It has been assumed here that O lies within the parabola so that $b < a$. When O is without the parabola, the equation will be

$$r_1 . OF_2 - r_2 . OF_1 \pm r . F_1F_2 \sqrt{\frac{X+b}{X+2b-a}} = 0,$$

and, if 2α be the angle between the two tangents drawn from O to the given parabola, $\cos 2\alpha = \frac{X+a}{a-2b-X}$, $2\sin^2\alpha = \frac{2(X+b)}{X+2b-a}$,

or the vector equation is

$$r_1 . OF_2 - r_2 . OF_1 \pm r . F_1F_2 \sin\alpha = 0.$$

When O lies on the parabola, $b=a$, $OF_1=0$, or F_1 coincides with O, which is then a triple focus (and cusp) of the pedal, $k_1 = 0$, $k_2 = -2$, and the coordinates of F_2, the only real single focus, will be $X-4a$, $-Y$; OF_2 being normal at F_2 to the parabola $y^2 = 4a(x+4a)$. When O lies on the axis, on the opposite side of the focus to the vertex, $b = 0$, OF_1, OF_2 are normals drawn from O to the parabola $y^2 = 4a(x+a)$, and the vector equation is $r_1 + r_2 = 2r$.

On the Calculation of Symmetric Functions.

By Mr. J. Hammond, M.A.

[*Read January* 12*th*, 1882.]

1. Consider the equation

$$(x-\mu)(x^n - p_1x^{n-1} + p_2x^{n-2} - p_3x^{n-3} + \dots) = 0 \quad\dots\dots\dots\dots(1),$$

suppose the roots of the equation of the n^{th} degree to be $\alpha, \beta, \gamma, \dots$,

and let $\quad \Sigma\alpha^q\beta^q \dots \gamma^r\delta^s \dots\dots = \phi(p_1, p_2, p_3, \dots\dots)$;

the introduction of the new root μ, on the one hand, changes $p_1, p_2, p_3, \dots$ into $p_1+\mu$, $p_2+\mu p_1$, $p_3+\mu p_2$, ..., and, on the other, introduces into the symmetric function new terms of the form

$$\mu^q\Sigma\alpha^q \dots \gamma^r\delta^s \dots\dots .$$

Hence

$$\Sigma\alpha^q\beta^q \dots \gamma^r\delta^s \dots\dots + \mu^q\Sigma\alpha^q \dots \gamma^r\delta^s \dots\dots + \mu^r\Sigma\alpha^q\beta^q \dots \delta^s \dots\dots + \mu^s\Sigma\alpha^q\beta^q \dots \gamma^r \dots\dots$$
$$= \phi(p_1+\mu, p_2+\mu p_1, p_3+\mu p_2, \dots\dots) \quad\dots\dots\dots\dots(2).$$

If now the expanded value of ϕ, in (2), is written

$$\phi + \mu\phi_1 + \mu^2\phi_2 + \mu^3\phi_3 + \dots\dots,$$

equating coefficients, we have

$$\left.\begin{array}{l}\phi_\kappa = 0 \text{ unless } \kappa = q, r, s \\ \phi_q = \Sigma\alpha^q \ldots \gamma^r\delta^s \ldots\ldots \\ \phi_r = \Sigma\alpha^q\beta^q \ldots \delta^s \ldots\ldots \\ \ldots \quad \ldots \quad \ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Or, the symmetric function equivalent to ϕ_κ is obtained by simply cancelling one of the letters in the original symmetric function that occurs with the index κ. If we wish to find the value of $\Sigma\alpha^n$,

$$\phi_1, \phi_2, \ldots \phi_{n-1} = 0,$$

and $\phi_n = 1$.

2. Several terms of a symmetric function can often be found by inspection from known formulæ.

Thus, let $$\phi = \Sigma\alpha^q\beta^r\gamma^s \ldots\ldots,$$

and suppose $\Sigma_m\alpha^q\beta^r\gamma^s \ldots$ to denote the value of the symmetric function when p_{m+1} and all succeeding coefficients in (1) are equated to zero; then

$$\begin{aligned}\phi\,(p_1, p_2, p_3, \ldots p_n) &= \phi\,(p_1, p_2, p_3, \ldots p_m, 0, 0, \ldots) \\ &\quad + \text{terms containing } p_{m+1}, p_{m+2}, p_{m+3}, \ldots p_n \\ &= \Sigma_m\alpha^q\beta^r\gamma^s \ldots\ldots + \text{other terms} \ldots\ldots\ldots\ldots\ldots(4).\end{aligned}$$

