that IJ is at right angles to AK. The line IJ is therefore given on

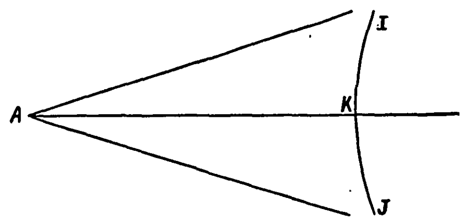

FIG. 6.

drawing the perpendicular from I on AK by the preceding construction.

A Theorem concerning the Special Systems of Point-Groups on a Particular Type of Base-Curve. By MISS FRANCES HARDCASTLE. Received November 22nd, 1897. Read December 9th, 1897.

The "special" systems of point-groups on a base-curve of deficiency p are known to be connected in pairs, viz. (employing the usual notation, originally introduced by Brill and Noether*), the special system g_R^r always exists simultaneously with a g_Q^q when the relations

$$Q+R = 2p-2, \quad \text{(i.)}$$

$$Q-R = 2(q-r) \quad \text{(ii.)}$$

hold among the positive integers Q, R, q, r.

Given p, these equations, as they stand, are capable of being satisfied by an infinite number of sets of values of Q, R, q, r,† but by means of a further relation, an inequality, namely, $Q \geqslant 2q$ (which also implies that $R \geqslant 2r$), the number of these sets, or, in

* *Math. Ann.*, Vol. VII., 1873, "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie."

† This is, of course, only true if we assume no knowledge of the application of the equations, and hence no connexion at all among the unknown integers.

other words, the number of different possible special systems of point-groups g^q_Q, g^r_R on a base-curve of given deficiency is limited. The equations (i.) and (ii.) are due to Brill and Noether, and are generally known as the Riemann-Roch equations. The inequality $Q \geqslant 2q$ was first established in connexion with the theory of Abelian functions, by Clifford,* but it is easily proved independently, directly from the definitions and properties of the special systems;† it applies to all base-curves of whatever nature their singularities may be; it is, in fact, the sole necessary restriction on all curves on the range of values of Q, R, q, r, when these quantities are expressed in terms of p, the deficiency of the curve (in particular, it holds, whether the given base-curve be "general" for the given value of p, or not). But for particular types of base-curves it appears not impossible to assign a narrower limitation to the range of values of Q, R, q, r, in terms of p, and, in the following pages, I state, and prove by an elementary method, such a limitation with reference to a particular type of base-curve.

The given base-curve is of order n, deficiency p, its singularity of highest order is an $(n-3)$-fold point, and its only other singularities are triple and double points, say T triple points and therefore $2n-5-p-3T$ double points. An adjoint curve of order $n-3$, C_{n-3} (in future, a curve of order n will be denoted by C_n) has, by definition, an $(n-4)$-fold point at the $(n-3)$-fold point of the base-curve; it can therefore have no other singularity, and, if $T>0$, every adjoint C_{n-3} must degenerate into T straight lines joining the $(n-3)$-fold point to the T triple points—these lines have no more points of intersection with the base-curve—together with a C_{n-3-T} with an $(n-4-T)$-fold point at the $(n-3)$-fold point of the base-curve, which passes once through each of the T triple points, and once through each of the $2n-5-p-3T$ double points—this C_{n-3-T} cuts the base-curve in $2p-2$ points, exclusive of the base-points.

Any two such curves C_{n-3-T} cut each other in

$$(n-3-T)^2-(n-4-T)^2-T-(2n-5-p-3T) = p-2$$

points, exclusive of the intersections at the base-points; these $p-2$ points are not necessarily on the base-curve, but it is clear that, whatever may be the number of intersections of two such curves which do actually lie on the base-curve, it can never, *provided the*

* "Classification of Loci," *Phil. Trans.*, 1878; *Coll. Works*, p. 329.
† *Cf.* Bertini, *Annali di Mat.*, XXII., 1894.

curves are non-degenerate, exceed $p-2$. If $T=0$, any two adjoints C_{n-3}—each of which cuts the base-curve in $2p-2$ points, exclusive of the base-points—cut each other in $(n-3)^2-(n-4)^2-(2n-5-p)$ points (exclusive of the base-points), which is again $= p-2$, and we shall therefore, in what follows, deal with a C_{n-3-T}, where $T \geqslant 0$ (with an $(n-4-T)$-fold point, passing once through each of the $2n-5-p-3T$ double points, and, if $T \geqslant 1$, once through the T triple points), as the exact equivalent, in association with the given base-curve, of the adjoint C_{n-3} associated, in general, with every base-curve.

