (*i.e.*, a sum of even powers of the natural numbers) to the coefficients in the series.

Form of the General Theorem if the Divisor unity be omitted. § 24.

24. It may be remarked that, if we omit from the divisors of n, $n-1$, $n-3$, ... the divisor unity which is common to them all, the only change introduced into the formulæ is the addition of a single term consisting of an uneven power of l, where $\frac{1}{2}l(l+1)$ is the triangular number nearest to, and not inferior to, n.

Thus, if $\sigma'_r(n)$ denotes the sum of the r^{th} powers of the divisors of n, unity being included, we have, corresponding to the general theorem in § 11,

$$\sigma'_m(n)-3\sigma'_m(n-1)+5\sigma'_m(n-3)-7\sigma'_m(n-6)+9\sigma'_m(n-10)-\&c.$$

$$\begin{aligned}
= \quad & 2\frac{m^{(2)}}{2!}\{\sigma'_{m-2}(n-1)-(1^2+2^2)\,\sigma'_{m-2}(n-3) \\
& \qquad\qquad +(1^2+2^2+3^2)\,\sigma'_{m-2}(n-6)-\&c.\} \\
& +2\frac{m^{(4)}}{4!}\{\sigma'_{m-4}(n-1)-(1^4+2^4)\,\sigma'_{m-4}(n-3) \\
& \qquad\qquad +(1^4+2^4+3^4)\,\sigma'_{m-2}(n-6)-\&c.\} \\
& \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
& +2m\{\sigma'(n-1)-(1^{m-1}+2^{m-1})\,\sigma'(n-3) \\
& \qquad\qquad +(1^{m-1}+2^{m-1}+3^{m-1})\,\sigma'(n-6)-\&c.\} \\
& +(-1)^{l}l^{m}+\left[(-1)^{\rho-1}(1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1})\right],
\end{aligned}$$

---

* In Euler's recurring formula

$$\sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)-\&c.=0,$$

he dispensed with an additional term which occurs when n is a pentagonal number by assigning to $\sigma(0)$ the value n (*Opera Minora Collecta*, Vol. I., p. 149).

where l has the meaning just assigned to it, so that, when n is a triangular number $\frac{1}{2}g\,(g+1)$, in which case the additional term appears, l is equal to g; otherwise $\frac{1}{2}l\,(l+1)$ is the triangular number next greater than n. It will be observed that l is always equal to the number of terms in the series on the left-hand side of the equation, not counting the term $\sigma_m\,(0)$, if it occurs. As examples, let $m=3$, and put $n=5$ and 6. In these two cases the formula becomes

$$\sigma_3'\,(5)-3\sigma_3'\,(4)+5\sigma_3'\,(2)=6\,\{\sigma'\,(4)-5\sigma'\,(2)\}-3^3,$$

and $$\sigma_3'\,(6)-3\sigma_3'\,(5)+5\sigma_3'\,(3)=6\,\{\sigma'\,(5)-5\sigma'\,(3)\}-3^3+1^4+2^4+3^4;$$

that is, $$125-3\times\ 72+5\times\ 8=6\,\{6-10\}-27,$$

and $$251-3\times 125+5\times 27=6\,\{5-15\}-27+98.$$

Proof of the Theorem relating to the actual Divisors. §§ 25–27.

25. Let $$\operatorname{zs} x=\frac{\Theta_1'\,(x)}{\Theta_1\,(x)}=\operatorname{zn} x+\frac{\operatorname{cn} x\operatorname{dn} x}{\operatorname{sn} x},$$

where $\Theta_1\,(x)$ and $\operatorname{zn} x$ are the same as Jacobi's $\mathrm{H}\,(x)$ and $\mathrm{Z}\,(x)$ respectively. Then, denoting $\dfrac{2K}{\pi}$ by ρ, we have

$$\rho\operatorname{zs}\rho x=\frac{2q^{\frac14}\cos x-6q^{\frac94}\cos 3x+10q^{\frac{25}{4}}\cos 5x-\&c.}{2q^{\frac14}\sin x-\,q^{\frac94}\sin 3x+2q^{\frac{25}{4}}\sin 5x-\&c.}.$$

Now, it can be shown that

$$\rho\operatorname{zs}\rho x=\frac{\cos x}{\sin x}+\frac{4q^2}{1-q^2}\sin 2x+\frac{4q^4}{1-q^4}\sin 4x+\frac{4q^6}{1-q^6}\sin 6x+\&c.$$

Let $\sigma\phi\,(n)$ denote the sum

$$\phi\,(d_1)+\phi\,(d_2)+\phi\,(d_3)+\ldots+\phi\,(d_f),$$

where $d_1, d_2, d_3, \ldots, d_f$ are all the divisors of n. Using this notation, the coefficient of q^{2n} in the above q-series is easily seen to be equal to $4\sigma\,(\sin 2nx)$, and the equation may be written

$$\rho\operatorname{zs}\rho x=\frac{\cos x}{\sin x}+4\Sigma_1^\infty\,\sigma\,(\sin 2nx)\,q^{2n}.*$$

* This equation was given in this form in the *Messenger of Mathematics*, Vol. XVIII., p. 6, as one of a system of sixteen similar formulæ representing the twelve elliptic and four zeta functions. It is to be observed that the symbol σ is supposed always to refer to the divisors of n (not $2n$). The sign of summation also refe to n.

Equating the two expressions for $\rho\,\mathrm{zs}\,\rho x$, we obtain the identical relation

$$\frac{\cos x}{\sin x}+4\Sigma_1^\infty\sigma(\sin 2nx)\,q^{2n}=\frac{\cos x-3q^3\cos 3x+5q^6\cos 5x-\&c.}{\sin x-q^3\sin 3x+q^6\sin 5x-\&c.}.$$

26. Replacing q^2 by q, this equation becomes

$$\frac{\cos x}{\sin x}+4\Sigma_1^\infty\sigma(\sin 2nx)\,q^n=\frac{\cos x-3q\cos 3x+5q^3\cos 5x-\&c.}{\sin x-q\sin 3x+q^3\sin 5x-\&c.};$$

whence, by multiplying up, we find

$$\begin{aligned}4(\sin x-q\sin 3x+q^3\sin 5x-q^6\sin 7x+\&c.)\,\Sigma_1^\infty\sigma(\sin 2nx)\,q^n\\ =\ &(\sin 3x\cot x-3\cos 3x)\,q\\ &-(\sin 5x\cot x-5\cos 5x)\,q^3\\ &+(\sin 7x\cot x-7\cos 7x)\,q^6\\ &-\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\end{aligned}$$

$$=-\sin x\frac{d}{dx}\frac{\sin 3x}{\sin x}q+\sin x\frac{d}{dx}\frac{\sin 5x}{\sin x}q^3-\sin x\frac{d}{dx}\frac{\sin 7x}{\sin x}q^6-\&c.$$

The equation may therefore be expressed in the form

$$4\left(1-\frac{\sin 3x}{\sin x}q+\frac{\sin 5x}{\sin x}q^3-\frac{\sin 7x}{\sin x}q^6+\&c.\right)\Sigma_1^\infty\sigma(\sin 2nx)\,q^n$$

$$=-\frac{d}{dx}\frac{\sin 3x}{\sin x}q+\frac{d}{dx}\frac{\sin 5x}{\sin x}q^3-\frac{d}{dx}\frac{\sin 7x}{\sin x}q^6+\&c.^*$$

Now $$\frac{\sin 3x}{\sin x}=1+2\cos 2x,$$

$$\frac{\sin 5x}{\sin x}=1+2\cos 2x+2\cos 4x,$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots;$$

and, in general, m being any uneven number,

$$\frac{\sin mx}{\sin x}=1+2\cos 2x+2\cos 4x+\ldots+2\cos(m-1)\,x.$$

---

* This equation may also be obtained very simply by writing the first equation in § 26 in the form

$$4\Sigma_1^\infty\sigma(\sin 2nx)\,q^n+\frac{d}{dx}\log\sin x=\frac{d}{dx}\log(\sin x-q\sin 3x+q^3\sin 5x-\&c.).$$

The equation therefore becomes

$$\{1-(1+2\cos 2x)\,q+(1+2\cos 2x+2\cos 4x)\,q^3-\&c.\}\,\Sigma_1^\infty\,\sigma(\sin 2nx)\,q^n$$
$$=\sin 2x.q-(\sin 2x+2\sin 4x)\,q^3+(\sin 2x+2\sin 4x+3\sin 6x)\,q^6-\&c.$$

27. Replacing $2x$ by x, we have finally

$$\{1-(1+2\cos x)\,q+(1+2\cos x+2\cos 2x)\,q^3$$
$$-(1+2\cos x+2\cos 2x+2\cos 3x)\,q^6+\&c.\}$$
$$\times\{\Sigma_1\sin dx\,.\,q+\Sigma_2\sin dx\,.\,q^2+\Sigma_3\sin dx\,.\,q^3+\&c.\}$$
$$=\sin x\,.\,q-(\sin x+2\sin 2x)\,q^3+(\sin x+2\sin 2x+3\sin 3x)\,q^6-\&c.,$$

where, in accordance with the notation employed in § 9, $\Sigma_n\phi\,(d)$ denotes $\phi\,(d_1)+\phi\,(d_2)+\ldots+\phi\,(d_f)$; $d_1, d_2, \ldots, d_f$ being all the divisors of n.

The coefficient of q^n on the left-hand side of this equation is

$$\Sigma_n\sin dx-\Sigma_{n-1}\{\sin dx+\sin\,(d-1)\,x+\sin\,(d+1)\,x\}$$
$$+\Sigma_{n-3}\{\sin dx+\sin(d-1)x+\sin\,(d+1)\,x+\sin\,(d-2)\,x+\sin\,(d+2)\,x\}$$
$$-\&c.$$

We see, therefore, by equating coefficients, that this expression must be equal to zero if n is not a triangular number, and that it is equal to

$$(-1)^{g-1}\,(\sin x+2\sin 2x+3\sin 3x+\ldots+g\sin gx)$$

when n is the triangular number $\frac{1}{2}g\,(g+1)$.

It is evident that this relation can only exist by virtue of the actual numbers d, $d-1$, $d+1$, &c., which occur in the arguments on the left-hand side, cancelling each other (with the exception of the one 1, two 2's, three 3's, ..., which are to remain when n is a triangular number); for no property of the sine, as distinguished from any other uneven function, can have any influence upon the mutual destruction of the terms. We may, therefore, either replace the sines by ϕ's, ϕ being any uneven function, thus obtaining the theorem in § 9, or we may pass directly from the sine-theorem to the general theorem relating to the actual divisors (§ 3).

Second Theorem relating to the Actual Divisors of the Numbers n, $n-1$, $n-3$, *&c.* §§ 28–30.

28. From the last formula in § 26, by multiplication by $\sin x$, we find

$$2(\sin x - q\sin 3x + q^3\sin 5x - q^6\sin 7x + \&c.)\ \Sigma_1^\infty \sigma(\sin 2nx)\, q^n$$

$$= (\cos x - \cos 3x)\, q - (\cos x + \cos 3x - 2\cos 5x)\, q^3$$
$$+ (\cos x + \cos 3x + \cos 5x - 3\cos 7x)\, q^6 - \&c.,$$

the general term on the right-hand side being

$$\{\cos x + \cos 3x + \cos 5x + \ldots + \cos(2g-1)x - g\cos(2g+1)x\}\ q^{\frac{1}{2}g(g+1)}.$$

The coefficient of q^n on the left-hand side is

$$\begin{aligned} &\Sigma_n \{\cos(2d-1)x - \cos(2d+1)x\} \\ &-\Sigma_{n-1}\{\cos(2d-3)x - \cos(2d+3)x\} \\ &+\Sigma_{n-3}\{\cos(2d-5)x - \cos(2d+5)x\} \\ &-\Sigma_{n-6}\{\cos(2d-7)x - \cos(2d+7)x\} \\ &+\&c.; \end{aligned}$$

this expression therefore is equal to zero, if n is not a triangular number, and is equal to

$$(-1)^{g-1}\{\cos x + \cos 3x + \cos 5x + \ldots + \cos(2g-1)x - g\cos(2g+1)x\},$$

when n is a triangular number $\frac{1}{2}g(g+1)$.