In the case $\Sigma\alpha^5\beta^4\gamma^3\delta^2\epsilon$, we have

$$\Sigma_5\alpha^5\beta^4\gamma^3\delta^2\epsilon = p_5\Sigma_4\alpha^4\beta^3\gamma^2\delta,$$

and, referring to tables of the 10th degree, we see that

$$\begin{aligned}\Sigma\alpha^5\beta^4\gamma^3\delta^2\epsilon = p_5\,(p_1p_2p_3p_4 &- 3p_3^2p_4 - 3p_1^2p_4^2 + 4p_2p_4^2 - 3p_1p_2^2p_5 + 4p_1^2p_3p_5 \\ &+ 5p_2p_3p_5 - 5p_5^2) \\ &+ \text{terms containing } p_6, p_7, p_8, \ldots p_{15}.\end{aligned}$$

If $\phi = \Sigma\alpha^n$, it is easily seen that

$$\begin{aligned}\Sigma_2\alpha^n = p_1^n - np_1^{n-2}p_2 &+ \frac{n\,(n-3)}{2\,!}\,p_1^{n-4}p_2^2 - \frac{n\,(n-4)\,(n-5)}{3\,!}\,p_1^{n-6}p_2^3 \\ &+ \frac{n\,(n-5)(n-6)(n-7)}{4\,!}\,p_1^{n-8}p_2^4 - \&c. \ \ldots\ldots(5).\end{aligned}$$

In fact, if we consider the equation

$$x^2 - 2x\cos\theta + 1 = 0,$$

in which $p_1 = 2\cos\theta$ and $p_2 = 1$, it is obvious that $\Sigma_2\alpha^n = 2\cos n\theta$, and hence that the coefficients in (5) are those of the development of $2\cos n\theta$ in descending powers of $2\cos\theta$.

This gives $\frac{n+1}{2}$ or $\frac{n}{2}+1$ of the leading terms of $\Sigma\alpha^n$ according as n is odd or even.

3. The use of ϕ_1 alone will sometimes enable us to calculate the value of a symmetric function, and in the case where Σ contains no letters with unity for their index, the determination will be independent of the values of symmetric functions of inferior degrees, since here $\phi_1 = 0$.

For example, if $\phi = \Sigma\alpha^2\beta^2\gamma^2......$, the index of the last letter being 1 or 2 according as n is odd or even.

First, when $n = 2m$, assume

$$\phi = p_m^2 + A_1 p_{m-1} p_{m+1} + A_2 p_{m-2} p_{m+2} + A_3 p_{m-3} p_{m+3} + \dots$$
$$\dots + A_{m-1} p_1 p_{2m-1} + A_m p_{2m};$$

operating with $\quad \dfrac{d}{dp_1} + p_1 \dfrac{d}{dp_2} + p_2 \dfrac{d}{dp_3} + ...,$

we have

$$\phi_1 = p_{m-1} p_m (2 + A_1) + p_{m-2} p_{m+1} (A_1 + A_2) + p_{m-3} p_{m+2} (A_2 + A_3) + \dots$$
$$\dots + p_{2m-1} (A_{m-1} + A_m),$$

where, since in this case $\phi_1 = 0$, it follows that

$$A_1 = -2,\ A_2 = 2,\ ......\ A_m = (-)^m 2,$$

i.e., $\quad \Sigma\alpha^2\beta^2\gamma^2 \dots = p_m^2 - 2p_{m-1} p_{m+1} + 2p_{m-2} p_{m+2} - \dots + (-)^m 2p_{2m}$ ...(6).

Secondly, if $n = 2m+1$, assume

$$\phi = p_m p_{m+1} + B_1 p_{m-1} p_{m+2} + B_2 p_{m-2} p_{m+3} + \dots + B_{m-1} p_1 p_{2m} + B_m p_{2m+1};$$

as before,

$$\phi_1 = p_m^2 + p_{m-1} p_{m+1} (1 + B_1) + p_{m-2} p_{m+2} (B_1 + B_2) + \dots + (B_{m-1} + B_m) p_{2m};$$

but here, since the index of the last letter in Σ is unity, this value of ϕ_1 must be equivalent to (6). Hence

$$1 + B_1 = -2,\ \ B_1 + B_2 = 2,\ ......\ B_{m-1} + B_m = (-)^m 2,$$

giving $\quad \Sigma\alpha^2\beta^2\gamma^2 \dots = p_m p_{m+1} - 3p_{m-1} p_{m+2} + 5p_{m-2} p_{m+3} - \dots$
$$\dots + (-)^m (2m+1) p_{2m+1} ......(7).$$