We proceed to prove that, *on the given base-curve, the only pairs of complementary special systems of point-groups* g^r_R, g^q_Q *which can be found satisfy one of the condition* $Q \geqslant 3q$, $R \geqslant 3r$.

The g^r_R being, by hypothesis, special and complete, is determined on the given base-curve by an r-ply infinite system of curves C_{n-3-T}, unconditioned, save by passage through the base-points, and by passage through any arbitrarily selected G^q_Q of the residual g^q_Q. If $r=1$, we have a singly infinite system of curves C_{n-3-T}, and two linearly independent curves of this system cut each other in Q points on the base-curve, as well as at the base-points; but, if non-degenerate, they have been shown not to cut each other in more than $p-2$ points, exclusive of the base-points; *if*, therefore, $Q > p-2$, *the* g'_R *is certainly not determined on the given base-curve by means of proper curves* C_{n-3-T}, *nor*, *a fortiori*, *can a* g^r_R, $r>1$, *be so determined.* Now either Q or R is always $>p-2$ (because $Q+R = 2p-2$); if, therefore, the systems $g^r_R g^q_Q$ are determined on the base-curve by *proper curves* C_{n-3-T}, one or other of the quantities r and q must be zero; hence *non-degenerate systems of curves* C_{n-3-T} *can only determine, on the given base-curve, systems of point-groups which satisfy the condition we have to prove* (for, if $q=0$, $Q \geqslant 3q$; and, if $r=0$, $R \geqslant 3r$). But this does not yet prove that no systems of point-groups contradicting this condition can be found on the given base-curve; it does not even prove that it is *impossible* to find a g'_R on the given base-curve, although $Q > p-2$, for the argument does not apply if the G^q_Q be so chosen that the system of curves C_{n-3-T} *degenerates* into systems of curves of lower order (the sum of whose orders is, of course, $n-3-T$) which together satisfy the necessary conditions of passage through the base-points. In fact, a single example suffices to show that a g'_R (whether $q \geqslant p-2$) can be determined by such a *degenerate* system of curves C_{n-3-T}. For let the G^q_Q be so chosen that the system of curves C_{n-3-T} degenerates into a straight line through the

$(n-3)$-fold point of the base-curve, and into a proper C_{n-4-T} (with an $(n-5-T)$-fold point at the $(n-3)$-fold point, and passing once through each of the T triple points, and of the $2n-5-p-3T$ double points); then we obtain a g_3^1 $(R=3)$ whose elements are the G_3's formed by the three points in which this straight line cuts the base-curve as it rotates about the $(n-3)$-fold point with one degree of freedom in the plane. Here $R=3r$, and this particular example falls under the rule we have to prove, but it still remains to show that even a degenerate system of curves C_{n-3-T} *cannot* determine on the base-curve a $g_Q^q\,g_R^r$, unless the combination of values of Q, q, R, r satisfies one of the condition $Q \geqslant 3q$, $R \geqslant 3r$.

For this purpose, consider the possible ways in which a C_{n-3-T} can degenerate.* It must have an $(n-4-T)$-fold point at the $(n-3)$-fold point of the base-curve, and, since a *proper* $C_{n-3-T-x}$ can have no singularity of higher order than an $(n-4-T-x)$-fold point, a C_x which together with this curve makes up a C_{n-3-T} must have an x-fold point at the $(n-3)$-fold point of the base-curve, *i.e.*, it consists of x straight lines through this point. These x straight lines may consist of:—

(1) t lines free to rotate about the $(n-3)$-fold point of the base-curve—each of these lines has then one degree of freedom in the plane, and cuts the base curve in three more points.

