On some Classes of Multiple Definite Integrals.
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[*Read November* 9*th*, 1876.]

1. In what follows, repeated use will be made of the facts that, a and a' being any positive constants,

$$\int_0^\infty \frac{\phi(ax)-\phi(a'x)}{x}\,dx = \int_{a'}^{a} da \int_0^\infty \phi'(ax)\,dx = \log\frac{a}{a'}\,\{\phi(\infty)-\phi(0)\} \qquad \ldots\ldots\ldots\ldots(1)$$

if the limits $\phi(\infty)$ and $\phi(0)$ are definite, and that

$$\int_{-\infty}^{0} \frac{\phi(ax)-\phi(a'x)}{x}\,dx = \int_{a'}^{a} da \int_{-\infty}^{0} \phi'(ax)\,dx = \log\frac{a}{a'}\,\{\phi(0)-\phi(-\infty)\} \qquad \ldots\ldots\ldots\ldots(2)$$

if the limits $\phi(0)$ and $\phi(-\infty)$ are both definite.

Consider first the double integral $\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy$, in which a, a', b, b' are any positive constants.

By (1), we have

$$\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(ax+b'y)}{xy}\,dx\,dy = \int_0^\infty \log\frac{b}{b'}\,\{\phi(\infty)-\phi(ax)\}\,\frac{dx}{x}$$

provided that neither $\phi(\infty)$ nor $\phi(ax)$ be infinite, and

$$\int_0^\infty\int_0^\infty \frac{\phi(ax+b'y)-\phi(a'x+b'y)}{xy}\,dx\,dy = \int_0^\infty \log\frac{a}{a'}\,\{\phi(\infty)-\phi(b'y)\}\,\frac{dy}{y}$$

under a like condition. Thus, adding,

$$\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy = \left\{\log\frac{a}{a'}+\log\frac{b}{b'}\right\}\phi(\infty)\int_0^\infty\frac{dx}{x}$$
$$-\log\frac{b}{b'}\int_0^\infty\frac{\phi(ax)}{x}\,dx-\log\frac{a}{a'}\int_0^\infty\frac{\phi(b'x)}{x}\,dx.$$

Therefore, in the case when $\log\frac{a}{a'}+\log\frac{b}{b'}=0$, *i.e.*, when $ab=a'b'$,

$$\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy = \log\frac{a}{a'}\int_0^\infty\frac{\phi(ax)-\phi(b'x)}{x}\,dx$$
$$= \log\frac{a}{a'}\log\frac{a}{b'}\,\{\phi(\infty)-\phi(0)\} \qquad \ldots\ldots\ldots\ldots\ldots(3),$$

the only restriction as to the nature of the function ϕ being, that it be not made infinite when its argument is zero or any positive quantity finite or infinite.

In precisely the same manner, using (2), we find that, if the function

ϕ be such as not to become infinite when its argument is zero or any negative quantity, then, under the same condition $ab = a'b'$,

$$\int_{-\infty}^{0}\int_{-\infty}^{0} \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy = \log\frac{a}{a'}\log\frac{a}{b'}\{\phi(0)-\phi(-\infty)\} \quad \text{.........(4).}$$

In the results (3) and (4), it is of course to be noticed that a may be either one of the two constants a and b.

2. Consider now the triple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz)-\phi(a'x+b'y+c'z)}{xyz}\,dx\,dy\,dz,$$

the function ϕ being of the same nature as in (3), and the coefficients a, b, c, a', b', c' all positive. We have, by three applications of (1),

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz)-\phi(ax+by+c'z)}{xyz}\,dx\,dy\,dz$$
$$= \log\frac{c}{c'}\int_0^\infty\int_0^\infty \frac{\phi(\infty)-\phi(ax+by)}{xy}\,dx\,dy,$$

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+c'z)-\phi(ax+b'y+c'z)}{xyz}\,dx\,dy\,dz$$
$$= \log\frac{b}{b'}\int_0^\infty\int_0^\infty \frac{\phi(\infty)-\phi(ax+c'z)}{xz}\,dx\,dz,$$

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+b'y+c'z)-\phi(a'x+b'y+c'z)}{xyz}\,dx\,dy\,dz$$
$$= \log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(\infty)-\phi(b'y+c'z)}{yz}\,dy\,dz.$$

If then $\log\frac{a}{a'}+\log\frac{b}{b'}+\log\frac{c}{c'}=0$, *i.e.*, if $abc = a'b'c'$, we have, by addition,