Again, if $\phi = \Sigma\alpha^2\beta\gamma\delta \dots = u_n$, the destruction of the last letter changes u_n into u_{n-1}, so that, if

$$u_n = p_1 p_{n-1} + A_n p_n,$$

$$\left(\frac{d}{dp_1} + p_{n-2} \frac{d}{dp_{n-1}} + p_{n-1} \frac{d}{dp_n}\right) u_n = u_{n-1} = p_1 p_{n-2} + (1 + A_n) p_{n-1}.$$

From this we obtain $\quad 1 + A_n = A_{n-1}$,

and finally $\quad \Sigma\alpha^2\beta\gamma\delta \dots = p_1 p_{n-1} - np_n$ ....(8).

4. But ϕ_1, which is of weight $n-1$, will not in many cases give enough relations among the coefficients of ϕ, which is of weight n, to

determine all of them. In such cases the missing relations can be supplied by using single terms of $\phi_2, \phi_3, \ldots$; the method of finding such terms is in every case the same as in the following example, where $n = 11$, and it is required to find the coefficient of $p_1 p_2 p_3^2$ in ϕ_2.

If we consider all the factors of $p_1 p_2 p_3^2$, not exceeding in this case the second degree, viz., $1, p_1, p_2, p_3, p_1 p_2, p_1 p_3, p_2 p_3, p_3^2$, and the corresponding terms of the operator, in this case

$$\frac{1}{2!}\left(\frac{d^2}{dp_1^2} + 2p_1 \frac{d^2}{dp_1\,dp_2} + \ldots\right);$$

we see that the term $p_1 p_2 p_3^2$ in ϕ_2 can only come from

$$\tfrac{1}{2}\frac{d^2}{dp_1^2}(p_1^3 p_2 p_3^2),\quad p_1 \frac{d^2}{dp_1\,dp_2}(p_1 p_2^2 p_3^2),\quad p_2 \frac{d^2}{dp_1\,dp_3}(p_1^2 p_3^3),$$

$$p_3 \frac{d^2}{dp_1\,dp_4}(p_1^2 p_2 p_3 p_4),\quad p_1 p_2 \frac{d^2}{dp_3\,dp_3}(p_2 p_3^3),\quad p_1 p_3 \frac{d^2}{dp_2\,dp_4}(p_2^2 p_3 p_4),$$

$$p_2 p_3 \frac{d^2}{dp_3\,dp_4}(p_1 p_3^2 p_4),\quad \tfrac{1}{2} p_3^2 \frac{d^2}{dp_4^2}(p_1 p_2 p_4^2);$$

and hence, if

$$\phi = \ldots\ldots + A p_1^3 p_2 p_3^2 + B p_1 p_2^2 p_3^2 + C p_1^2 p_3^3 + D p_1^2 p_2 p_3 p_4$$
$$+ E p_2 p_3^3 + F p_2^2 p_3 p_4 + G p_1 p_3^2 p_4 + H p_1 p_2 p_4^2 + \ldots\ldots,$$

$$\phi_2 = \ldots\ldots + (3A + 2B + 6C + 2D + 3E + 2F + 2G + H)\, p_1 p_2 p_3^2 + \ldots\ldots,$$

and if we have already found all but one of the quantities $A, B, \ldots H$, the remaining one is found from ϕ_2.

The remaining article consists of applications of this principle, chiefly to the case $\Sigma\alpha^n$.

5. Suppose

$$\phi = A_0 p_1^n + A_2 p_1^{n-2} p_2 + A_3 p_1^{n-3} p_3 + \ldots$$
$$\ldots + A_{n-2} p_1^2 p_{n-2} + A_{n-1} p_1 p_{n-1} + A_n p_n + \text{other terms},$$

then, if $\kappa < n$ the last terms of $\phi_1, \phi_2, \ldots \phi_\kappa$ are respectively

$$(A_{n-1} + A_n)\, p_{n-1},\quad (A_{n-2} + A_{n-1})\, p_{n-2},\quad \ldots\quad (A_{n-\kappa} + A_{n-\kappa+1})\, p_{n-\kappa};$$