(2) s lines joining the $(n-3)$-fold point to s of the double points—each of these lines is *fixed* in position, and cuts the base-curve in one more point.

(3) T' lines which coincide with T' of those lines which join the $(n-3)$-fold point to the T triple points—each of these lines is *fixed* in position and cuts the base-curve in *no* more points.

The remaining curve, which makes up the C_{n-3-T}, and, together with these x straight lines, fulfils the necessary conditions of passage through the base-points, is a *proper* $C_{n-3-T-x}$ with an $(n-4-T-x)$-fold point at the $(n-3)$-fold point of the base-curve, which passes once through such of the triple and double points of the base-curve as are not on the x straight lines. But, if this $C_{n-3-T-x}$ is a conic (*i.e.*, if $x=n-5-T$), and finally degenerates into two straight lines, one of

* We are not concerned with the conditions which may cause the system of C_{n-3-T} to degenerate, but we show that, even if it must degenerate, no one of all possible degenerate systems can determine complementary systems of point-groups in which Q, q, R, r contradict the above condition.

these may possibly not fall under any one of the three categories already considered, for it may be—

(*a*) A line not passing through any base-point. It then has two degrees of freedom in the plane, and cuts the base-curve in n points ;

(*b*) A line joining two triple points, or two double points, or a triple point to a double point. It is then fixed in position, and cuts the base-curve in $n-6$, $n-4$, $n-5$ more points respectively ; or

(*c*) A line passing through a triple point or a double point. It then has one degree of freedom in the plane, and cuts the base-curve in $n-3$ or $n-2$ more points respectively.

But, in the line of argument which now follows, it is assumed that the $C_{n-3-T-x}$ is a proper curve of order higher than one—the case in which it is a straight line is considered separately in a foot-note; t, s, T' have any initially assigned values (compatible with the values of n, p, T), and the proof consists in demonstrating that, if $Q < 3q$ *and* $R < 3r$, we are always led to two contradictory inequalities among these numbers, the known number p and another number m, also of initially assigned value.

If possible, then, let $Q = 3q-k$, $R = 3r-l$, where k and l are positive integers connected by the equation $2p-2 = 3(q+r)-k-l$.

We begin with the $g^q_{Q=3q-k}$, and inquire under what conditions an element G^q_Q of this system can obtain q degrees of freedom on the base-curve, *i.e.*, how the Q points can be placed among the intersections of the base-curve with $x = t+s+T'$ straight lines and the proper $C_{n-3-T-x}$ in order that G_Q may have exactly q degrees of freedom. Now, if all the Q points are on straight lines, it is clear that G_Q can have at most $\left[\frac{Q}{3}\right]$ $\left(\text{i.e., the greatest integer in } \frac{Q}{3}\right)$ degrees of freedom, even in the most favourable case when the points of G_Q are in triplets on some of the t straight lines which have each one degree of freedom,* and $\left[\frac{Q}{3}\right] = \left[q-\frac{k}{3}\right]$ is less than q; hence, to obtain q

* This is also true in the case we excepted above from the general argument. For it clearly applies if the last straight line (when the whole C_{n-3-T} consists of straight lines) falls under the heading (*b*), for *no* degrees of freedom are then acquired if some of the Q points lie on it. It also applies if this line falls under heading (*c*), for then one degree of freedom is added only if $n-3$ or $n-2$ of the Q points lie on the line, and then the total number of degrees of freedom is $\left[\frac{Q-(n-3)}{3}\right]+1$ at most, and this is not greater than $\left[\frac{Q}{3}\right]$, unless $n<6$ [where an $(n-3)$-fold point, being only a double point, is not a base-curve of the type here

degrees of freedom, it is necessary to assume that some, or all, of the Q points, say m of them, are among the intersections of the base-curve with the proper $C_{n-3-T-x}$.

Similarly, some of the R points of an element G_R^r of the g_R^r must be on the proper $C_{n-3-T-x}$, for $\left[\frac{R}{3}\right]$ is less than r. Now, the total number of points on the x straight lines is $3t+s$; the remaining $2p-2-(3t+s)$ are therefore on the $C_{n-3-T-x}$, and of these m belong to the G_Q^q. Hence, $2p-2-3t-s-m$ (the number of points belonging to the G_R^r which are on the $C_{n-3-T-x}$) *must be a positive integer*, whatever may be the values of t, s, T', m.