29. Using the notation of § 5 so that $\alpha, \beta, \ldots$, are the divisors of n, $\alpha_1, \beta_1, \ldots$, those of $n-1$, $\alpha_3, \beta_3, \ldots$, of $n-3$, &c., it follows that the numbers given by the formula,

$$\begin{Bmatrix} 2\alpha+1, & 2\beta+1, & \ldots \\ -(2\alpha-1), & -(2\beta-1), & \ldots \end{Bmatrix} - \begin{Bmatrix} 2\alpha_1+3, & 2\beta_1+3, & \ldots \\ -[2\alpha_1-3], & -[2\beta_1-3], & \ldots \end{Bmatrix}$$
$$+ \begin{Bmatrix} 2\alpha_3+5, & 2\beta_3+5, & \ldots \\ -[2\alpha_3-5], & -[2\beta_3-5], & \ldots \end{Bmatrix} - \&c.,$$

all cancel each other, unless n is a triangular number $\frac{1}{2}g(g+1)$, in which case there are left remaining one 1, one 3, ..., one $2g-1$, all having the same sign, and also the number $2g+1$ occurring g times, and having the opposite sign. If g be even, the g different numbers have the positive sign; if uneven, the negative sign.

The numbers $2\alpha_1-3$, $2\beta_1-3$, ... are enclosed in square brackets to indicate that their absolute values are to be taken (irrespective of sign), *i.e.* $[a]$ denotes a, if a is positive, and $-a$ if a is negative, so that $[a]$ is always a positive quantity. It is unnecessary to enclose $2\alpha-1$, $2\beta-1$, ... in square brackets, as they are necessarily positive.*

As an example, let $n=6$. The doubles of the divisors of 6, 5, 3 are 2, 4, 6, 12; 2, 10; 2, 6 respectively, and, since $g=3$, the theorem asserts that the numbers

$$\begin{Bmatrix} 3, & 5, & 7, & 13 \\ -1, & -3, & -5, & -11 \end{Bmatrix} - \begin{Bmatrix} 5, & 13 \\ -1, & -7 \end{Bmatrix} + \begin{Bmatrix} 7, & 11 \\ -3, & -1 \end{Bmatrix}$$

cancel each other with the exception of -1, -3, -5, 7, 7, 7.

30. The theorem may be exhibited conveniently in the following manner.

Taking the above example, we first write down in a central line the doubles of the divisors 6, 5, 3. We add 1 to each of the divisors in the first set, writing the numbers so obtained above them; we then add 3 to the second set, writing the numbers below; then 5 to the next set, writing the numbers above. We have thus formed the scheme:

$$\begin{array}{ccc} 3,\ 5,\ 7,\ 13, & & 7,\ 11, \\ (2,\ 4,\ 6,\ 12), & (2,\ 10), & (2,\ 6), \\ & 5,\ 13. & \end{array}$$

To complete the scheme, we subtract 1 from each of the divisors in the first set, 3 from the second set, 5 from the third set, writing the numbers below and above alternately, and attending only to the absolute values of the numbers (for example, in the second set, subtracting 3 from 2 and 10 we obtain -1 and 7, and we enter 1

---

* In the notation of § 3, the theorem asserts that the numbers given by the formula,

$$G_n\begin{pmatrix} 2d+1 \\ -[2d-1] \end{pmatrix} - G_{n-1}\begin{pmatrix} 2d+3 \\ -[2d-3] \end{pmatrix} + G_{n-3}\begin{pmatrix} 2d+5 \\ -[2d-5] \end{pmatrix} - G_{n-6}\begin{pmatrix} 2d+7 \\ -[2d-7] \end{pmatrix} + \&c.,$$

all cancel each other if n is not a triangular number, and reduce to

$$(-1)^g \left\{1,\ 3,\ 5,\ \dots,\ 2g-1, \text{ and } -(2g+1) \text{ occurring } g \text{ times}\right\},$$

if n is the g^{th} triangular number $\tfrac{1}{2}g\,(g+1)$.

and 7). We thus obtain the system of numbers

$$\begin{array}{ccc} 3,5,7,13, & 1,\ 7, & 7,11, \\ (2,4,6,12), & (2,10), & (2,\ 6), \\ 1,3,5,11, & 5,13, & 3,\ 1. \end{array}$$

The theorem then asserts that, if we cancel the numbers which occur both in the upper and lower lines (ignoring the middle line), we shall have left three 7's in the upper line, and the three uneven numbers inferior to 7 (*i.e.*, 1, 3, 5) in the lower line.

In general, if the numbers derived from the divisors be written down according to this method, the theorem asserts that, if we regard the numbers occurring both in the upper and lower lines as cancelling each other, then this cancellation is complete if n is not a triangular number, but that when n is equal to a triangular number $\frac{1}{2}g(g+1)$, then there are left, after the cancelling, $g\ (2g+1)$'s in one line, and the g uneven numbers inferior to $2g+1$ in the other. If g is even, the $(2g+1)$'s are left in the lower line; if uneven, in the upper line.

Taking as examples the numbers 9 and 10 (as in § 4), and forming the systems of numbers, we obtain, for $n = 9$,

$$\begin{array}{cccc} 3,7,19, & 1,1,\ 5,13, & 7,9,11,17, & 5,\ 1, \\ (2,6,18), & (2,4,\ 8,16), & (2,4,\ 6,12), & (2,\ 6); \\ 1,5,17, & 5,7,11,19, & 3,1,\ 1,\ 7, & 9,13, \end{array}$$

in which all the numbers in the upper and lower lines cancel each other, and, for $n = 10$,

$$\begin{array}{cccc} 3,5,11,21, & 1,3,15, & 7,19, & 5,\ 3,\ 1, \\ (2,4,10,20), & (2,6,18), & (2,14), & (2,\ 4,\ 8), \\ 1,3,\ 9,19, & 5,9,21, & 3,\ 9, & 9,11,15, \end{array}$$

in which, after cancelling all the numbers common to the upper and lower lines, we have left four 9's in the lower line, and the four uneven numbers inferior to 9, viz., 1, 3, 5, 7, in the upper line.

Numerical Theorems relating to the Sums of Powers of the Divisors of n, $n-1$, $n-3$, *&c.* §§ 31–33.

31. By replacing each of the numbers in the theorem of § 29 by the same arbitrary function of itself, and adding together the members of

each group, as in § 9, we find that, if ψ be any even function, then

$$\Sigma_n\{\psi(2d+1)-\psi(2d-1)\}-\Sigma_{n-1}\{\psi(2d+3)-\psi(2d-3)\}$$
$$+\Sigma_{n-3}\{\psi(2d+5)-\psi(2d-5)\}-\Sigma_{n-6}\{\psi(2d+7)-\psi(2d-7)\}$$
$$+\&c.$$

is equal to zero, if n is not a triangular number, and equal to

$$(-1)^{g-1}\{g\psi(2g+1)-\psi(1)-\psi(3)-\psi(5)-\ldots-\psi(2g-1)\}$$

if n is a triangular number $\frac{1}{2}g(g+1)$.

This result may of course be deduced directly from the theorem in § 28, by replacing the cosines by arbitrary functions.

32. As a particular case, let

$$\psi(2d+1) = (2d+1)^2.$$

We thus find

$$\Sigma_n 8d-\Sigma_{n-1}24d+\Sigma_{n-3}40d-\Sigma_{n-6}56d+\&c.$$
$$= 0, \text{ or } (-1)^{g-1}\{g(2g+1)^2-1^2-3^2-5^2-\ldots-(2g-1)^2\},$$

according as n is not a triangular number, or is equal to $\frac{1}{2}g(g+1)$.

Now $\quad 1^2+3^2+5^2+\ldots+(2g-1)^2 = \frac{1}{3}g(2g-1)(2g+1),$

so that $g(2g+1)^2-1^2-3^2-5^2-\ldots-(2g-1)^2 = \frac{4}{3}g(g+1)(2g+1),$

and the formula becomes

$$\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+\&c.$$
$$= 0, \text{ or } (-1)^{g-1}\tfrac{1}{6}g(g+1)(2g+1),$$

according as n is not a triangular number, or is equal to $\frac{1}{2}g(g+1)$. This result is equivalent to the recurring formula quoted in the first section of this paper.

33. Putting $\qquad \psi(2d+1) = (2d+1)^{m+1},$

m being any uneven number, we obtain the general theorem

$$\sigma_m(n)-3\sigma_m(n-1)+5\sigma_m(n-3)-7\sigma_m(n-6)+\&c.$$
$$+\frac{m^{(2)}}{2^2.3!}\{\sigma_{m-2}(n)-3^3\sigma_{m-2}(n-1)+5^3\sigma_{m-2}(n-3)-7^3\sigma_{m-2}(n-6)+\&c.\}$$
$$+\frac{m^{(4)}}{2^4.5!}\{\sigma_{m-4}(n)-3^5\sigma_{m-4}(n-1)+5^5\sigma_{m-4}(n-3)-7^5\sigma_{m-4}(n-6)+\&c.\}$$

... ...

$$+\frac{m^{(2)}}{2^{m-3}.3!}\{\sigma_3(n)-3^{m-2}\sigma_3(n-1)+5^{m-2}\sigma_3(n-3)-7^{m-2}\sigma_3(n-6)+\&c.\}$$

$$+\frac{1}{2^{m-1}}\{\sigma(n)-3^m\sigma(n-1)+5^m\sigma(n-3)-7^m\sigma(n-6)+\&c.\}$$

$$=\Big[(-1)^{g-1}\frac{1}{2^{m+1}(m+1)}\{g(2g+1)^{m+1}-1^{m+1}-3^{m+1}-5^{m+1}-\ldots$$
$$\ldots-(2g-1)^{m+1}\}\Big],$$

where (as in § 11) the square brackets denote that the term enclosed by them only appears when n is equal to $\frac{1}{2}g(g+1)$. Otherwise the right-hand side of the equation is zero.

This result corresponds to the general theorem in § 11. Comparing the two formulæ, we see that in § 11 all the series except the first have one term less, and that the additional term is somewhat simpler in form. On the other hand, the coefficients in the above series are much simpler than in § 11, consisting each of a single power, and all the series are of exactly the same form, whereas in § 11 the first series, on the left-hand side of the equation, is not included in the general form of those on the right-hand side.

The particular cases of $m = 3$ and 5. §§ 34–37.

34. By putting $m = 3$ and 5, we find

$$\sigma_3(n)-3\sigma_3(n-1)+5\sigma_3(n-3)-7\sigma_3(n-6)+\&c.$$
$$+\tfrac{1}{4}\{\sigma(n)-3^3\sigma(n-1)+5^3\sigma(n-3)-7^3\sigma(n-6)+\&c.\}$$
$$=\Big[(-1)^{g-1}\frac{1}{2^4.4}\{g(2g+1)^3-1^4-3^4-5^4-\ldots-(2g-1)^4\}\Big],$$

and

$$\sigma_5(n)-3\sigma_5(n-1)+5\sigma_5(n-3)-7\sigma_5(n-6)+\&c.$$
$$+\tfrac{5}{8}\{\sigma_3(n)-3^3\sigma_3(n-1)+5^3\sigma_3(n-3)-7^3\sigma_3(n-6)+\&c.\}$$
$$+\tfrac{1}{16}\{\sigma(n)-3^5\sigma(n-1)+5^5\sigma(n-3)-7^5\sigma(n-6)+\&c.\}$$
$$=\Big[(-1)^{g-1}\frac{1}{2^6.6}\{g(2g+1)^5-1^6-3^6-5^6-\ldots-(2g-1)\}^6\Big].$$

