Note on the Algebraic Properties of Pfaffians. By J. BRILL, M.A.
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1. My object is, in the first instance, to obtain a generalization of the ordinary theorem for the expansion of a Pfaffian, viz.,

$$[123 \dots (2m)] = [12][345 \dots (2m)] - [13][245 \dots (2m)] + \dots$$
$$\dots + (-1)^{2m}[1\,(2m)][234 \dots (2m-1)]; \quad (1)$$

and afterwards to develop some of the consequences of the formulæ so obtained.

Equation (1) gives the expansion of a Pfaffian of the m-th type in terms of certain Pfaffians of the first type and their complementaries. It would seem probable that similar formulæ should exist for the development of a Pfaffian of the m-th type in terms of Pfaffians of any lower type and their complementaries. We will endeavour to obtain such formulæ by the method of induction.

Now we have

$$\begin{aligned}[123 \dots (2m)] &= -[2134 \dots (2m)] \\ &= -[21][345 \dots (2m)] + [23][145 \dots (2m)] \\ &\quad -[24][135 \dots (2m)] + \dots \\ &\qquad \dots + (-1)^{2m-1}[2\,(2m)][134 \dots (2m-1)] \\ &= [12][345 \dots (2m)] + [23][145 \dots (2m)] \\ &\quad -[24][135 \dots (2m)] + \dots \\ &\qquad \dots + (-1)^{2m-1}[2\,(2m)][134 \dots (2m-1)].\end{aligned}$$

Similarly, we should obtain

$$\begin{aligned}[123 \dots (2m)] &= -[13][245 \dots (2m)] + [23][145 \dots (2m)] \\ &\quad + [34][1256 \dots (2m)] - [35][1246 \dots (2m)] + \dots \\ &\quad \dots + (-1)^{2m-2}[3\,(2m)][1245 \dots (2m-1)].\end{aligned}$$

We have, in all, $2m$ equations of this type, including equation (1).

Adding these together, and dividing the result by 2, we obtain

$$\begin{aligned} m\,[123\ldots(2m)] = {} & [12][345\ldots(2m)]-[13][245\ldots(2m)]+\ldots \\ & \ldots+(-1)^{2m}[1\,(2m)][234\ldots(2m-1)] \\ & +[23][145\ldots(2m)]-[24][135\ldots(2m)] \\ & \ldots+(-1)^{2m-1}[2\,(2m)][134\ldots(2m-1)] \\ & +[34][1256\ldots(2m)]-[35][1246\ldots(2m)]+\ldots \\ & \ldots+(-1)^{2m-2}[3\,(2m)][1245\ldots(2m-1)] \\ & +\ldots. \end{aligned}$$

Taking account of equation (1), this reduces to

$$\begin{aligned} (m-1)\,[123\ldots(2m)] = {} & [23][145\ldots(2m)]-[24][135\ldots(2m)]+\ldots \\ & \ldots+(-1)^{2m-1}[2\,(2m)][134\ldots(2m-1)] \\ & +[34][1256\ldots(2m)-[35][1246\ldots(2m)]+\ldots \\ & \ldots+(-1)^{2m-2}[3\,(2m)][1245\ldots(2m-1)]+\ldots. \end{aligned}$$

This equation may be written in the form

$$\begin{aligned} & (m-1)\,[123\ldots(2m)] \\ & \quad = \Sigma\pm[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(2m)][pq], \qquad (2) \end{aligned}$$

where p and q denote two numbers, standing in their natural order, abstracted from the set

$$2,\ 3,\ 4,\ \ldots,\ 2m.$$

The Σ denotes that all possible products of the given type are to be included, the sign of any given product being positive or negative according as the number of displacements required to restore the series

$$p,\ q,\ 2,\ 3,\ 4,\ \ldots,\ (p-1),\ (p+1),\ \ldots,\ (q-1),\ (q+1),\ \ldots,\ (2m)$$

to its natural numerical order is even or odd.

Now the theorem expressed by equation (2) may be applied to each of the coefficients in that equation. Thus we have

$$\begin{aligned} & (m-2)[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(2m)] \\ & \quad = \Sigma\pm[12\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots \\ & \qquad\qquad \ldots(s-1)(s+1)\ldots(2m)][rs]. \end{aligned}$$

Therefore we obtain

$$(m-1)(m-2)[123\ldots(2m)]$$
$$=2\Sigma\pm[12\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots$$
$$\ldots(s-1)(s+1)\ldots(2m)]$$
$$\times\{[pq][rs]-[pr][qs]+[ps][qr]\},$$

which reduces to

$$\tfrac{1}{2}(m-1)(m-2)[123\ldots(2m)]$$
$$=\Sigma\pm[12\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots$$
$$\ldots(s-1)(s+1)\ldots(2m)][pqrs]. \quad (3)$$

In equation (3), p, q, r, s denote four numbers, standing in their natural order, abstracted from the set

$$2,\ 3,\ 4,\ \ldots,\ 2m.$$

A rule similar to that we gave in the case of equation (2) obtains for the determination of the sign of any particular product included under the Σ.