The assigned positions of the Q points of the G_Q^q and the R points of the G_R^r may be shown by writing them thus

$$G_Q^q = G^q_{(Q-m)+(m)}, \quad G_R^r = G^r_{(3t+s-Q+m)+(2p-2-3t-s-m)},$$

where the numbers in the first brackets of each subscript are the numbers of points on the x straight lines, and those in the second brackets are the numbers on the $C_{n-3-T-x}$. We have to consider how the q and r degrees of freedom are respectively obtained.

The greatest number of degrees of freedom which the $Q-m$ points on straight lines can have is, by a previous argument, $\left[\frac{Q-m}{3}\right]$, and, since $q = \frac{Q+k}{3}$, this is always *less than* q. Hence the q degrees of freedom of the element G_Q^q cannot arise wholly from the degrees of freedom of the $Q-m$ points on straight lines; *the m points on the proper $C_{n-3-T-x}$ must have at least one degree of freedom.* But this necessitates that the number of the remaining points on the $C_{n-3-T-x}$, viz., the $2p-2-3t-s-m$ belonging to the G_R^r, should certainly not exceed the number of intersections of two curves $C_{n-3-T-x}$, exclusive of those at the base-point (*cf.* similar reasoning on p. 134). And this

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considered]. Similarly it applies if the line falls under heading (a), for then one or two degrees of freedom are added only if $n-1$ or n of the Q points lie on the line, and then the total number of degrees of freedom is

$$\left[\frac{Q-(n-1)}{3}\right]+1 \quad \text{or} \quad \left[\frac{Q-n}{3}\right]+2,$$

at most, which are not greater than $\left[\frac{Q}{3}\right]$, unless $n<4$ or $n<6$.

number is

$$(n-3-T-s-t-T')^2-(n-4-T-s-t-T')^2$$
$$-(2n-5-2p-3T-s)-T+T$$
$$=p-2-2t-s-T'$$

(*cf.* p. 133). Thus the condition that the element G_Q should have as many as q degrees of freedom on the base-curve, where $Q=3q-k$, is that it should be possible to choose t, s, T', m such that

$$2p-2-3t-s-m \leqslant p-2-2t-s-T'$$

or

$$p \leqslant t+m-T'. \qquad \text{(A)}$$

The same line of argument can be applied to the case of the G^r_R, and shows that the $3t+s-Q+m$ points of the x straight lines can have at most *one third* of this number of degrees of freedom; but, since $Q=3q-k$ and $k-3q=3r-l-(2p-2)$, this number of degrees of freedom is $\frac{1}{3}[3r-l-(2p-2-3t-s-m)]$. But this number is always *less than* r, *provided* l *is a positive integer*, for $2p-2-3t-s-m$ was seen, with the same proviso, to be a positive integer. Hence the r degrees of freedom of the element G^r_R cannot arise wholly from the degrees of freedom of the points on straight lines; the $2p-2-3t-s-m$ points on the proper $C_{n-3-T-x}$ must have at least one degree of freedom. And, as in the similar case of the G^q_Q, this necessitates that m, the number of remaining points on the $C_{n-3-T-x}$, should not exceed the number of intersections of two curves $C_{n-3-T-x}$, exclusive of those at the base-points, *i.e.*, that

$$m \leqslant p-2t-2-s-T'$$

or

$$p \geqslant m+2t+s+2+T'. \qquad \text{(B)}$$

But, by (A), $p=m+t-T'-y$ $(y \geqslant 0)$; therefore (B) becomes

$$0 \geqslant 2+t+s+2T'+y,$$

which is impossible, since t, s, T', and y are, by hypothesis, positive integers, or zero. If, therefore, $Q<3q$ and, at the same time, $R<3r$, we are led to a logical absurdity on the assumption that the G^q_Q was so chosen that the system of curves C_{n-3-T} *degenerates* into systems of lower order; and we have already seen that non-degenerate systems can only determine systems for which either $Q \geq 3q$ or $R \geq 3r$. Our initial statement is, therefore, completely proved, viz., on the particular type of base-curve in question, it is necessary that either $Q \geq 3q$ or $R \geq 3r$, in order that a pair of complementary systems $g_R^r g_Q^q$ should exist.