35. Since

$$1^4+3^4+5^4+\ldots+(2g-1)^4=\tfrac{16}{5}g^5-\tfrac{8}{3}g^3+\tfrac{7}{15}g,$$

we find that the additional term in the first of these two equations is equal to

$$(-1)^{g-1}\left\{\tfrac{1}{5}g^5+\tfrac{1}{2}g^4+\tfrac{5}{12}g^3+\tfrac{1}{8}g^2+\tfrac{1}{120}g\right\},$$

which $$=(-1)^{g-1}\frac{(2g+1)(g^2+g)(12g^2+12g+1)}{120}.$$

In a similar manner, since

$$1^6+3^6+5^6+\ldots+(2g-1)^6=\tfrac{64}{7}g^7-16g^5+\tfrac{28}{3}g^3-\tfrac{31}{21}g,$$

we find that the additional term in the second equation

$$=(-1)^{g-1}\left\{\tfrac{1}{7}g^7+\tfrac{1}{2}g^6+\tfrac{2}{3}g^5+\tfrac{5}{12}g^4+\tfrac{19}{144}g^3+\tfrac{1}{32}g^2+\tfrac{13}{2016}g\right\}$$

$$=(-1)^{g-1}\frac{(2g+1)(g^2+g)\left\{144\,(g^2+g)^2+24\,(g^2+g)+13\right\}}{2016}.$$

36. Expressed in the notation of §§ 16 and 17, the formulæ become

$$\Sigma\,(-1)^i(2i+1)\,\sigma_3\left\{n-\tfrac{1}{2}i\,(i+1)\right\}+\tfrac{1}{4}\Sigma\,(-1)^i(2i+1)^3\sigma\left\{n-\tfrac{1}{2}i(i+1)\right\}$$
$$=\left[(-1)^{g-1}\tfrac{1}{120}g\,(g+1)(2g+1)(12g^2+12g+1)\right],$$

and

$$\Sigma\,(-1)^i(2i+1)\,\sigma_5\left\{n-\tfrac{1}{2}i(i+1)\right\}+\tfrac{5}{8}\Sigma\,(-1)^i(2i+1)^3\sigma_3\left\{n-\tfrac{1}{2}i\,(i+1)\right\}$$
$$+\tfrac{1}{16}\Sigma\,(-1)^i(2i+1)^5\,\sigma\left\{n-\tfrac{1}{2}i\,(i+1)\right\}$$
$$=\left[(-1)^{g-1}\tfrac{1}{2016}\,g\,(g+1)(2g+1)\left\{144\,(g^2+g)^2+24\,(g^2+g)+13\right\},\right.$$

the summations extending from $i=0$ to $i=h$, where $\frac{1}{2}h\,(h+1)$ is the triangular number next inferior to n.

We may evidently dispense with the additional term (the right-hand member of the equations being then zero for all values of n) if we put, in the first formula,

$$\sigma\,(0)=0,\quad \sigma_3\,(0)=\tfrac{2}{5}n^2+\tfrac{1}{60}n;$$

and, in the second formula,

$$\sigma\,(0)=0,\quad \sigma_3\,(0)=0,\quad \sigma_5\,(0)=\tfrac{4}{7}n^3+\tfrac{1}{21}n^2+\tfrac{13}{1008}n.$$

If these terms of zero argument are included, the summations are to extend from $i=0$ to $i=\frac{1}{2}k\,(k+1)$, where $\frac{1}{2}k\,(k+1)$ is the triangular number, nearest to, and not exceeding, n.

37. We may, however, represent the additional term in a more elegant manner (as in §§ 16 and 17) by means of values of $\sigma\,(0)$, $\sigma_3\,(0)$, $\sigma_5\,(0)$, ..., all of which are simply proportional to n. These

values are found to be, for the first formula,

$$\sigma(0) = \tfrac{1}{5}n, \quad \sigma_3(0) = -\tfrac{1}{30}n,$$

and, for the second formula,

$$\sigma(0) = \tfrac{1}{7}n, \quad \sigma_3(0) = -\tfrac{1}{70}n, \quad \sigma_5(0) = \tfrac{1}{63}n.$$

The same Theorems expressed in terms of c and t. §§ 38–40.

38. In the notation of § 18, in which

$$c = 2i+1, \quad t = \tfrac{1}{2}i(i+1),$$

we may express the theorems, in the cases of $m = 1, 3, 5$, in the forms

$$\Sigma\left(\frac{-1}{c}\right) c\sigma(n-t) = 0,$$

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_3(n-t) + \tfrac{1}{4}\Sigma\left(\frac{-1}{c}\right) c^3\sigma(n-t) = 0,$$

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_5(n-t) + \tfrac{5}{8}\Sigma\left(\frac{-1}{c}\right) c^3\sigma_3(n-t) + \tfrac{1}{16}\Sigma\left(\frac{-1}{c}\right) c^5\sigma(n-t) = 0;$$

where, in the first formula, $\sigma(0) = \frac{1}{3}n$; in the second, $\sigma(0) = \frac{1}{5}n$, $\sigma_3(n) = -\frac{1}{30}n$; and in the third, $\sigma(0) = \frac{1}{7}n$, $\sigma_3(0) = -\frac{1}{70}n$, $\sigma_5(0) = \frac{1}{63}n$.

39. It may be observed that the formulæ contained in the general theorems of §§ 11 and 33 are not independent. Thus, writing the formulæ with the additional term, we have, from §§ 16 and 35,

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_3(n-t) + 2\Sigma\left(\frac{-1}{c}\right) ct\sigma(n-t)$$
$$= \left[(-1)^{g-1}\left\{\tfrac{1}{5}g^5 + \tfrac{1}{2}g^4 + \tfrac{1}{3}g^3 - \tfrac{1}{30}g\right\}\right],$$

$$\Sigma\left(\frac{-1}{c}\right) c\sigma_3(n-t) + \tfrac{1}{4}\Sigma\left(\frac{-1}{c}\right) c^3\sigma(n-t)$$
$$= \left[(-1)^{g-1}\left\{\tfrac{1}{5}g^5 + \tfrac{1}{2}g^4 + \tfrac{5}{12}g^3 + \tfrac{1}{8}g^2 + \tfrac{1}{120}g\right\}\right];$$

whence, by subtraction, since $c^2 = 8t+1$, we obtain

$$\tfrac{1}{4}\Sigma\left(\frac{-1}{c}\right) c\sigma(n-t) = \left[(-1)^{g-1}\left\{\tfrac{1}{12}g^3 + \tfrac{1}{8}g^2 + \tfrac{1}{24}g\right\}\right];$$

that is, $$\Sigma\left(\frac{-1}{c}\right) c\sigma(n-t) = \left[(-1)^{g-1}\left\{\tfrac{1}{3}g^3 + \tfrac{1}{2}g^2 + \tfrac{1}{6}g\right\}\right],$$

which is the original recurring formula of § 1.

40. Similarly, from §§ 17 and 19, we have

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_5(n-t)+\tfrac{20}{3}\Sigma\left(\frac{-1}{c}\right)ct\,\sigma_3(n-t)+4\Sigma\left(\frac{-1}{c}\right)(ct^2-\tfrac{1}{6}ct)\,\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{7}g^7+\tfrac{1}{2}g^6+\tfrac{1}{2}g^5-\tfrac{1}{6}g^3+\tfrac{1}{42}g\right\}\right],$$

and, from §§ 35 and 38,

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_5(n-t)+\tfrac{5}{6}\Sigma\left(\frac{-1}{c}\right)c\,(8t+1)\,\sigma_3(n-t)$$
$$+\tfrac{1}{16}\Sigma\left(\frac{-1}{c}\right)c\,(64t^2+16t+1)\,\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{7}g^7+\tfrac{1}{2}g^6+\tfrac{2}{3}g^5+\tfrac{5}{12}g^4+\tfrac{19}{144}g^3+\tfrac{1}{32}g^2+\tfrac{13}{1008}g\right\}\right];$$

giving, by subtraction,

$$\tfrac{5}{6}\Sigma\left(\frac{-1}{c}\right)c\sigma_3(n-t)+\tfrac{5}{3}\Sigma\left(\frac{-1}{c}\right)(ct+\tfrac{3}{80}c)\,\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{6}g^5+\tfrac{5}{12}g^4+\tfrac{43}{144}g^3+\tfrac{1}{32}g^2-\tfrac{5}{288}g\right\}\right];$$

that is,

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_3(n-t)+2\Sigma\left(\frac{-1}{c}\right)(ct+\tfrac{3}{80}c)\,\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{5}g^5+\tfrac{1}{2}g^4+\tfrac{43}{120}g^3+\tfrac{3}{80}g^2-\tfrac{1}{48}g\right\}\right].$$

Now $\tfrac{3}{40}\Sigma\left(\frac{-1}{c}\right)c\sigma(n-t)=\left[(-1)^{g-1}\left\{\tfrac{1}{40}g^3+\tfrac{3}{80}g^2+\tfrac{1}{80}g\right\}\right];$

whence, subtracting, we find

$$\Sigma\left(\frac{-1}{c}\right)c\sigma_3(n-t)+2\Sigma\left(\frac{-1}{c}\right)ct\,\sigma(n-t)$$
$$=\left[(-1)^{g-1}\left\{\tfrac{1}{5}g^5+\tfrac{1}{2}g^4+\tfrac{1}{3}g^3-\tfrac{1}{30}g\right\}\right]$$
$$=\left[(-1)^{g-1}\left\{1^4+2^4+3^4+\ldots+g^4\right\}\right],$$

which is equivalent to the second formula of § 19.

Method of representing the Additional Term by constant values of $\sigma(0)$, $\sigma_3(0)$, *&c.* §§ 41–43.

41. The additional term in the general theorem of § 33 may be represented in the following manner:—

Let $f = 2g+1$, so that f is the coefficient of $\sigma_m(0)$, when it occurs. The additional term may then be written

$$\left[(-1)^{\frac{1}{2}(f+1)}\frac{1}{2^{m+1}(m+1)}\left\{\tfrac{1}{2}(f+1)f^{m+1}-1^{m+1}-3^{m+1}-5^{m+1}-\ldots-f^{m+1}\right\}\right].$$

Now the sum of the series $1^{m+1}+3^{m+1}+5^{m+1}+\ldots+f^{m+1}$ is (see § 53)

$$\tfrac{1}{2}\frac{f^{m+2}}{m+2}+\tfrac{1}{2}f^{m-1}+(m+1)^{(1)}B_1f^m-2^2\frac{(m+1)^{(3)}}{3\,!}\frac{B_2}{2}f^{m-2}$$
$$+2^4\frac{(m+1)^{(5)}}{5\,!}\frac{B_3}{3}f^{m-4}-\ldots\pm2^{m-1}(m+1)^{(1)}\frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}f.$$

The additional term is therefore equal to

$$\left[(-1)^{\frac{1}{2}(f+1)}\left\{\frac{f^{m+2}}{2^{m+2}(m+2)}-B_1\frac{f^m}{2^{m+1}}+\frac{m^{(2)}}{3\,!}\frac{B_2}{2}\frac{f^{m-2}}{2^{m-1}}\right.\right.$$
$$\left.\left.-\frac{m^{(4)}}{5\,!}\frac{B_3}{3}\frac{f^{m-4}}{2^{m-3}}-\ldots\pm\frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}\frac{f}{2^2}\right\}\right].$$

By comparing this expression with the left-hand member of the formula in § 33, it is evident that we may represent all the terms after the first by putting

$$\sigma(0)=-\tfrac{1}{4}B_1,\quad\sigma_3(0)=\tfrac{1}{4}\frac{B_2}{2},\quad\sigma_5(0)=-\tfrac{1}{4}\frac{B_3}{3},\ldots,$$
$$\sigma_m(0)=(-1)^{\frac{1}{2}(m+1)}\tfrac{1}{4}\frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)},$$

the general value being

$$\sigma_{2r-1}(0)=(-1)^r\tfrac{1}{4}\frac{B_r}{r}.$$

When these constant values are assigned to $\sigma(0)$, $\sigma_3(0),\ldots,\sigma_m(0)$, the additional term reduces to the monomial expression

$$\left[(-1)^{g-1}\frac{(2g+1)^{m+2}}{2^{m+2}(m+2)}\right].$$

It is to be noticed that the above values of $\sigma(0)$, $\sigma_3(0),\ldots,\sigma_m(0)$ are absolute numerical constants, being independent both of n and m.

42. We may cause the additional term to disappear entirely by assigning to one of the quantities $\sigma(0)$, $\sigma_3(0)$, ... a value depending upon f and m, all the others retaining their constant values.