for the factors of $p_{n-\kappa}$ are simply 1 and $p_{n-\kappa}$, so that the last term of ϕ_κ is obtained from

$$\frac{1}{\kappa!}\frac{d^\kappa}{dp_1^\kappa}(A_{n-\kappa} p_1^\kappa p_{n-\kappa}),$$

and $$\frac{1}{(\kappa-1)!}\, p_{n-\kappa} \frac{d^\kappa}{dp_1^{\kappa-1}\,dp_{n-\kappa+1}}(A_{n-\kappa+1} p_1^{\kappa-1} p_{n-\kappa+1}),$$

and is therefore $$(A_{n-\kappa} + A_{n-\kappa+1})\, p_{n-\kappa}.$$

When $\phi = \Sigma\alpha^n$, since from (5), or otherwise, we have $A_2 = -n$, it follows that $A_3 = n,\ A_4 = -n,\ \dots\ A_n = (-)^{n+1} n.$

But if $\phi = \Sigma\alpha^q\beta^r$, where $q > r$ and $q+r = n$, we have

$$\phi_r = \Sigma\alpha^q, \quad \phi_q = \Sigma\alpha^r, \quad \phi_\kappa = 0,$$

for other values of κ; thus

$$A_q + A_{q+1} = \text{coeff. } p_q \text{ in } \Sigma\alpha^q = (-)^{q+1} q,$$

and $$A_r + A_{r+1} = (-)^{r+1} r.$$

Here, then, $$A_2,\ A_3,\ \dots\ A_r = 0,$$

$$A_{r+1} = (-)^{r+1} r,\ \ A_{r+2} = (-)^{r+2} r,\ \dots\dots\ A_q = (-)^q r,$$

$$A_{q+1} = (-)^{q+1}(q+r) = (-)^{q+1} n,\ \ A_{q+2} = (-)^{q+2} n,\ \dots\dots\ A_n = (-)^n n.$$

These values must, however, be modified in the case

$$q = r = \frac{n}{2},$$

when $$A_{q+1} = (-)^{q+1}\frac{n}{2},\ \dots\dots\ A_n = (-)^n \frac{n}{2}.$$

Other symmetric functions may be similarly treated.

Again, suppose

$$\phi = \dots\dots + A_m p_1^{n-m} p_m + B_m p_1^{n-m-2} p_2 p_m + A_{m+1} p_1^{n-m-1} p_{m+1} + \dots\dots,$$

and consider the term $p_1^{n-m-1} p_m$ in ϕ_1; factors $1, p_1, p_m$.

This term is

$$\frac{d}{dp_1}(A_m p_1^{n-m} p_m) + p_1 \frac{d}{dp_2}(B_m p_1^{n-m-2} p_2 p_m) + p_m \frac{d}{dp_{m+1}}(A_{m+1} p_1^{n-m-1} p_{m+1})$$
$$= \{(n-m) A_m + B_m + A_{m+1}\} p_1^{n-m-1} p_m, \text{ if } m > 2.$$

When $m = 2$,

$$\phi = \dots\dots + A_2 p_1^{n-2} p_2 + B_2 p_1^{n-4} p_2^2 + A_3 p_1^{n-3} p_3 + \dots\dots,$$

and $$\phi_1 = \dots\dots + \{(n-2) A_2 + 2B_2 + A_3\} p_1^{n-3} p_2 + \dots\dots.$$

Hence $A_2, A_3, \dots A_n$ having been found from the last terms of $\phi_1, \phi_2, \dots$; $B_2, B_3, \dots$, can be found from ϕ_1 only.

If $\phi = \Sigma\alpha^n$, since

$$A_2 = -n,\ \dots\ A_m = (-)^{m+1} n,$$

it follows that $$B_2 = \frac{n(n-3)}{2},$$

which agrees with (5), and

$$B_m = (-)^m n(n-m-1),\ m > 2 \ \dots\dots\dots\dots\dots\dots (8),$$

the last of the series being

$$B_{n-2} = (-)^n n = \text{coeff. } p_2 p_{n-2} \text{ in } \Sigma a^n, \; n > 4.$$