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz)-\phi(a'x+b'y+c'z)}{xyz}\,dx\,dy\,dz$$
$$= -\log\frac{c}{c'}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)}{xy}\,dx\,dy + \left(\log\frac{c}{c'}+\log\frac{a}{a'}\right)\int_0^\infty\int_0^\infty \frac{\phi(ax+c'z)}{xz}\,dx\,dz$$
$$-\log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(b'y+c'z)}{yz}\,dy\,dz$$
$$= \log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(ax+c'y)-\phi(b'x+c'y)}{xy}\,dx\,dy$$
$$-\log\frac{c}{c'}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(ax+c'y)}{xy}\,dx\,dy$$
$$= \log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty \frac{\phi(\infty)-\phi(c'y)}{y}\,dy$$
$$-\log\frac{c}{c'}\log\frac{b}{c'}\int_0^\infty \frac{\phi(\infty)-\phi(ax)}{x}\,dx, \text{ by (1);}$$

and so, provided that $\log\frac{a}{a'}\log\frac{a}{b'}=\log\frac{c}{c'}\log\frac{b}{c'}$,

$$=\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\frac{\phi(ax)-\phi(c'x)}{x}\,dx$$

$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(\infty)-\phi(0)\}\ldots\ldots\ldots\ldots(5).$$

The two conditions in the six constants formed above for the truth of this may be written

$$abc=a'b'c',\quad \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}=\log\frac{a}{c}\log\frac{b}{c'}\log\frac{c}{c'},$$

In precisely the same manner, using (2), we find that, ϕ being of the same nature as in (4), under the same two conditions in the coefficients,

$$\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^0\frac{\phi(ax+by+cz)-\phi(a'x+b'y+c'z)}{xyz}\,dx\,dy\,dz$$

$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(0)-\phi(-\infty)\}\ldots\ldots(6).$$

It is to be noticed that, in these conditions and results, a denotes any one of the three a, b, c, and c' any one of a', b', c'.

3. Again, the quadruple integral, in which ϕ is of the same nature as in (3) and (5), and the constants a, b, c, d, a', b', c', d' all positive,

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{\phi(ax+by+cz+du)-\phi(a'x+b'y+c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{\phi(ax+by+cz+du)-\phi(ax+by+c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{\phi(ax+by+c'z+d'u)-\phi(a'x+b'y+c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$

$$=\log\frac{c}{c'}\log\frac{c}{d'}\int_0^\infty\int_0^\infty\frac{\phi(\infty)-\phi(ax+by)}{xy}\,dx\,dy$$

$$+\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\int_0^\infty\frac{\phi(\infty)-\phi(c'z+d'u)}{zu}\,dz\,du,$$

by (3), provided that $ab=a'b'$ and $cd=c'd'$. If, in addition, $\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}=0$, this becomes

$$=\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\int_0^\infty\frac{\phi(ax+by)-\phi(c'x+d'y)}{xy}\,dx\,dy$$

$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\}\ldots\ldots\ldots\ldots(7),$$

if the additional condition $ab=c'd'$ is satisfied. Thus there are altogether four conditions

$$ab = cd = a'b' = c'd', \quad \log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} = 0.$$

In like manner, under the same four conditions, ϕ being now of the same nature as in (4) and (6),

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0} \frac{\phi(ax+by+cz+du)-\phi(a'x+b'y+c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$
$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(0)-\phi(-\infty)\} \text{........(8).}$$

Like methods may now be employed to calculate the values of the analogous multiple integrals of higher orders under conditions which are obtained in the course of the work.

4. Again, consider the double integral

$$\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy,$$

the function ϕ being such as in (3), (5), and (7). We have

$$\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)-\phi(ax)\,\phi(b'y)}{xy}\,dx\,dy$$
$$= \int_0^\infty \frac{\phi(ax)}{x}\,dx \int_0^\infty \frac{\phi(by)-\phi(b'y)}{y}\,dy$$
$$= \log\frac{b}{b'}\{\phi(\infty)-\phi(0)\}\int_0^\infty \frac{\phi(ax)}{x}\,dx, \text{ by (1),}$$

and

$$\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(b'y)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy$$
$$= \log\frac{a}{a'}\{\phi(\infty)-\phi(0)\}\int_0^\infty \frac{\phi(b'y)}{y}\,dy$$

in like manner. Thus, adding, under the condition $\log\frac{a}{a'}+\log\frac{b}{b'} = 0$, *i.e.*, $ab = a'b'$, we have that in that case

$$\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy$$
$$= -\log\frac{a}{a'}\{\phi(\infty)-\phi(0)\}\int_0^\infty \frac{\phi(ax)-\phi(b'x)}{x}\,dx$$
$$= -\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}^2 \text{........................(9).}$$

In precisely the same way, by (2),

$$\int_{-\infty}^{0}\int_{-\infty}^{0} \frac{\phi(ax)\,\phi(by)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy$$
$$= -\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(0)-\phi(-\infty)\}^2 \text{.................. (10),}$$

under the same condition $ab = a'b'$, ϕ in this case being of the same nature as in (4), (6), and (8).