For example, the additional term disappears (the right-hand member of the equation being then always zero) if we put

$$\sigma(0) = -\tfrac{1}{4}B_1 + \frac{f^2}{8(m+2)} = \tfrac{1}{8}\left(\frac{f^2}{m+2} - \tfrac{1}{3}\right).$$

Since $f^2 - 1 = 4(g^2+g) = 8n$, this value of $\sigma(0)$ may be written

$$\sigma(0) = \frac{n}{m+2} - \tfrac{1}{24}\frac{m-1}{m+2}.$$

Thus, for $m = 1$, $\sigma(0) = \tfrac{1}{3}n$;

" $m = 3$, $\sigma(0) = \tfrac{1}{5}n - \tfrac{1}{60}$;

" $m = 5$, $\sigma(0) = \tfrac{1}{7}n - \tfrac{1}{42}$;

" $m = 7$, $\sigma(0) = \tfrac{1}{9}n - \tfrac{1}{36}$;

&c., &c.

43. Taking the particular cases of $m = 3$ and 5 (§§ 34 and 37), we have, therefore, for all values of n,

$$\sigma_3(n) - 3\sigma_3(n-1) + 5\sigma_3(n-3) - 7\sigma_3(n-6) + \&c.$$

$$+\tfrac{1}{4}\{\sigma(n) - 3^3\sigma(n-1) + 5^3\sigma(n-3) - 7^3\sigma(n-6) + \&c.\} = 0,$$

if $\sigma(0) = \tfrac{1}{5}n - \tfrac{1}{60}$, $\sigma_3(0) = \tfrac{1}{240}$,

and $\sigma_5(n) - 3\sigma_5(n-1) + 5\sigma_5(n-3) - 7\sigma_5(n-6) + \&c.$

$$+\tfrac{5}{6}\{\sigma_3(n) - 3^3\sigma_3(n-1) + 5^3\sigma_3(n-3) - 7^3\sigma_3(n-6) + \&c.\}$$

$$+\tfrac{1}{16}\{\sigma(n) - 3^5\sigma(n-1) + 5^5\sigma(n-3) - 7^5\sigma(n-6) + \&c.\} = 0,$$

if $\sigma(0) = \tfrac{1}{7}n - \tfrac{1}{42}$, $\sigma_3(0) = \tfrac{1}{240}$, $\sigma_5(0) = -\tfrac{1}{504}$.*

Formula connecting the Sums of the Divisors of the First n Numbers. § 44.

44. In a paper† in the fifth volume of the *Proceedings of the Cambridge Philosophical Society*, I gave the following formula which

* The general theorem in § 33 and the method of representing the additional term in §§ 41 and 42 are given without proof in the *Messenger*, Vol. xx., pp. 176–191, in connexion with the investigation referred to in the note to § 53.

† "On the Sum of the Divisors of a Number" (1884, p. 108). The formula is given on p. 112. It also occurs in *Proc. Lond. Math. Soc.*, Vol. xv., p. 118 (1884).

connects together the sums of the divisors of all the numbers from unity to n:

$$\sigma(n)-2\sigma(n-1)-2\sigma(n-2)+3\sigma(n-3)+3\sigma(n-4)+3\sigma(n-5)$$
$$-4\sigma(n-6)-4\sigma(n-7)-4\sigma(n-8)-4\sigma(n-9)+5\sigma(n-10)+\ldots$$
$$\ldots+5\sigma(n-14)-\ldots+(-1)^{r-1}r\sigma(1)=(-1)^{r}\tfrac{1}{6}(s^{3}-s).$$

In this formula the first term $\sigma(n)$ has the coefficient unity, the next two terms have 2 as coefficient, the next three have 3, and so on. The letter s denotes what the coefficient of $\sigma(0)$ would be if the series were continued one term further. Thus s is equal to r, unless $r\sigma(1)$ is the last term of the group of r terms having r as coefficient, in which case s is equal to $r+1$.

I now proceed to investigate a relation connecting together the actual divisors of all the numbers from unity to n, which includes the above formula as a particular case.

General Theorem relating to the actual Divisors of the First n Numbers. §§ 45–50.

45. From § 28,

$$2(\sin x-q\sin 3x+q^{3}\sin 5x-q^{6}\sin 7x+\&c.)\,\Sigma_1^\infty\sigma(\sin 2nx)\,q^{n}$$
$$=(\cos x-\cos 3x)\,q-(\cos x+\cos 3x-2\cos 5x)\,q^{3}$$
$$+(\cos x+\cos 3x+\cos 5x-3\cos 7x)\,q^{6}-\&c.$$

Now, by multiplying the series

$$\sin x-q\sin 3x+q^{3}\sin 5x-\&c.$$

by
$$1+q+q^{2}+q^{3}+q^{4}+q^{5}+\&c.,$$

we obtain the series

$$\sin x-(\sin 3x-\sin x)(q+q^{2})+(\sin 5x-\sin 3x+\sin x)(q^{3}+q^{4}+q^{5})$$
$$-(\sin 7x-\sin 5x+\sin 3x-\sin x)(q^{6}+q^{7}+q^{8}+q^{9})+\&c.,$$

which is equal to

$$\frac{1}{2\cos x}\{\sin 2x-(q+q^{2})\sin 4x+(q^{3}+q^{4}+q^{5})\sin 6x$$
$$-(q^{6}+\ldots+q^{9})\sin 8x+\&c.\};$$

and, by multiplying the series

$$(\cos x - \cos 3x)\, q - (\cos x + \cos 3x - 2\cos 5x)\, q^3 + \&c.$$

by $$1 + q + q^2 + q^3 + q^4 + q^5 + \&c.,$$

we obtain the series

$$(\cos x - \cos 3x)(q + q^2) - (2\cos 3x - 2\cos 5x)(q^3 + q^4 + q^5)$$
$$+ (\cos x - \cos 3x + 3\cos 5x - 3\cos 7x)(q^6 + q^7 + q^8 + q^9)$$
$$- (2\cos 3x - 2\cos 5x + 4\cos 7x - 4\cos 9x)(q^{10} + \dots + q^{14})$$
$$+ \&c.$$

Thus, by multiplying both sides of the above equation by

$$1 + q + q^2 + q^3 + \&c.,$$

we obtain the equation

$$\{\sin 2x - (q + q^2)\sin 4x + (q^3 + q^4 + q^5)\sin 6x - \&c.\}\, \Sigma_1^\infty \sigma(\sin 2nx)\, q^n$$
$$= \cos x\{(\cos x - \cos 3x)(q + q^2) - (2\cos 3x - 2\cos 5x)(q^3 + q^4 + q^5) - \&c.\},$$

the right-hand member of which

$$= \tfrac{1}{2}(1 - \cos 4x)(q + q^2) - \tfrac{1}{2}(2\cos 2x - 2\cos 6x)(q^3 + q^4 + q^5)$$
$$+ \tfrac{1}{2}(1 + 2\cos 4x - 3\cos 8x)(q^6 + q^7 + q^8 + q^9) - \&c.$$

We thus find, by putting $\tfrac{1}{2}x$ for x,

$$2\{\sin x - (q + q^2)\sin 2x + (q^3 + q^4 + q^5)\sin 3x - (q^6 + \dots + q^9)\sin 4x + \&c.\}$$
$$\times \Sigma_1^\infty \sigma(\sin nx)\, q^n$$
$$= (1 - \cos 2x)(q + q^2) - (2\cos x - 2\cos 3x)(q^3 + q^4 + q^5)$$
$$+ (1 + 2\cos 2x - 3\cos 4x)(q^6 + \dots + q^9)$$
$$- \&c.,$$

the general term on the right-hand side being

$$\{1 + 2\cos 2x + 2\cos 4x + \dots + 2\cos (r-2)\, x - (r-1)\cos rx\}$$
$$\times (q^{\frac{1}{2}r(r-1)} + \dots + q^{\frac{1}{2}r(r+1)-1}),$$

or $$-\{2\cos x + 2\cos 3x + \dots + 2\cos (r-2)\, x - (r-1)\cos rx\}$$
$$\times (q^{\frac{1}{2}r(r-1)} + \dots + q^{\frac{1}{2}r(r+1)-1}),$$

according as r is even or uneven.

46. The coefficient of q^n on the left-hand side of this equation is

$$\Sigma_n \{\cos(d-1)x - \cos(d+1)x\}$$
$$-(\Sigma_{n-1}+\Sigma_{n-2})\{\cos(d-2)x-\cos(d+2)x\}$$
$$+(\Sigma_{n-3}+\Sigma_{n-4}+\Sigma_{n-5})\{\cos(d-3)x-\cos(d+3)x\}$$
$$-\&c.;$$

so that, using the notation of § 29, in which $[a]$ denotes the absolute magnitude of a, irrespective of sign, we find that the numbers given by the formula,

$$G_n\begin{pmatrix} d+1 \\ -[d-1]\end{pmatrix}-(G_{n-1}+G_{n-2})\begin{pmatrix} d+2 \\ -[d-2]\end{pmatrix}$$
$$+(G_{n-3}+G_{n-4}+G_{n-5})\begin{pmatrix} d+3 \\ -[d-3]\end{pmatrix}-\&c.,$$

all cancel one another with the exception of

-0, two (-2)'s, two (-4)'s, ..., two $\{-(p-2)\}$'s, and $(p-1)$ p's,

if p be even, and

two 1's, two 3's, ..., two $(p-2)$'s and $(p-1)(-p)$'s,

if p be uneven, where $\frac{1}{2}p(p+1)$ is the triangular number next superior to n.* Zeros are to be retained and treated in the same manner as other numbers.

---

* It may be convenient to distinguish between the meanings of the letters h (§ 15), k (§ 16), l (§ 24), and p, which are used in this paper in connexion with the triangular numbers which are adjacent to, or equal, to n. The number $\frac{1}{2}h(h+1)$ is the triangular number next inferior to, n and $\frac{1}{2}p(p+1)$ is the triangular number next superior to n, whether n is a triangular number or not; $\frac{1}{2}k(k+1)$ is the triangular number next inferior to n, if n is not a triangular number, and is equal to n if n is a triangular number; $\frac{1}{2}l(l+1)$ is the triangular number next superior to n, if n is not a triangular number, and is equal to n if n is a triangular number. Thus, if n is not a triangular number,

$$h=k \text{ and } l=p=h+1=k+1,$$

and, if n is a triangular number, $\frac{1}{2}g(g+1)$,

$$h=g-1,\quad k=l=g,\quad p=g+1.$$

In the notation of §§ 5 and 29, the above formula may be written

$$\begin{Bmatrix} \alpha+1, & \beta+1, & \dots \\ -(\alpha-1), & -(\beta-1), & \dots \end{Bmatrix}$$

$$-\begin{Bmatrix} \alpha_1-2, & \beta_1+2, & \dots; & \alpha_2+2, & \beta_2+2, & \dots \\ -[\alpha_1-2], & -[\beta_1-2], & \dots; & -[\alpha_2-2], & -[\beta_2-2], & \dots \end{Bmatrix}$$

$$+\begin{Bmatrix} \alpha_3+3, & \beta_3+3, & \dots; & \alpha_4+3, & \beta_4+3, & \dots; \\ -[\alpha_3-3], & -[\beta_3-3], & \dots; & -[\alpha_4-3], & -[\beta_4-3], & \dots; \end{Bmatrix}$$

$$\begin{matrix} \alpha_5+3, & \beta_5+3, & \dots \\ -[\alpha_5-3], & -[\beta_5-3], & \dots \end{matrix}\Bigg\} - \&c.$$