Thursday, January 13th, 1898.

Professor E. B. ELLIOTT, F.R.S., President, in the Chair.

Nine members present.

Miss Mildred E. Barwell; Mr. Bertram Hopkinson, B.A., Scholar of Trinity College, Cambridge; Mr. Robert Hamilton Pinkerton, B.A. Balliol College, Oxford, Mathematical Lecturer, University College, Cardiff; and Mr. Algernon Charles Legge Wilkinson, B.A., Scholar of Trinity College, Cambridge, Mathematical Lecturer, University College, Cardiff, were elected members.

The President informed the members present of the death, on December 13th, 1897, of their foreign member Signor Brioschi, and dwelt upon the work the deceased member had done and the loss experienced by the mathematical world. He then read a letter from Professor R. Meldola, F.R.S., drawing attention to the proposed "Sylvester Memorial" Fund, circulars relating to which were laid upon the table.

Mr. Love communicated a paper by Mr. B. Hopkinson "On Discontinuous Fluid Motion," and Mr. S. H. Burbury gave a sketch of his paper "On the General Theory of Stationary Motion in a System of Molecules."

The following papers were briefly communicated by the President:—

Note on a Property of Pfaffians: H. F. Baker.

On the Intersections of Two Conics of a given type, and on the Intersections of Two Cubics: H. M. Taylor.

On the Continuous Group defined by any given Group of Finite Order: Professor W. Burnside.

On those Transformations of Coordinates which lead to new Solutions of Laplace's Equation: Professor Forsyth.

Mr. Mathews exhibited copies of a figure connected with Professor F. Morley's paper read at the November meeting.

The following presents to the Library were received:—

"The Japan Imperial University Calendar," 1896-7.

"Proceedings of the Royal Society," Vol. LXII., No. 381.

Zeuthen, H. G.—"Notes sur l'Histoire des Mathématiques," pamphlet.

Maddison, Isabel.—"On Singular Solutions of Differential Equations of the First Order in Two Variables, and the Geometrical Properties of certain Invariants and Covariants of their complete Primitives," pamphlet.

"Report of the Superintendent of the U.S. Naval Observatory," for the years ending June 30th, 1894, and 1897; Washington, 1895, 1897.

"Year-Book of the Royal Society for 1896–7."

"Biblioteca Boncompagni," Pt. 1, 8vo; Roma, 1898. (Sale from 27th January to 12th February.)

"Bulletin of the American Mathematical Society," Series 2, Vol. IV., No. 3; December, 1897.

"Bulletin des Sciences Mathématiques," Tome XXI.; November, 1897.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. III., Fasc. 11; Napoli.

Darboux, G.—"Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes," Tome I., 8vo; Paris, 1898.

"Rendiconti del Circolo Matematico di Palermo," Tome XI., Fasc. 6.

"Atti della reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. VI., Fasc. 11; Roma, 1897.

"Sachregister der Abhandlungen und Berichte der Math.-Phys. Classe der K. Sachs. Gesellschaft der Wissenschaften zu Leipzig," roy. 8vo; Leipzig, 1897.

"Acta Mathematica," XX., 3, 4, XXI.; Stockholm, 1897.

"Educational Times," January, 1898.

Gordan, P.—"Resultanten ternärer Formen," "Der Hermitesche Reciprocitätssatz," "Le résultant de trois formes ternaires quadratiques," pamphlets.

"Annales de l'Ecole Polytechnique de Delft," Tome VIII., Livr. 3, 4.

"G. Lejeune Dirichlets Werke," herausgegeben von L. Kronecker fortzesetz von L. Fuchs, Band II., 4to; Berlin, 1897.

"Collected Mathematical Papers of Arthur Cayley," Vol. XIII., 4to; Cambridge, 1897.

"Indian Engineering," Vol. XXII., Nos. 21-25; Nov. 20-Dec. 18, 1897.