We may again apply the theorem expressed by equation (2) to the coefficients of the Pfaffians of the type $[pqrs]$ contained in equation (3). Thus suppose p, q, r, s, t, u to be six numbers, standing in their natural numerical order, abstracted from the set

$$2,\ 3,\ 4,\ \ldots,\ 2m.$$

We will denote the Pfaffian formed from the numbers

$$1,\ 2,\ 3,\ 4,\ \ldots,\ 2m$$

by leaving out $\quad p,\ q,\ r,\ s,\ t,\ u$

by the symbol $\quad C[pqrstu]$.

Then we have

$$\tfrac{1}{2}(m-1)(m-2)(m-3)[123\ldots(2m)]$$
$$\begin{aligned}=\Sigma\pm C[pqrstu]\{\ &[pq][rstu]-[pr][qstu]+[ps][qrtu]\\ &-[pt][qrsu]+[pu][qrst]+[qr][pstu]\\ &-[qs][prtu]+[qt][prsu]-[qu][prst]\\ &+[rs][pqtu]-[rt][pqsu]+[ru][pqst]\\ &+[st][pqru]-[su][pqrt]+[tu][pqrs]\}.\end{aligned}$$

But, in virtue of what we have proved above, we see that

$$[pq][rstu]-[pr][qstu]+\ldots+[tu][pqrs] = 3\,[pqrstu].$$

Thus we obtain

$$\frac{(m-1)(m-2)(m-3)}{1.2.3}[123\ldots(2m)] = \Sigma\pm C[pqrstu][pqrstu]. \quad (4)$$

The method of procedure is now evident, as it is easily seen that the same process may be repeated. By its repeated application we arrive at a theorem which may be written in the form

$$\frac{(m-1)(m-2)\ldots(m-n+1)}{(n-1)!}[123\ldots(2m)]$$
$$= \Sigma\pm[1p_1p_2\ldots p_{2n-1}]\,C[1p_1p_2\ldots p_{2n-1}]. \quad (5)$$

In this formula $p_1, p_2, \ldots, p_{2n-1}$

denote $2n-1$ numbers, standing in their natural order, abstracted from the set $2, 3, 4, \ldots, 2m$.

The sign of any particular product, included under the Σ, is decided by the number of displacements required to restore the series of numbers

$$p_1, p_2, \ldots, p_{2n-1}, 2, 3, 4, \ldots, (p_1-1), (p_1+1), \ldots$$

to its natural numerical order, according to the customary rule.

2. Reverting to the original definition of a Pfaffian as the square root of a skew-symmetrical determinant of even order, we see that it is easy to justify that the repetition of any one of the numbers occurring within its symbolical expression necessitates the evanescence of the said Pfaffian.

Thus, if in equation (1) we replace $2m$ by unity, we obtain the known theorem

$$[12][1345\ldots(2m-1)]-[13][1245\ldots(2m-1)]+\ldots$$
$$\ldots+(-1)^{2m-1}[1\,(2m-1)][1234\ldots(2m-2)] = 0, \quad (6)$$

which is also capable of an easy direct proof.

In a similar manner we can deduce a new theorem from any special case of the general theorem of the preceding article. Thus we have

$$\Sigma\pm[1pqr][123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots$$
$$\ldots(2m-1)] = 0, \quad (7)$$

and so on.

Now consider the expression

$$[12xy][1345 \ldots (2m-1)\,xy]-[13xy][1245 \ldots (2m-1)\,xy]+\ldots$$
$$\ldots+(-1)^{2m-1}[1\,(2m-1)\,xy][1234 \ldots (2m-2)\,xy].$$

On expanding the Pfaffians of the second type we obtain for the coefficient of $[1x]$ in the above expression

$$-\{[2y][1345 \ldots (2m-1)\,xy]-[3y][1245 \ldots (2m-1)\,xy]+\ldots$$
$$\ldots+(-1)^{2m-1}[(2m-1)\,y][1234\,(2m-2)\,xy]\}$$
$$=[xy][1234 \ldots (2m-1)\,y]-[1y][2345 \ldots (2m-1)\,xy],$$

by an application of the theorem contained in equation (6).