47. As an example, putting $n=6$, so that $p=4$, the theorem asserts that

$$\begin{Bmatrix} 2, & 3, & 4, & 7 \\ -0, & -1, & -2, & -5 \end{Bmatrix} - \begin{Bmatrix} 3, & 7; & 3, & 4, & 6 \\ -1, & -3; & -1, & -0, & -2 \end{Bmatrix}$$

$$+\begin{Bmatrix} 4, & 6; & 4, & 5; & 4 \\ -2, & -0; & -2, & -1; & -2 \end{Bmatrix}$$

all cancel each other excepting only

$$-0, \ -2, \ -2, \ 4, \ 4, \ 4.$$

Again, putting $n=7$, so that $p=4$ as before, the theorem asserts that

$$\begin{Bmatrix} 2, & 8 \\ -0, & -6 \end{Bmatrix} - \begin{Bmatrix} 3, & 4, & 5, & 8; & 3, & 7 \\ -1, & -0, & -1, & -4; & -1, & -3 \end{Bmatrix}$$

$$+\begin{Bmatrix} 4, & 5, & 7; & 4, & 6; & 4, & 5 \\ -2, & -1, & -1; & -2, & -0; & -2, & -1 \end{Bmatrix} - \begin{Bmatrix} 5 \\ -3 \end{Bmatrix}$$

all cancel each other excepting only

$$-0, \ -2, \ -2, \ 4, \ 4, \ 4.$$

48. The theorem may be most conveniently exhibited in a form similar to that by which the theorem of § 29 was expressed in § 30. We first write down in the central line all the divisors of the numbers $n, n-1, \dots, 3, 2, 1$. We then add 1 to each of the divisors of the first number n, writing the numbers so obtained above them; we add 2 to the divisors of the next two numbers, writing the numbers below; we add 3 to the next three sets of divisors, writing the numbers

above; and so on. Thus, in the case of $n=6$, we first form the scheme

2, 3, 4, 7			4, 6,	4, 5,	4,
(1, 2, 3, 6)	(1, 5),	(1, 2, 4)	(1, 3),	(1, 2),	(1).
	3, 7,	3, 4, 6			

To complete the system of numbers, we subtract 1 from each of the divisors of n, writing the numbers below; we subtract 2 from the next two sets, writing the numbers above; 3 from the next three sets, writing the numbers below; and so on: in all cases attending only to the absolute values of the numbers (*i.e.*, ignoring negative signs). Thus, in the above example, we obtain the completed scheme

2, 3, 4, 7	1, 3,	1, 0, 2	4, 6,	4, 5,	4,
(1, 2, 3, 6)	(1, 5),	(1, 2, 4)	(1, 3),	(1, 2),	(1),
0, 1, 2, 5	3, 7,	3, 4, 6	2, 0,	2, 1,	2.

The theorem asserts that, in general, if $\frac{1}{2}p(p+1)$ be the triangular number next superior to n, then, after cancelling the numbers which occur both in the upper and lower lines (ignoring the middle line which contains the divisors themselves), there will be left remaining, if p be even, $(p-1)$ p's in the upper line, and one 0, two 2's, two 4's, ..., and two $(p-2)$'s in the lower line, and, if p be uneven, two 1's, two 3's, ..., two $(p-2)$'s in the upper line, and $(p-1)\,p$'s in the lower line.

Thus, in the above scheme, for which $p=4$, there remain uncancelled three 4's in the upper line, and 0, 2, 2 in the lower line.

49. As additional examples, let $n=8$ and 9, the value of p being therefore 4 in each case. The schemes are

2, 3, 5, 9	1, 5,	1, 0, 1, 4	4, 8,	4, 5, 7,	4, 6	3, 2,	3
(1, 2, 4, 8)	(1, 7),	(1, 2, 3, 6)	(1, 5),	(1, 2, 4),	(1, 3)	(1, 2),	(1)
0, 1, 3, 7	3, 9,	3, 4, 5, 8	2, 2,	2, 1, 1,	2, 0	5, 6,	5

and

2, 4, 10	1, 0, 2, 6,	1, 5	4, 5, 6, 9,	4, 8,	4, 5, 7
(1, 3, 9)	(1, 2, 4, 8),	(1, 7)	(1, 2, 3, 6),	(1, 5),	(1, 2, 4)
0, 2, 8	3, 4, 6, 10,	3, 9	2, 1, 0, 3,	2, 2,	2, 1, 1

3, 1,	3, 2,	3
(1, 3),	(1, 2),	(1)
5, 7,	5, 6,	5.

In each of these schemes, after the cancelling, three 4's are left in the upper line, and 0, 2, 2 in the lower.

For $n = 10$, the scheme is

2, 3, 6, 11	1, 1, 7,	1, 0, 2, 6	4, 10,	4, 5, 6, 9,	4, 8
(1, 2, 5, 10)	(1, 3, 9),	(1, 2, 4, 8)	(1, 7),	(1, 2, 3, 6),	(1, 5)
0, 1, 4, 9	3, 5, 11,	3, 4, 6, 11	2, 4,	2, 1, 0, 3,	2, 2

3, 2, 0,	3, 1,	3, 2,	3
(1, 2, 4),	(1, 3),	(1, 2),	(1)
5, 6, 8,	5, 7,	5, 6,	5;

and, after the cancelling, 1, 1, 3, 3 are left in the upper line, and four 5's in the lower line, as should be the case since $p = 5$.

50. In forming the schemes we divide the numbers beginning with n and proceeding downwards into sets of one, two, three, &c. The last set may be incomplete, as in the cases of $n = 8$ and 9 (when it consists of 2 and 3 numbers respectively), or complete, as in the case of $n = 10$ (when it consists of the full number 4). In the former case (*i.e.*, when the set is incomplete) p is equal to the full number of numbers which would belong to the set; but when the set is complete, p is equal to the number of terms belonging to the set, increased by unity. Thus, for $n = 8$ or 9, $p = 4$; but for $n = 10$, $p = 5$.

The preceding theorem (§§ 46–48) seems to me to be the most interesting of the results contained in this paper, as it connects together in so simple a manner all the actual divisors of the numbers from 1 to n.

Numerical Theorems relating to the Sums of Powers of the Divisors of the First n Numbers. §§ 51–56.

51. By proceeding as in § 31, we see that, if ψ be any even function,

$$\Sigma_n \{\psi(d+1)-\psi(d-1)\} - (\Sigma_{n-1}+\Sigma_{n-2}) \{\psi(d+2)-\psi(d-2)\}$$
$$+ (\Sigma_{n-3}+\Sigma_{n-4}+\Sigma_{n-5}) \{\psi(d+3)-\psi(d-3)\} - \&c.$$

is equal to

$$-\ \psi(0) - 2\psi(2) - 2\psi(4) - \dots - 2\psi(p-2) + (p-1)\,\psi(p),$$

or $$2\psi(1) + 2\psi(3) + 2\psi(5) + \dots + 2\psi(p-2) - (p-1)\,\psi(p),$$

according as p is even or uneven, $\frac{1}{2}p(p+1)$ being the triangular number next superior to n.

We may also define p as the number of terms in the complete set or group of terms to which Σ_0 would belong, if the series were continued one term further (so as to include Σ_0). It is evident that Σ_0 will belong to the same set as Σ_1 except when Σ_1 is the last term of its set. As the series is supposed to be continued to Σ_0 merely to determine p, the value zero is, of course, to be assigned to this term.

52. As a particular case, putting

$$\psi(d+1) = (d+1)^2,$$

we obtain the theorem

$$\Sigma_n 4d - (\Sigma_{n-1} + \Sigma_{n-2})\, 8d + (\Sigma_{n-3} + \Sigma_{n-4} + \Sigma_{n-5})\, 24d - \&c.$$

$$= -2\{2^2+4^2+6^2+\ldots+(p-2)^2\} + (p-1)\,p^2,$$

or

$$2\{1^2+3^2+5^2+\ldots+(p-2)^2\} - (p-1)\,p^2,$$

according as p is even or uneven.

The right-hand member of this equation may be written

$$-2\,(2^2+4^2+6^2+\ldots+p^2)+p^3+p^2,$$

or

$$2\,(1^2+3^2+5^2+\ldots+p^2)-p^3-p^2,$$

according as p is even or uneven.

Now, p being even,

$$2^2+4^2+6^2+\ldots+p^2 = \tfrac{1}{6}p^3+\tfrac{1}{2}p^2+\tfrac{1}{3}p,$$

and, p being uneven,

$$1^2+3^2+5^2+\ldots+p^2 = \tfrac{1}{6}p^3+\tfrac{1}{2}p^2+\tfrac{1}{3}p\,;$$

so that, whether p be even or uneven, we find

$$\Sigma_n d - 2\,(\Sigma_{n-1}+\Sigma_{n-2})\, d + 3\,(\Sigma_{n-3}+\Sigma_{n-4}+\Sigma_{n-5})\, d - \&c. = (-1)^p\,\tfrac{1}{6}\,(p^3-p),$$

which is the same as the formula quoted in § 44.

53. In general, putting

$$\psi(d+1) = (d+1)^m,$$

m being any uneven number, the left-hand member of the equation becomes

$$2(m+1)^{(1)}\Big[\sigma_m(n)-2\{\sigma_m(n-1)+\sigma_m(n-2)\}$$
$$+3\{\sigma_m(n-3)+\sigma_m(n-4)+\sigma_m(n-5)\}$$
$$-4\{\text{next four}\}+5\{\ldots\}-\&c.\Big]$$
$$+2\frac{(m+1)^{(3)}}{3!}\Big[\sigma_{m-2}(n)-2^3\{\sigma_{m-2}(n-1)+\sigma_{m-2}(n-2)\}$$
$$+3^3\{\text{next three}\}-4^3\{\ldots\}+\&c.\Big]$$
$$+2\frac{(m+1)^{(5)}}{5!}\Big[\sigma_{m-4}(n)-2^5\{\sigma_{m-4}(n-1)+\sigma_{m-4}(n-2)\}+3^5\{\ldots\}-\&c.\Big]$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$+2(m+1)^{(1)}\Big[\sigma(n)-2^m\{\sigma(n-1)+\sigma(n-2)\}+3^m\{\ldots\}-\&c.\Big],$$

where, as before, $m^{(r)}$ denotes the factorial $m(m-1)\ldots(m-r+1)$.

The right-hand member of the equation is

$$-2\{2^{m+1}+4^{m+1}+6^{m+1}+\ldots+p^{m+1}\}+p^{m+2}+p^{m+1},$$

or $$2\{1^{m+1}+3^{m+1}+5^{m+1}+\ldots+p^{m+1}\}-p^{m+2}-p^{m+1},$$

according as p is even or uneven.

Now it can be shown* that the expression

$$\tfrac{1}{2}\frac{p^{m+2}}{m+2}+\tfrac{1}{2}p^{m+1}+(m+1)^{(1)}B_1p^m-2^2\frac{(m+1)^{(3)}}{3!}\frac{B_2}{2}p^{m-2}$$
$$+2^4\frac{(m+1)^{(5)}}{5!}\frac{B_3}{3}p^{m-4}-\ldots\pm 2^{m-1}(m+1)^{(1)}\frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}p$$

represents the value of the series

$$2^{m+1}+4^{m+1}+6^{m+1}+\ldots+p^{m+1},$$

if p be even, and the value of the series

$$1^{m+1}+3^{m+1}+5^{m+1}+\ldots+p^{m+1},$$

if p be uneven.

---

* The proof of this theorem forms the subject of a paper, "Note on the Sums of Even Powers of Even and Uneven Numbers" (*Messenger*, Vol. xx., pp. 172–176), which contains the investigations to which I was led in obtaining the values of the series which form the right-hand member of the above equation.