If, as before, $m > 2$, the term $p_1^{n-m-2} p_m$ in ϕ_2, whose factors are 1, p_1, p_m, p_1^2, $p_1 p_m$, can only originate from the five terms

$$\tfrac{1}{2} \frac{d^2}{dp_1^2} (p_1^{n-m} p_m), \quad p_1 \frac{d^2}{dp_1 dp_2} (p_1^{n-m-2} p_2 p_m),$$

$$p_m \frac{d^2}{dp_1 dp_{m+1}} (p_1^{n-m-1} p_{m+1}), \quad \frac{p_1^2}{2} \frac{d^2}{dp_2^2} (p_1^{n-m-4} p_2^2 p_m),$$

and
$$p_1 p_m \frac{d^2}{dp_2 dp_{m+1}} (p_1^{n-m-3} p_2 p_{m+1});$$

so that with coefficients $A_m \ldots B_m \ldots$ as before, if the coefficient of $p_1^{n-m-4} p_2^2 p_m$ in ϕ be denoted by C_m, the term in ϕ_2 is

$$\left\{ \frac{(n-m)(n-m-1)}{2} A_m + (n-m-2) B_m + (n-m-1) A_{m+1} + C_m + B_{m+1} \right\} p_1^{n-m-2} p_m,$$

whence the coefficients C_m are determined.

In the case Σa^n, substituting for the A's and B's their values found above, we obtain

$$C_m = (-)^{m+1} \frac{n(n-m-2)(n-m-3)}{2}, \quad m > 2 \quad \ldots\ldots\ldots(9),$$

the last term of the series being

$$C_{n-4} = (-)^{n+1} n.$$

The calculation of C_2, like that of B_2, will only reproduce a term of (5). Coefficients of other terms of Σa^n may be found in the same way, without the slightest variation in the method.

Thursday, February 9th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. J. H. Tompson, F.C.S., F.G.S., Science Master in the Auckland College, New Zealand, was elected a Member, and Mrs. Bryant was admitted into the Society.

Mr. Tucker read a short abstract of a paper by Mr. H. M. Jeffery, F.R.S., "On certain Quartic Curves having a Tacnode at Infinity at which the Line at Infinity is the Multiple Tangent."

Mr. J. W. L. Glaisher, F.R.S., having taken the Chair *pro tem.*, the President communicated some results connected with Euler's Formula connecting the sum of the Divisors of a Number with the Pentagonal Numbers, and remarked that the formula really expressed the equality of the sum of the divisors to the sum of the m^{th} powers of the roots of a certain equation.

Mr. Hammond and Mr. Tucker also made brief communications.

The following presents were received:—

Carte-de-visite likeness from Mr. C. E. Bickmore.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 2e Série, Tome iv., 3e Cahier; Paris, 1881.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Fasc. 3°, 4°, 5°.

"Educational Times," February, 1882.

"Proceedings of Royal Society," Vol. xxxiii., No. 217.

"Tidsskrift for Mathematik.... Fjerde Række, Femte Aargang, Første, Andet, Tredie, Fjerde, Femte, Sjette Hefte;" Kjøbenhavn, 1881.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xvi., 3me, 4me, 5me Livraisons; Harlem, 1881.

"Jahrbuch über die Fortschritte der Mathematik," elfter Band, Heft 3; Jahrgang 1879; Berlin, 1882.

Donations to the Bodleian Library during the year ending November 8, 1881; Oxford, 1881.

"The Scientific Proceedings of the Royal Dublin Society," Vol. ii., Pt. vii. (November, 1880), Vol. iii., Pt. i. (January, 1881), Pt. ii. (April, 1881), Pt. iii. (July, 1881), Pt. iv. (October, 1881).

"The Scientific Transactions of the Royal Dublin Society," Vol. i., Series ii., Pt. xiii., "On the Possibility of Originating Wave Disturbances in the Ether by means of Electric Forces," Pt. 2, by G. F. Fitzgerald (read May 19th, 1880). Pt. xiv., "Explorations in the Bone Cave of Ballynamintra, near Cappagh, County Waterford," by A. L. Adams, G. H. Kinahan, and R. J. Ussher (read November 15th, 1880).

The following three Mémoires (des xii. Bandes der "Abhandlungen der Mathematisch-Physischen Classe der Königl.-sächsischen Gesellschaft der Wissenschaften"):—

C. Neumann: "Uber die Peripolaren Coordinaten," November (Leipzig, 1880).

C. Neumann: "Die Vertheilung der Elektricität auf einer Kugelcalotte," November 1 (Leipzig, 1880).

W. G. Hankel: "Elektrische Untersuchungen, fünfzehnte Abhandlung über die aktino- und Piezo-elektrischen Eigenschaften des Bergkrystalles und ihre Beziehung zu den Thermo-elektrischen," No. vii. (Leipzig, 1881).