5. The triple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)\,\phi(cz)-\phi(a'x)\,\phi(b'y)\,\phi(c'z)}{xyz}\,dx\,dy\,dz$$

$$= \int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)}{xy}\,dx\,dy\int_0^\infty \frac{\phi(cz)-\phi(c'z)}{z}\,dz$$
$$+\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(c'z)}{xz}\,dx\,dz\int_0^\infty \frac{\phi(by)-\phi(b'y)}{y}\,dy$$
$$+\int_0^\infty\int_0^\infty \frac{\phi(b'y)\,\phi(c'z)}{yz}\,dy\,dz\int_0^\infty \frac{\phi(ax)-\phi(a'x)}{x}\,dx$$

$$= \{\phi(\infty)-\phi(0)\}\left\{\log\frac{c}{c'}\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)}{xy}\,dx\,dy\right.$$
$$\left.+\log\frac{b}{b'}\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(c'z)}{xz}\,dx\,dz+\log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(b'y)\,\phi(c'z)}{yz}\,dy\,dz\right\},$$

which, if $\log\frac{a}{a'}+\log\frac{b}{b'}+\log\frac{c}{c'}=0$, *i.e.*, if $abc = a'b'c'$, may be written

$$= \{\phi(\infty)-\phi(0)\}\left\{\log\frac{a}{c'}\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)-\phi(ax)\,\phi(c'y)}{xy}\,dx\,dy\right.$$
$$\left.-\log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(c'y)-\phi(b'x)\,\phi(c'y)}{xy}\,dx\,dy\right\}$$

$$= \{\phi(\infty)-\phi(0)\}^2\left\{\log\frac{c}{c'}\log\frac{b}{c'}\int_0^\infty \frac{\phi(ax)}{x}\,dx\right.$$
$$\left.-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty \frac{\phi(c'y)}{y}\,dy\right\},$$

which, if $\log\frac{a}{a'}\log\frac{c}{b'}=\log\frac{c}{c'}\log\frac{b}{c'}$, may be written

$$= \{\phi(\infty)-\phi(0)\}^2\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty \frac{\phi(ax)-\phi(c'x)}{x}\,dx$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(\infty)-\phi(0)\}^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(11)}.$$

So, too, under the same two conditions,

$$abc = a'b'c',\quad \log\frac{a}{a'}\log\frac{a}{b'} = \log\frac{b}{c'}\log\frac{c}{c'},$$

which latter may be written more symmetrically

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'} = \log\frac{a}{c'}\log\frac{b}{c'}\log\frac{c}{c'},$$

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0}\frac{\phi(ax)\,\phi(by)\,\phi(cz)-\phi(a'x)\,\phi(b'y)\,\phi(c'z)}{xyz}\,dx\,dy\,dz$$
$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(0)-\phi(-\infty)\}^3 \text{........ (12).}$$

6. The corresponding quadruple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{\phi(ax)\phi(by)\phi(cz)\phi(du)-\phi(a'x)\phi(b'y)\phi(c'z)\phi(d'u)}{xyzu}\,dx\,dy\,dz\,du$$
$$=\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)}{xy}\,dx\,dy\int_0^\infty\int_0^\infty\frac{\phi(cz)\,\phi(du)-\phi(c'z)\,\phi(d'u)}{zu}\,dz\,du$$
$$+\int_0^\infty\int_0^\infty\frac{\phi(c'z)\,\phi(d'u)}{zu}\,dz\,du\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy$$
$$=-\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)}{xy}\,dx\,dy\,.\log\frac{c}{c'}\log\frac{c}{d'}\{\phi(\infty)-\phi(0)\}^2$$
$$-\int_0^\infty\int_0^\infty\frac{\phi(c'z)\,\phi(d'u)}{zu}\,dz\,du\,.\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}^2,$$

by (9), provided that $cd=c'd'$ and $ab=a'b'$. If in addition

$$\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}=0,$$

this becomes

$$=\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}^2\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)-\phi(c'x)\,\phi(d'y)}{xy}\,dx\,dy$$
$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\}^4 \text{........................ (13),}$$

provided that also $ab=c'd'$.