We find, therefore, that, both for even and uneven values of p, the right-hand member of the equation is equal to

$$(-1)^p \left\{ \frac{m+1}{m+2} p^{m+2} - 2(m+1)^{(1)} B_1 p^m + 2^3 \frac{(m+1)^{(3)}}{3!} \frac{B_2}{2} p^{m-2} \right.$$
$$\left. - 2^5 \frac{(m+1)^{(5)}}{5!} \frac{B_3}{3} p^{m-4} + \&c. \right\}.$$

54. By dividing throughout by $2(m+1)$, we thus obtain the theorem

$$\sigma_m(n) - 2\{\sigma_m(n-1) + \sigma_m(n-2)\}$$
$$+ 3\{\sigma_m(n-3) + \sigma_m(n-4) + \sigma_m(n-5)\} - \&c.$$
$$+ \frac{m^{(2)}}{3!}\left[\sigma_{m-2}(n) - 2^3\{\sigma_{m-2}(n-1) + \sigma_{m-2}(n-2)\} + 3^3\{\dots\} - \&c.\right]$$
$$+ \frac{m^{(4)}}{5!}\left[\sigma_{m-4}(n) - 2^5\{\sigma_{m-4}(n-1) + \sigma_{m-4}(n-2)\} + 3^5\{\dots\} - \&c.\right]$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$+ \frac{m^{(2)}}{3!}\left[\sigma_3(n) - 2^{m-2}\{\sigma_3(n-1) + \sigma_3(n-2)\} + 3^{m-2}\{\dots\} - \&c.\right]$$
$$+ \sigma(n) - 2^m\{\sigma(n-1) + \sigma(n-2)\}$$
$$+ 3^m\{\sigma(n-3) + \sigma(n-4) + \sigma(n-5)\} - \&c.$$
$$= (-1)^p \left\{ \frac{p^{m+2}}{2(m+2)} - B_1 p^m + 2^2 \frac{m^{(2)}}{3!} \frac{B_2}{2} p^{m-2} - 2^4 \frac{m^{(4)}}{5!} \frac{B_3}{3} p^{m-4} + \dots \right.$$
$$\left. \dots \pm 2^{m-1} \frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)} p \right\}.$$

55. As in § 41, we may represent all the terms on the right-hand side of the equation, with the exception of the first, by assigning to $\sigma(0)$, $\sigma_3(0)$, ..., $\sigma_m(0)$ numerical values which are independent of both n and m. These values are

$$\sigma(0) = -B_1, \quad \sigma_3(0) = 2^2 \frac{B_2}{2}, \quad \sigma_5(0) = -2^4 \frac{B_3}{3}, \dots,$$

$$\sigma_{2r-1}(0) = (-1)^r 2^{2r-2} \frac{B_r}{r}, \dots, \sigma_m(0) = (-1)^{\frac{1}{2}(m+1)} 2^{m-1} \frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}.$$

Assigning these values to $\sigma(0)$, $\sigma_3(0)$, ..., $\sigma_m(0)$, we may write the theorem in the form:

$$\sigma_m(n)-2\{\sigma_m(n-1)+\sigma_m(n-2)\}+3\{\ldots\}-\ldots$$
$$\ldots+(-1)^{p-1}p\{\ldots+\sigma_m(0)\}$$
$$+\frac{m^{(2)}}{3!}\left[\sigma_{m-2}(n)-2^3\{\sigma_{m-2}(n-1)+\sigma_{m-2}(n-2)\}\right.$$
$$\left.+3^3\{\ldots\}-\ldots+(-1)^{p-1}p^3\{\ldots+\sigma_{m-2}(0)\}\right]$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+\frac{m^{(2)}}{3!}\left[\sigma_3(n)-2^{m-2}\{\sigma(n-1)+\sigma(n-2)\}\right.$$
$$\left.+3^{m-2}\{\ldots\}-\ldots+(-1)^{p-1}p^{m-2}\{\ldots+\sigma_3(0)\}\right]$$
$$+\sigma(n)-2^m\{\sigma(n-1)+\sigma(n-2)\}+3^m\{\ldots\}-\ldots$$
$$\ldots+(-1)^{p-1}p^m\{\ldots+\sigma(0)\}$$
$$=(-1)^p\frac{p^{m+2}}{2(m+2)}.^*$$

56. As examples of this general theorem, putting $m=3$ and 5, we have

$$\sigma_3(n)-2\{\sigma_3(n-1)+\sigma_3(n-2)\}+3\{\ldots\}-\ldots$$
$$\ldots+(-1)^{p-1}p\{\ldots+\sigma_3(0)\}$$
$$+\sigma(n)-2^3\{\sigma(n-1)+\sigma(n-2)\}+3^3\{\ldots\}-\ldots$$
$$\ldots+(-1)^{p-1}p^3\{\ldots+\sigma(0)\}$$
$$=(-1)^p\tfrac{1}{10}p^5,$$

where $\sigma(0)=-\frac{1}{6}$, $\sigma_0(0)=\frac{1}{15}$,

and

$$\sigma_5(n)-2\{\sigma_5(n-1)+\sigma_5(n-2)\}+3\{\ldots\}-\ldots$$
$$\ldots+(-1)^{p-1}p\{\ldots+\sigma_5(0)\}$$
$$+\tfrac{10}{3}\left[\sigma_3(n)-2^3\{\sigma_3(n-1)+\sigma_3(n-2)\}+3^3\{\ldots\}-\ldots\right.$$
$$\left.\ldots+(-1)^{p-1}p^3\{\ldots+\sigma(0)\}\right]$$
$$+\sigma(n)-2^5\{\sigma(n-1)+\sigma(n-2)\}+3^5\{\ldots\}-\ldots$$
$$\ldots+(-1)^{p-1}p^5\{\ldots+\sigma(0)\}$$
$$=(-1)^p\tfrac{1}{14}p^7,$$

where $\sigma(0)=-\frac{1}{6}$, $\sigma_3(0)=\frac{1}{15}$, $\sigma_5(0)=-\frac{8}{63}$.

---

* We may evidently dispense with this term by putting

$$\sigma(0)=\frac{p^2}{2(m+2)}-\tfrac{1}{6};$$

$\sigma_3(0)$, $\sigma_5(0)$, ..., retaining their constant values as above. We may regard p as defined by the fact that p^m is the coefficient of $\sigma(0)$.

Second Theorem connecting the actual Divisors of the First n Numbers.
§§ 57–60.

57. The series

$$\sin x-(\sin 3x-\sin x)(q+q^2)$$
$$+(\sin 5x-\sin 3x+\sin x)(q^3+q^4+q^5)-\&c.,$$

which was obtained (§ 45) by multiplying $\sin x-q\sin 3x+q^3\sin 5x-\&c.$ by $1+q+q^2+q^3+\&c.$, may be expressed in the form

$$\sin x\{1-2\cos 2x\,(q+q^2)+(1+2\cos 4x)(q^3+q^4+q^5)$$
$$-(2\cos 2x+2\cos 6x)(q^6+\ldots+q^9)+\&c.\},$$

and the series

$$(\cos x-\cos 3x)(q+q^2)-(2\cos 3x-2\cos 5x)(q^3+q^4+q^5)+\&c.,$$

which was obtained (§ 45) by multiplying

$$(\cos x-\cos 3x)\,q-(\cos x+\cos 3x-2\cos 5x)\,q^3+\&c.$$

by $$1+q+q^2+q^3+\&c.,$$

may be expressed in the form

$$2\sin x\{\sin 2x\,(q+q^2)-2\sin 4x\,(q^3+q^4+q^5)$$
$$+(\sin 2x+3\sin 6x)(q^6+\ldots+q^9)-\&c.\}.$$

Putting $\frac{1}{2}x$ for x, we thus obtain the equation

$$\{1-2\cos x\,(q+q^2)+(1+2\cos 2x)(q^3+q^4+q^5)$$
$$-(2\cos x+2\cos 3x)(q^6+\ldots+q^9)+(1+2\cos 2x+2\cos 4x)(q^{10}+\ldots+q^{14})$$
$$-(2\cos x+2\cos 3x+2\cos 5x)(q^{15}+\ldots+q^{20})$$
$$+(1+2\cos 2x+2\cos 4x+2\cos 6x)(q^{21}+\ldots+q^{27})-\&c.\}\,\Sigma_1^\infty\,\sigma(\sin nx)\,q^n$$
$$=\sin x\,(q+q^2)-2\sin 2x\,(q^3+q^4+q^5)+(\sin x+3\sin 3x)(q^6+\ldots+q^9)$$
$$-(2\sin 2x+4\sin 4x)(q^{10}+\ldots+q^{14})$$
$$+(\sin x+3\sin 3x+5\sin 5x)(q^{15}+\ldots+q^{20})-\&c.$$

58. The coefficient of q^n on the left-hand side is

$$\Sigma_n\sin dx-(\Sigma_{n-1}+\Sigma_{n-2})\,\{\sin (d\pm 1)\,x\}$$
$$+(\Sigma_{n-3}+\Sigma_{n-4}+\Sigma_{n-5})\,\{\sin dx+\sin (d\pm 2)\,x\}$$
$$-(\Sigma_{n-6}+\ldots+\Sigma_{n-9})\,\{\sin (d\pm 1)\,x+\sin (d\pm 3)\,x\}$$
$$+(\Sigma_{n-10}+\ldots+\Sigma_{n-14})\,\{\sin dx+\sin (d\pm 2)\,x+\sin (d\pm 4)\,x\}$$
$$-\&c.,$$

and we see, therefore, that the numbers given by the formula

$$\begin{aligned} G_n(d) &- (G_{n-1}+G_{n-2})(d\pm 1) + (G_{n-3}+G_{n-4}+G_{n-5})(d,\ d\pm 2) \\ &- (G_{n-6}+\dots+G_{n-9})(d\pm 1,\ d\pm 3) \\ &+ (G_{n-10}+\dots+G_{n-14})(d,\ d\pm 2,\ d\pm 4) - \&c. \end{aligned}$$

all cancel each other, with the exception of

one 1, three 3's, five 5's, ..., $(p-1)(p-1)$'s, if p be even,

and

two (-2)'s, four (-4)'s, six (-6)'s,..., $(p-1)\{-(p-1)\}$'s, if p be uneven,

where $\frac{1}{2}p\,(p+1)$ is the triangular number next superior to n. Zeros are not to be taken account of.

59. As an example, putting $n = 9$, the theorem asserts that the numbers

$$\{1,\ 3,\ 9\} - \left\{\begin{array}{ll} 2,\ 3,\ 5,\ 9, & 2,\ 8 \\ (1,\ 2,\ 4,\ 8), & (1,\ 7) \\ 0,\ 1,\ 3,\ 7, & 0,\ 6 \end{array}\right\} + \left\{\begin{array}{lll} 3,\ 4,\ 5,\ 8, & 3,\ 7, & 3,\ 4,\ 6 \\ 1,\ 2,\ 3,\ 6, & 1,\ 5, & 1,\ 2,\ 4 \\ -1,\ 0,\ 1,\ 4, & -1,\ 3, & -1,\ 0,\ 2 \end{array}\right\}$$

$$- \left\{\begin{array}{llll} 4,\ 6, & 4, & 5, & 4 \\ 2,\ 4, & 2, & 3, & 2 \\ (1,\ 3), & (1, & 2), & (1) \\ 0,\ 2, & 0, & 1, & 0 \\ -2,\ 0, & -2, & -1, & -2 \end{array}\right\}$$

all cancel each other, excepting only 1, 3, 3, 3. The divisors in round brackets are shown for convenience, but are not to be included among the numbers. Thus the numbers given by the formula are

$$\begin{array}{llllll} & & & & & -4,\ -6,\ -4,\ -5,\ -4 \\ & & 3,\ 4,\ 5,\ 8, & 3,\ 7, & 3,\ 4,\ 6, & \\ & -2,\ -3,\ -5,\ -9,\ -2,\ -8, & & & & -2,\ -4,\ -2,\ -3,\ -2 \\ 1,\ 3,\ 9, & & 1,\ 2,\ 3,\ 6, & 1,\ 5, & 1,\ 2,\ 4, & \\ & 0,\ -1,\ -3,\ -7,\ \ 0,\ -6, & & & & 0,\ -2,\ \ 0,\ -1,\ \ 0 \\ & & -1,\ 0,\ 1,\ 4, & -1,\ 3, & -1,\ 0,\ 2, & \\ & & & & & 2,\ \ 0,\ \ 2,\ \ 1,\ \ 2 \end{array}$$

and it is easily seen that, after cancelling, 1, 3, 3, 3 are alone left. Putting $n = 10$, we form the scheme

$$\{1,\ 2,\ 5,\ 10\} - \left\{\begin{array}{cc} 2,\ 4,\ 10, & 2,\ 3,\ 5,\ 9 \\ (1,\ 3,\ 9), & (1,\ 2,\ 4,\ 8) \\ 0,\ 2,\ 8, & 0,\ 1,\ 3,\ 7 \end{array}\right\} + \left\{\begin{array}{ccc} 3,\ 9, & 3,\ 4,\ 5,\ 8, & 3,\ 7 \\ 1,\ 7, & 1,\ 2,\ 3,\ 6, & 1,\ 5 \\ -1,\ 5, & -1,\ 0,\ 1,\ 4, & -1,\ 3 \end{array}\right\}$$

$$-\left\{\begin{array}{cccccccc} 4, & 5, & 7, & 4, & 6, & 4, & 5, & 4 \\ 2, & 3, & 5, & 2, & 4, & 2, & 3, & 2 \\ (1, & 2, & 4), & (1, & 3), & (1, & 2), & (1) \\ 0, & 1, & 3, & 0, & 2, & 0, & 1, & 0 \\ -2, & -1, & 1, & -2, & 0, & -2, & -1, & -2 \end{array}\right\};$$

the numbers given by the formula are therefore

$$\begin{array}{lcr} & & -4,-5,-7,-4,-6,-4,-5,-4 \\ & 3,9,\ \ 3,4,5,8,\ \ 3,7, & \\ -2,-4,-10,-2,-3,-5,-9, & & -2,-3,-5,-2,-4,-2,-3,-2 \\ 1,2,5,10, & 1,7,\ \ 1,2,3,6,\ \ 1,5, & \\ 0,-2,\ -8,\ \ 0,-1,-3,-7, & & 0,-1,-3,\ \ 0,-2,\ \ 0,-1,\ \ 0 \\ & -1,5,-1,0,1,4,-1,3, & \\ & & 2,\ \ 1,-1,\ \ 2,\ \ 0,\ \ 2,\ \ 1,\ \ 2 \end{array}$$

which reduce to

$$-2,\ -2,\ -4,\ -4,\ -4,\ -4,$$

as should be the case, since $p = 5$.