Hence the above expression becomes

$$[xy]\,\{[12][1345 \ldots (2m-1)\,xy]-[13][1245 \ldots (2m-1)\,xy]+\ldots$$
$$\ldots+(-1)^{2m-1}[1\,(2m-1)][1234 \ldots (2m-2)\,xy]$$
$$+[1x][1234 \ldots (2m-1)\,y]-[1y][1234 \ldots (2m-1)\,x]\}=0,$$

by a further application of the theorem contained in equation (6). Thus we obtain

$$[12xy][1345 \ldots (2m-1)\,xy]-[13xy][1245 \ldots (2m-1)\,xy]+\ldots$$
$$\ldots+(-1)^{2m-1}[1\,(2m-1)\,xy][1234 \ldots (2m-2)\,xy]=0. \quad (8)$$

Consider next the expression

$$[12x_1y_1x_2y_2][1345 \ldots (2m-1)\,x_1y_1x_2y_2]$$
$$-[13x_1y_1x_2y_2][1245 \ldots (2m-1)\,x_1y_1x_2y_2]+\ldots$$
$$\ldots+(-1)^{2m-1}[1\,(2m-1)\,x_1y_1x_2y_2][1234 \ldots (2m-2)\,x_1y_1x_2y_2].$$

Expanding the Pfaffians of the third type, we obtain for the coefficient of $[1x_1]$ in this expression

$$-\{[2y_1x_2y_2][1345 \ldots (2m-1)\,x_1y_1x_2y_2]$$
$$-[3y_1x_2y_2][1245 \ldots (2m-1)\,x_1y_1x_2y_2]+\ldots$$
$$\ldots+(-1)^{2m-1}[(2m-1)\,y_1x_2y_2][1234 \ldots (2m-2)\,x_1y_1x_2y_2]\}$$
$$=\quad [x_1y_1x_2y_2][1234 \ldots (2m-1)\,y_1x_2y_2]$$
$$-[1y_1x_2y_2][2345 \ldots (2m-1)\,x_1y_1x_2y_2],$$

by an application of the theorem contained in equation (8). Hence, since we have

$$[1x_1][1y_1x_2y_2]-[1y_1][1x_1x_2y_2]+[1x_2][1x_1y_1y_2]-[1y_2][1x_1y_1x_2]=0,$$

it follows that our expression reduces to

$$\begin{aligned}[x_1y_1x_2y_2]\{\ &[12][1345\ldots(2m-1)\,x_1y_1x_2y_2]\\ &-[13][1245\ldots(2m-1)\,x_1y_1x_2y_2]+\ldots\\ &\ldots+(-1)^{2m-1}[1\,(2m-1)][1234\ldots(2m-2)\,x_1y_1x_2y_2]\\ &+[1x_1][1234\ldots(2m-1)\,y_1x_2y_2]\\ &-[1y_1][1234\ldots(2m-1)\,x_1x_2y_2]\\ &+[1x_2][1234\ldots(2m-1)\,x_1y_1y_2]\\ &-[1y_2][1234\ldots(2m-1)\,x_1y_1x_2]\}=0,\end{aligned}$$

by an application of the theorem contained in equation (6). Thus we have

$$\begin{aligned}&[12x_1y_1x_2y_2][1345\ldots(2m-1)\,x_1y_1x_2y_2]\\ &-[13x_1y_1x_2y_2][1245\ldots(2m-1)\,x_1y_1x_2y_2]+\ldots\\ &\ldots+(-1)^{2m-1}[1\,(2m-1)\,x_1y_1x_2y_2][1234\ldots(2m-2)\,x_1y_1x_2y_2]=0.\quad(9)\end{aligned}$$

This process is clearly capable of indefinite repetition. Thus we eventually obtain, as the generalization of the theorems contained in equations (8) and (9), the equation

$$\begin{aligned}&[12x_1y_1\ldots x_ny_n][1345\ldots(2m-1)\,x_1y_1\ldots x_ny_n]\\ &-[13x_1y_1\ldots x_ny_n][1245\ldots(2m-1)\,x_1y_1\ldots x_ny_n]+\ldots\\ &\ldots+(-1)^{2m-1}[1(2m-1)\,x_1y_1\ldots x_ny_n][1234\ldots(2m-2)\,x_1y_1\ldots x_ny_n]=0.\end{aligned}\quad(10)$$