Numerical Theorems relating to the sums of Powers of the First n Numbers. §§ 60–62.

60. From the theorem relating to the actual divisors it follows immediately that, if ϕ be any uneven function,

$$\Sigma_n \phi(d) - (\Sigma_{n-1} + \Sigma_{n-2})\{\phi(d+1) + \phi(d-1)\}$$
$$+ (\Sigma_{n-3} + \Sigma_{n-4} + \Sigma_{n-5})\{\phi(d) + \phi(d+2) + \phi(d-2)\}$$
$$-(\Sigma_{n-6} + \ldots + \Sigma_{n-9})\{\phi(d+1) + \phi(d-1) + \phi(d+3) + \phi(d-3)\}$$
$$+ \&c.$$

is equal to

$$\phi(1) + 3\phi(3) + 5\phi(5) + \ldots + (p-1)\,\phi(p-1), \text{ if } p \text{ be even,}$$

and to

$$-2\phi(2)-4\phi(4)-6\phi(6)-\ldots-(p-1)\,\phi(p-1), \text{ if } p \text{ be uneven,}$$

where, as before, $\frac{1}{2}p(p+1)$ is the triangular number next superior to n.

61. By putting $\phi(d) = d$ we obtain the original formula quoted in § 44, which was also obtained in § 52 by putting $\psi(d+1) = (d+1)^2$.

If we put $\phi(d) = d^m$, m being uneven, we obtain the general theorem

$$\sigma_m(n)-2\{\sigma_m(n-1)+\sigma_m(n-2)\}$$
$$+3\{\sigma_m(n-3)+\sigma_m(n-4)+\sigma_m(n-5)\}-4\{\text{next four}\}+\&c.$$
$$= \quad 2\frac{m^{(2)}}{2!}\Big[\sigma_{m-2}(n-1)+\sigma_{m-2}(n-2)-2^2\{\text{next three}\}$$
$$+(1^2+3^2)\{\text{next four}\}-(2^2+4^2)\{\text{next five}\}+\&c.\Big]$$
$$+2\frac{m^{(4)}}{4!}\Big[\sigma_{m-4}(n-1)+\sigma_{m-4}(n-2)-2^4\{\text{next three}\}$$
$$+(1^4+3^4)\{\text{next four}\}-\&c.\Big]$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+2m\Big[\sigma(n-1)+\sigma(n-2)-2^{m-1}\{\text{next three}\}$$
$$+(1^{m+1}+3^{m-1})\{\text{next four}\}-\&c.\Big]$$
$$+\begin{cases} \quad 1^{m+1}+3^{m+1}+5^{m+1}+\ldots+(p-1)^{m+1}, \text{ if } p \text{ be even,} \\ \text{or} \quad -2^{m+1}-4^{m+1}-6^{m+1}-\ldots-(p-1)^{m+1}, \text{ if } p \text{ be uneven,} \end{cases}$$

p being defined as in the preceding section.

62. The coefficients in the series on the right-hand side are of the forms $\quad 1^{2r}+3^{2r}+5^{2r}+\ldots \quad$ and $\quad 2^{2r}+4^{2r}+6^{2r}+\ldots$

alternately, according as the number of terms in the group is even or uneven. This apparent alternation of form disappears if we express each coefficient as a function of the number of terms in the group. For (by § 53) the function

$$\frac{1}{2}\frac{x^{2r+1}}{2r+1}+\frac{1}{2}x^{2r}+(2r)^{(1)}B_1x^{2r-1}-2^2\frac{(2r)^{(3)}}{3!}\frac{B_2}{2}x^{2r-3}+2^4\frac{(2r)^{(5)}}{5!}\frac{B_3}{3}x^{2r-5}-\ldots$$
$$\ldots \pm 2^{2r-2}(2r)^{(1)}\frac{B_r}{r}x$$

is equal to

$$2^{2r}+4^{2r}+6^{2r}+\ldots+x^{2r} \quad \text{or} \quad 1^{2r}+3^{2r}+5^{2r}+\ldots+x^{2r},$$

according as r is even or uneven, so that the coefficient of the s^{th} group of terms in the r^{th} right-hand series, *i.e.*, the coefficient of

$$\sigma_{m-2r}(n-t)+\sigma_{m-2r}(n-t-1)+\ldots+\sigma_{m-2r}(n-t-s),$$

where t denotes $\frac{1}{2}s(s+1)$, is equal to

$$(-1)^{s-1}\left\{\tfrac{1}{2}\frac{s^{2r+1}}{2r+1}+\tfrac{1}{2}s^{2r}+(2r)^{(1)}B_1 s^{2r-1}-2^3\frac{(2r)^{(3)}}{3\,!}\frac{B_2}{2}s^{2r-3}+\&\text{c.}\right\},$$

and the last term is equal to

$$(-1)^{l-1}\left\{\tfrac{1}{2}\frac{l^{m+2}}{m+2}+\tfrac{1}{2}l^{m+1}+(m+1)^{(1)}B_1 l^{m}-2^3\frac{(m+1)^{(3)}}{3\,!}\frac{B_2}{2}l^{m-2}+\&\text{c.}\right\}.$$

The letter s may be regarded as denoting either the number of the group, or the number of terms in the group diminished by unity; and $l=p-1$, so that $\frac{1}{2}l(l+1)$ is the triangular number next inferior to n or equal to n, according as n is not, or is, a triangular number.

Other Formulæ relating to the actual Divisors. §§ 63–67.

63. The algebraical formulæ from which the four theorems relating to the actual divisors (§§ 3, 29, 46, 58) have been deduced are

(i.)

$$\{1-(1+2\cos x)\,q+(1+2\cos x+2\cos 2x)\,q^3-\&\text{c.}\}\,\Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$
$$=\sin x\,.\,q-(\sin x+2\sin 2x)\,q^3+(\sin x+2\sin 2x+3\sin 3x)\,q^6-\&\text{c.}\ (\S\ 27.)$$

(ii.)

$$2\,\{\sin x-q\sin 3x+q^3\sin 5x-q^6\sin 7x+\&\text{c.}\}\,\Sigma_1^\infty\,\sigma\,(\sin 2nx)\,q^n$$
$$=(\cos x-\cos 3x)\,q-(\cos x+\cos 3x-2\cos 5x)\,q^3$$
$$+(\cos x+\cos 3x+\cos 5x-3\cos 7x)\,q^6-\&\text{c.}\quad(\S\ 28.)$$

(iii.)

$$2\,\{\sin x-(q+q^2)\sin 2x+(q^3+q^4+q^5)\sin 3x-\&\text{c.}\}\,\Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$
$$=(1-\cos 2x)(q+q^2)-(2\cos x-2\cos 3x)(q^3+q^4+q^5)$$
$$+(1+2\cos 2x-3\cos 4x)(q^6+\ldots+q^9)$$
$$-(2\cos x+2\cos 3x-4\cos 5x)(q^{10}+\ldots+q^{14})+\&\text{c.}\quad(\S\ 45.)$$

(iv.)

$$\{1-2\cos x\,(q+q^2)+(1+2\cos 2x)(q^3+q^4+q^5)$$
$$-(2\cos x+2\cos 3x)(q^6+\ldots+q^9)$$
$$+(1+2\cos 2x+2\cos 4x)(q^{10}+\ldots+q^{14})-\&\text{c.}\}\,\Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$
$$=\sin x\,(q+q^2)-2\sin 2x\,(q^3+q^4+q^5)+(\sin x+3\sin 3x)(q^6+\ldots+q^9)$$
$$-(2\sin 2x+4\sin 4x)(q^{10}+\ldots+q^{14})+\&\text{c.}\quad(\S\ 57.)$$

The second of these formulæ is derivable from the first by multiplying by $\sin\frac{1}{2}x$ and replacing x by $2x$, and the third from the fourth by multiplying by $\sin x$.

64. We may obtain other formulæ relating to the actual divisors by multiplying these equations by $\sin x$ or $\cos x$, or other trigonometrical expressions, and equating coefficients.

Thus, for example, multiplying (ii.) by $\sin x$ and replacing x by $\frac{1}{2}x$, we find that

$$2\left\{1-\cos x-(\cos x-\cos 2x)\,q+(\cos 2x-\cos 3x)\,q^3\right.$$
$$\left.-(\cos 3x-\cos 4x)\,q^6+\&c.\right\}\Sigma_1^\infty\,\sigma\,(\sin nx)\,q^n$$
$$=(2\sin x-\sin 2x)\,q-(3\sin 2x-2\sin 3x)\,q^3+(4\sin 3x-3\sin 4x)\,q^6-\&c.,$$

the general term on the right-hand side being

$$(-1)^{g-1}\left\{(g+1)\sin gx-g\sin(g+1)\,x\right\}q^{\frac{1}{2}g(g+1)}.$$

65. Equating the coefficients of q^n, we see that the numbers given by the formula

$$G_n\left\{d,\,d,\,-(d\pm1)\right\}-G_{n-1}\left\{d\pm1,\,-(d\pm2)\right\}+G_{n-3}\left\{d\pm2,\,-(d\pm3)\right\}$$
$$-G_{n-6}\left\{d\pm3,\,-(d\pm4)\right\}+\&c.$$

all cancel each other, unless n is a triangular number $\frac{1}{2}g\,(g+1)$, in which case

$$(g+1)(-g)\text{'s} \quad\text{and}\quad g\,(g+1)\text{'s},$$

or

$$(g+1)\,g\text{'s} \quad\text{and}\quad g\left\{-(g+1)\right\}\text{'s},$$

remain uncancelled, according as g is even or uneven. Zeros are not to be taken account of.

As an example, putting $n=6$, so that $g=3$, the formula gives the numbers

$$\left\{\begin{matrix}1, & 2, & 3, & 6\\ 1, & 2, & 3, & 6\\ -2, & -3, & -4, & -7\\ 0, & -1, & -2, & -5\end{matrix}\right\}-\left\{\begin{matrix}0, & 4\\ 2, & 6\\ -3, & -7\\ 1, & -3\end{matrix}\right\}+\left\{\begin{matrix}-1, & 1\\ 3, & 5\\ -4, & -6\\ 2, & 0\end{matrix}\right\},$$

which cancel each other, excepting only

$$3,\ 3,\ 3,\ 3,\ -4,\ -4,\ -4.$$

Again, putting $n = 10$, so that $g = 4$, the formula gives

$$\begin{Bmatrix} 1, & 2, & 5, & 10 \\ 1, & 2, & 5, & 10 \\ -2, & -3, & -6, & -11 \\ 0, & -1, & -4, & -9 \end{Bmatrix} - \begin{Bmatrix} 0, & 2, & 8 \\ 2, & 4, & 10 \\ -3, & -5, & -11 \\ 1, & -1, & -7 \end{Bmatrix} + \begin{Bmatrix} -1, & 5 \\ 3, & 9 \\ -4, & -10 \\ 2, & -4 \end{Bmatrix}$$

$$- \begin{Bmatrix} -2, & -1, & 1 \\ 4, & 5, & 7 \\ -5, & -6, & -8 \\ 3, & 2, & 0 \end{Bmatrix},$$

which cancel each other, excepting only

$$-4,\ -4,\ -4,\ -4,\ -4,\ 5,\ 5,\ 5,\ 5.$$

This theorem is noticeable on account of the curious form of the group of numbers which is left uncancelled, $(g+1)$ occurring g times, and g occurring $(g+1)$ times, with opposite signs.