We are now in a position to obtain an interesting general theorem which possesses an important application. Thus, consider the expression

$$\begin{aligned}&[12x_1y_1\ldots x_ny_n][345\ldots(2m)\,x_1y_1\ldots x_ny_n]\\ &-[13x_1y_1\ldots x_ny_n][245\ldots(2m)\,x_1y_1\ldots x_ny_n]+\ldots\\ &\ldots+(-1)^{2m}[1\,(2m)\,x_1y_1\ldots x_ny_n][234\ldots(2m-1)\,x_1y_1\ldots x_ny_n].\end{aligned}$$

Expanding the Pfaffians of the $(n+1)$-th type, we obtain for the coefficient of $[1x_1]$ in this expression

$$\begin{aligned}-\{\ &[2y_1\ldots x_ny_n][345\ldots(2m)\,x_1y_1\ldots x_ny_n]\\ &-[3y_1\ldots x_ny_n][245\ldots(2m)\,x_1y_1\ldots x_ny_n]+\ldots\\ &\ldots+(-1)^{2m}[(2m)\,y_1\ldots x_ny_n][234\ldots(2m-1)\,x_1y_1\ldots x_ny_n]\}\\ =-&[x_1y_1\ldots x_ny_n][234\ldots(2m)\,y_1x_2y_2\ldots x_ny_n],\end{aligned}$$

by an application of the theorem contained in equation (10).* Thus our expression becomes

$$\begin{aligned}[x_1y_1 \dots x_ny_n] \{ \; & [12][345 \dots (2m)\, x_1y_1 \dots x_ny_n] \\ & -[13][245 \dots (2m)\, x_1y_1 \dots x_ny_n] + \dots \\ & \dots + (-1)^{2m}[1\,(2m)][234 \dots (2m-1)\, x_1y_1 \dots x_ny_n] \\ & -[1x_1][234 \dots (2m)\, y_1 x_2y_2 \dots x_ny_n] \\ & +[1y_1][234 \dots (2m)\, x_1 x_2y_2 \dots x_ny_n] - \dots \\ & \dots + [1y_n][234 \dots (2m)\, x_1y_1 \dots x_n] \} \\ = \; & [x_1y_1 \dots x_ny_n][123 \dots (2m)\, x_1y_1 \dots x_ny_n].\end{aligned}$$

Thus we obtain the general theorem

$$\begin{aligned} & [12x_1y_1 \dots x_ny_n][345 \dots (2m)\, x_1y_1 \dots x_ny_n] \\ & -[13x_1y_1 \dots x_ny_n][245 \dots (2m)\, x_1y_1 \dots x_ny_n] + \dots \\ & \dots + (-1)^{2m}[1\,(2m)\, x_1y_1 \dots x_ny_n][234 \dots (2m-1)\, x_1y_1 \dots x_ny_n] \\ & = [x_1y_1 \dots x_ny_n][123 \dots (2m)\, x_1y_1 \dots x_ny_n]. \qquad (11)\end{aligned}$$

3. It is evident that the process which we have employed in Art. 1 to obtain equations (2), (3), (4), &c., from equation (1) may with equal facility be applied to equation (11). The equation corresponding to (2) is obtained without difficulty. On the left-hand side of the equation corresponding to (3), we shall have

$$(m-1)(m-2)[123 \dots (2m)\, x_1y_1 \dots x_ny_n]\{[x_1y_1 \dots x_ny_n]\}^2.$$

Corresponding to the expression

$$[pq][rs]-[pr][qs]+[ps][qr]$$

on the right-hand side of the same equation, we shall have

$$\begin{aligned}[pq\,x_1y_1 \dots x_ny_n][rs\,x_1y_1 \dots x_ny_n] - [pr\,x_1y_1 \dots x_ny_n][qs\,x_1y_1 \dots x_ny_n] \\ + [ps\,x_1y_1 \dots x_ny_n][qr\,x_1y_1 \dots x_ny_n],\end{aligned}$$

* To obtain this particular application, write out the theorem on the supposition that the numbers involved are

$$1,\ 2,\ 3,\ \dots,\ (2m+1),\ x_2,\ y_2,\ \dots,\ x_n,\ y_n;$$

and then replace unity by y_1 and $(2m+1)$ by x_1 and transpose the numbers within the square brackets.

which, by means of the theorem contained in equation (11), reduces to

$$[x_1 y_1 \dots x_n y_n]\,[pqrs\, x_1 y_1 \dots x_n y_n].$$

Dividing both sides of the equation by $[x_1 y_1 \dots x_n y_n]$, we obtain the required result.

Similar remarks apply to equation (3). Thus, if we make use of the symbol

$$K\,[1 p_1 p_2 \dots p_{2l-1}]$$

to denote the Pfaffian obtained from the set of numbers

$$1,\ 2,\ 3,\ \dots,\ 2m,\ x_1,\ y_1,\ \dots,\ x_n,\ y_n$$

by leaving out the set $\quad 1,\ p_1,\ p_2,\ \dots,\ p_{2l-1},$

which are all supposed to be abstracted from the set formed by the first $2m$ of these numbers, we arrive at the theorem

$$\Sigma \pm [1 p_1 p_2 \dots p_{2l-1} x_1 y_1 \dots x_n y_n]\, K\,[1 p_1 p_2 \dots p_{2l-1}]$$
$$= \frac{(m-1)(m-2)\dots(m-l+1)}{(l-1)!}\,[123\dots(2m)\, x_1 y_1 \dots x_n y_n]\,[x_1 y_1 \dots x_n y_n]. \tag{12}$$

It is evident that we may treat this equation in a similar manner to that in which we treated the equations of Art. 1 in order to obtain equations (6) and (7). We shall thus arrive at a general theorem containing that expressed by equation (10) as a particular case. Thus, supposing

$$p_1,\ p_2,\ \dots,\ p_{2l-1}$$

to be any set of numbers abstracted from the set

$$2,\ 3,\ 4,\ \dots,\ 2m-1,$$

and denoting the Pfaffian formed from the set

$$1,\ 2,\ 3,\ 4,\ \dots,\ 2m-1,\ x_1,\ y_1,\ \dots,\ x_n,\ y_n$$

by leaving out these by the symbol

$$K\,[p_1 p_2 \dots p_{2l-1}],$$

we have $\quad \Sigma \pm [1 p_1 p_2 \dots p_{2l-1} x_1 y_1 \dots x_n y_n]\, K\,[p_1 p_2 \dots p_{2l-1}] = 0. \qquad (13)$

4. In the theory of the reduction of a Pfaffian expression, we come across Pfaffians of a special form. This special form, though giving rise to certain differential relations of great importance, in no way modifies the form of the algebraic relations by which the Pfaffians are connected. From the theorems developed in the present com-

munication, we can deduce certain relations that are of importance in the said theory. Thus, consider the theorem

$$[12x_1y_1 \ldots x_ny_n][34x_1y_1 \ldots x_ny_n]-[13x_1y_1 \ldots x_ny_n][24x_1y_1 \ldots x_ny_n]$$
$$+[14x_1y_1 \ldots x_ny_n][23x_1y_1 \ldots x_ny_n]$$
$$= [x_1y_1 \ldots x_ny_n][1234x_1y_1 \ldots x_ny_n].$$

As particular cases of this we have a set of theorems of the type

$$[1256 \ldots (2m)][3456 \ldots (2m)]-[1356 \ldots (2m)][2456 \ldots (2m)]$$
$$+[1456 \ldots (2m)][2356 \ldots (2m)] = [56 \ldots (2m)][123 \ldots (2m)], \quad (14)$$

which are of importance in the theory referred to.

On Burmann's Theorem. By A. C. Dixon.

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The following method gives a proof of Burmann's form of Lagrange's theorem, and also an extension of it which is curious and may possibly be useful.

Let Fx, fx be two functions of the complex variable x, and C a simple contour in the x-plane such that Fx, fx are analytical within and on C, and that $|fx| = k$, a constant, along C.

Let $a_1, a_2, a_3, \ldots$ be the points within C at which fx vanishes, and $b_1, b_2, b_3, \ldots$ those at which fx has a value c such that $|c| < k$. Denote $fx \div (x-a_r)$ by f_rx ($r = 1, 2, \ldots$), and suppose $f_1a_1, f_2a_2, \ldots$ not to vanish.

The value of
$$\int_{(C)} \frac{Fxf'xdx}{fx-c}$$
is
$$2\iota\pi\{Fb_1+Fb_2+Fb_3+\ldots\}.$$

But the subject of integration may be expanded in ascending powers of c, since $|c| < |f|$ along C. Hence

$$2\iota\pi\{Fb_1+Fb_2+\ldots\}$$
$$= \int_{(C)} \frac{Fxf'x}{fx}dx + c\int_{(C)} \frac{Fxf'x}{(fx)^2}dx + \ldots + c^n\int_{(C)} \frac{Fxf'x}{(fx)^{n+1}}dx + \ldots.$$