66. By multiplying (ii.) by $\cos x$, and replacing x by $\frac{1}{2}x$, we find that

$$2\left\{\sin x-(\sin x+\sin 2x)\,q+(\sin 2x+\sin 3x)\,q^3 - (\sin 3x+\sin 4x)\,q^6+\&c.\right\}\Sigma_1^\infty \sigma\,(\sin nx)\,q^n$$

$$= (1-\cos 2x)\,q-(1+2\cos x-\cos 2x-2\cos 3x)\,q^3 + (1+2\cos x+2\cos 2x-2\cos 3x-3\cos 4x)\,q^6+\&c.,$$

the coefficient of $(-1)^{g-1}q^{\frac{1}{2}g(g+1)}$ in the series on the right-hand side being

$$1+2\cos x+2\cos 2x+\ldots+2\cos(g-1)x-(g-1)\cos gx-g\cos(g+1)x.$$

It follows, therefore, that the numbers given by the formula

$$G_n\begin{pmatrix} d+1 \\ -[d-1] \end{pmatrix} - G_{n-1}\begin{pmatrix} d+1, & d+2 \\ -[d-1], & -[d-2] \end{pmatrix}$$

$$+ G_{n-3}\begin{pmatrix} d+2, & d+3 \\ -[d-2], & -[d-3] \end{pmatrix} - \&c.$$

all cancel each other, unless n is a triangular number $\frac{1}{2}g(g+1)$, in which case the numbers left uncancelled are

0, two 1's, two 2's, ..., two $(g-1)$'s, $(g-1)(-g)$'s, $g\{-(g+1)\}$'s if g is even, and these numbers with their signs reversed if g is uneven. Zeros are to be treated in the same manner as other numbers.

Thus, putting $n = 6$, so that $g = 3$, the formula gives

$$\left\{\begin{matrix} 2, & 3, & 4, & 7 \\ -0, & -1, & -2, & -5 \end{matrix}\right\} - \left\{\begin{matrix} 3, & 7 \\ 2, & 6 \\ -0, & -4 \\ -1, & -3 \end{matrix}\right\} + \left\{\begin{matrix} 4, & 6 \\ 3, & 5 \\ -1, & -1 \\ -2, & -0 \end{matrix}\right\},$$

in which the uncancelled numbers are

$$-0,\ -1,\ -1,\ -2,\ -2,\ 3,\ 3,\ 4,\ 4,\ 4.$$

Again, putting $n = 10$, so that $g = 4$, the formula gives

$$\left\{\begin{matrix} 2, & 3, & 6, & 11 \\ -0, & -1, & -4, & -9 \end{matrix}\right\} - \left\{\begin{matrix} 3, & 5, & 11 \\ 2, & 4, & 10 \\ -0, & -2, & -8 \\ -1, & -1, & -7 \end{matrix}\right\} + \left\{\begin{matrix} 4, & 10 \\ 3, & 9 \\ -1, & -5 \\ -2, & -4 \end{matrix}\right\}$$

$$- \left\{\begin{matrix} 5, & 6, & 8 \\ 4, & 5, & 7 \\ -2, & -1, & -1 \\ -3, & -2, & -0 \end{matrix}\right\},$$

in which the uncancelled numbers are

$$0,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ -4,\ -4,\ -4,\ -5,\ -5,\ -5,\ -5.$$

67. As another example, multiplying (iii.) by $\cos x$, we find that

$$\begin{aligned} 2\,\{\sin 2x &- (\sin x + \sin 3x)(q+q^3) + (\sin 2x + \sin 4x)(q^3+q^4+q^5) \\ &- (\sin 3x + \sin 5x)(q^6 + \dots + q^9) \\ &+ (\sin 4x + \sin 6x)(q^{10} + \dots + q^{14}) - \&c.\}\ \Sigma_1^\infty \sigma\,(\sin nx) \\ &= (\cos x - \cos 3x)(q+q^2) - (2 - 2\cos 4x)(q^3+q^4+q^5) \\ &+ (4\cos x - \cos 3x - 3\cos 5x)(q^6 + \dots + q^9) \\ &- (2 + 4\cos 2x - 2\cos 4x - 4\cos 6x)(q^{10} + \dots + q^{14}) + \&c., \end{aligned}$$

the coefficient of the group of terms, on the right-hand side, beginning with $q^{\frac{1}{2}r(r+1)}$, being

$$-\{2+4\cos 2x+4\cos 4x+\ldots+4\cos(r-2)x \\ -(r-2)\cos rx - r\cos(r+2)x\},$$

or $4\cos x+4\cos 3x+\ldots+4\cos(r-2)x-(r-2)\cos rx-r\cos(r+2)x$, according as r is even or uneven.

It follows, therefore, that the numbers given by the formula

$$G_n\begin{pmatrix} d+2 \\ -[d-2]\end{pmatrix}-(G_{n-1}+G_{n-2})\begin{pmatrix} d+1, & d+3 \\ -[d-1], & -[d-3]\end{pmatrix}$$
$$+(G_{n-3}+G_{n-4}+G_{n-5})\begin{pmatrix} d+2, & d+4 \\ -[d-2], & -[d-4]\end{pmatrix}$$
$$-(G_{n-6}+G_{n-7}+G_{n-8}+G_{n-9})\begin{pmatrix} d+3, & d+5 \\ -[d-3], & -[d-5]\end{pmatrix}+\&c.$$

all cancel each other, excepting

two 0's, four 2's, ..., four $(k-2)$'s, $(k-2)(-k)$'s, $k\{-(k+2)\}$'s

when k is even, and

four (-1)'s, four (-3)'s, ..., four $\{-(k-2)\}$'s, $(k-2)\,k$'s, $k(k+2)$'s

when k is uneven, $\frac{1}{2}k(k+1)$ being the triangular number which is next inferior to n, or equal to n, according as n is not, or is, a triangular number. Zeros are to be treated in the same manner as other numbers.

Thus, putting $n=6$, so that $k=3$, the theorem asserts that the numbers

$$\begin{Bmatrix} 3, & 4, & 5, & 8 \\ -1, & -0, & -1, & -4\end{Bmatrix}-\begin{Bmatrix} 4, & 8, & 4, & 5, & 7 \\ 2, & 6, & 2, & 3, & 5 \\ -0, & -4, & -0, & -1, & -3 \\ -2, & -2, & -2, & -1, & -1\end{Bmatrix}$$
$$+\begin{Bmatrix} 5, & 7, & 5, & 6, & 5 \\ 3, & 5, & 3, & 4, & 3 \\ -1, & -1, & -1, & -0, & -1 \\ -3, & -1, & -3, & -2, & -3\end{Bmatrix}$$

all cancel each other, with the exception of

$$-1,\ -1,\ -1,\ -1,\ 3,\ 5,\ 5,\ 5.$$

If $n = 10$, so that $k = 4$, the numbers are

$$\left\{\begin{matrix} 3, & 4, & 7, & 12 \\ -1, & -0, & -3, & -8 \end{matrix}\right\} - \left\{\begin{matrix} 4, & 6, & 12, & 4, & 5, & 7, & 11 \\ 2, & 4, & 10, & 2, & 3, & 5, & 9 \\ -0, & -2, & -8, & -0, & -1, & -3, & -7 \\ -2, & -0, & -6, & -2, & -1, & -1, & -5 \end{matrix}\right\}$$

$$+ \left\{\begin{matrix} 5, & 11, & 5, & 6, & 7, & 10, & 5, & 9 \\ 3, & 9, & 3, & 4, & 5, & 8, & 3, & 7 \\ -1, & -5, & -1, & -0, & -1, & -4, & -1, & -3 \\ -3, & -3, & -3, & -2, & -1, & -2, & -3, & -1 \end{matrix}\right\}$$

$$- \left\{\begin{matrix} 6, & 7, & 9, & 6, & 8, & 6, & 7, & 6 \\ 4, & 5, & 7, & 4, & 6, & 4, & 5, & 4 \\ -2, & -1, & -1, & -2, & -0, & -2, & -1, & -2 \\ -4, & -3, & -1, & -4, & -2, & -4, & -3, & -4 \end{matrix}\right\},$$

which cancel each other, with the exception of

$$0,\ 0,\ 2,\ 2,\ 2,\ 2,\ -4,\ -4,\ -6,\ -6,\ -6,\ -6.$$

Theorems derived from the other Zeta Functions. § 68.

68. In this paper I have limited myself to results which may be regarded as generalizations of the formula

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) + 7\sigma(n-6) - 9\sigma(n-10) + \&c. = 0,$$

and of the corresponding formula (§ 44) which connects together the σ's of the first n numbers.

In a second paper, I propose to give the similar theorems which stand in a corresponding relation to the recurring formulæ

$$\zeta(n) + \zeta(n-1) + \zeta(n-3) + \zeta(n-6) + \zeta(n-10) + \&c. = 0,^{*}$$

$$\Delta'(n) - 2\Delta'(n-1) + 2\Delta'(n-4) - 2\Delta'(n-9) + 2\Delta'(n-16) - \&c. = 0,^{\dagger}$$

---

* *Proc. Lond. Math. Soc.*, Vol. xv., p. 110, or *Proc. Camb. Phil. Soc.*, Vol. v., p. 116. The value of $\zeta(0)$ is supposed to be $-n$.

On pp. 118 and 119 of the former paper, and pp. 117 and 119 of the latter, the ζ-theorem corresponding to the σ-theorem in § 44 is given.

† The value of $\Delta'(0)$ is supposed to be $\frac{1}{2}n$. The corresponding theorem involving all the numbers from 1 to n as arguments is

$$\Delta'(n) - \Delta'(n-1) - \Delta'(n-2) - \Delta'(n-3) + \Delta'(n-4) + \Delta'(n-5) + \Delta'(n-6) + \Delta'(n-7)$$
$$+ \Delta'(n-8) - \Delta'(n-9) - \ldots - \Delta'(n-15) + \Delta'(n-16) + \ldots \pm \Delta'(0) = 0,$$

where $\Delta'(0)$ is to have the value $\frac{1}{2}r(r+1)$, $2r+1$ being the number of terms in the completed group in which $\Delta'(0)$ falls.

in which $\zeta(n)$ denotes the excess of the sum of the uneven divisors of n above the sum of the even divisors; and $\Delta'(n)$ denotes the sum of those divisors of n whose conjugates are uneven.

The general theorems relating to the actual divisors are derived from the other Zeta functions by a method similar to that employed in §§ 25–27.

On a certain Riemann's Surface. By W. Burnside.

[*Read May 14th*, 1891.]

The present note was suggested by the paper of Mr. R. A. Roberts, pp. 28–34 of the current volume of the Society's *Proceedings*, dealing with a particular case of Abel's theorem. The special interest of the case that Mr. Roberts treats lies in the degradation of the apparently Abelian integrals to ordinary elliptic integrals; and though, as Mr. Greenhill has pointed out to me, this property, for the form of integral in question, is sufficiently well known, being originally due to Legendre, still its consideration from quite another point of view, namely, that of Riemann's theory, is perhaps not altogether superfluous.

The equation $$s^3 = \frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}$$

defines a three-sheeted Riemann's surface. The only branch points are α, β, γ, δ, and at each of these the three sheets are connected together. The points at infinity on each sheet are ordinary points. The "Verzweigungs-schnitte," or lines of passage, will consist of a line from α to γ, connecting sheets I. and II., a second from α to γ, connecting II. and III., and two similar ones from β to δ. The lines may be taken straight, so that on crossing $\alpha\gamma$ or $\beta\delta$ from the left to the right a point will pass from I. to II., or from II. to III., or from III. to I.; the separate sheets being denoted by I., II., and III. The number of sheets being three, while there are four lines of passage, Riemann's number "p" for the surface will be 2, and there are, therefore, according to Riemann's theory, two independent integrals of the first kind upon it.