Similarly, under the same four conditions $ab=cd=a'b'=c'd'$ and $\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}=0$, the value of the quadruple integral of the like expression, each integration being between lower limit $-\infty$ and upper limit 0, is

$$-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(0)-\phi(-\infty)\}^4 \text{........ (14).}$$

Like methods may now be used to evaluate the corresponding multiple integrals of higher orders, under conditions in the constants obtained in the course of the work.

It is to be noticed that the conditions for the truth of the theorems (9) to (14) are in each case precisely the same as for that of the corresponding ones of the class (3) to (8).

7. Consider now the quadruple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by)\,\phi(cz+du)-\phi(a'x+b'y)\,\phi(c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du.$$

It may be written

$$= \int_0^\infty\int_0^\infty \frac{\phi(ax+by)}{xy}\,dx\,dy \int_0^\infty\int_0^\infty \frac{\phi(cz+du)-\phi(c'z+d'u)}{zu}\,dz\,du$$

$$+\int_0^\infty\int_0^\infty \frac{\phi(c'z+d'u)}{zu}\,dz\,du \int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty \frac{\phi(ax+by)}{xy}\,dx\,dy\,.\,\log\frac{c}{c'}\log\frac{c}{d'}\,\{\phi(\infty)-\phi(0)\}$$

$$+\int_0^\infty\int_0^\infty \frac{\phi(c'z+d'u)}{zu}\,dz\,du\,.\,\log\frac{a}{a'}\log\frac{a}{b'}\,\{\phi(\infty)-\phi(0)\},$$

by (3), provided that $cd=c'd'$ and $ab=a'b'$. If also

$$\log\frac{c}{c'}\log\frac{c}{d'}+\log\frac{a}{a'}\log\frac{a}{b'}=0,$$

this becomes

$$-\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(c'x+d'y)}{xy}\,dx\,dy$$

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\}^2 \ldots\ldots\ldots\ldots(15),$$

provided that also $ab=c'd'$.

In just the same way, under the same four conditions in the constants, viz.,

$$ab=cd=a'b'=c'd',\quad\text{and}\quad \log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}=0,$$

$$\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^0 \frac{\phi(ax+by)\,\phi(cz+du)-\phi(a'x+b'y)\,\phi(c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(0)-\phi(-\infty)\}^2 \ldots\ldots(16).$$

Observe that the conditions in the constants for the truth of (15) and (16) are the same as for that of the allied identities (7) and (8), and (13) and (14).

8. To take one more example of the same class: by three applications of (3),

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\phi(ax+by)\,\phi(cz+du)\,\phi(ev+fw)-\phi(a'x+b'y)\,\phi(c'z+d'u)\,\phi(e'v+f'w)}{xyzuvw}$$
$$\times\, dx\,dy\,dz\,du\,dv\,dw$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by)\,\phi(cz+du)}{xyzu}\,dx\,dy\,dz\,du$$
$$\times \log\frac{e}{e'}\log\frac{e}{f'}\{\phi(\infty)-\phi(0)\}$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by)\,\phi(e'v+f'w)}{xyvw}\,dx\,dy\,dv\,dw$$
$$\times \log\frac{c}{c'}\log\frac{c}{d'}\{\phi(\infty)-\phi(0)\}$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(c'z+d'u)\,\phi(e'v+f'w)}{zuvw}\,dz\,du\,dv\,dw$$
$$\times \log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\},$$

provided that $ef=e'f'$, $cd=c'd'$, and $ab=a'b'$. If then, in addition,

$$\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}+\log\frac{e}{e'}\log\frac{e}{f'}=0,$$

this becomes, by two more applications of (3),

$$= \log\frac{e}{e'}\log\frac{e}{f'}\{\phi(\infty)-\phi(0)\}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)}{xy}\,dx\,dy$$
$$\times \log\frac{c}{e'}\log\frac{c}{f'}\{\phi(\infty)-\phi(0)\}$$

$$-\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}\int_0^\infty\int_0^\infty \frac{\phi(e'v+f'w)}{vw}\,dv\,dw$$
$$\times \log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\},$$

provided that also $cd=e'f'$ and $ab=c'd'$,

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\}^2$$
$$\times\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(e'x+f'y)}{xy}\,dx\,dy,$$

if the additional condition

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}=\log\frac{c}{e'}\log\frac{c}{f'}\log\frac{e}{e'}\log\frac{e}{f'} \text{ holds,}$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}\{\phi(\infty)-\phi(0)\}^3 \ldots (17),$$

if also $ab=e'f'$.

In the same way the value of the like integral, with $-\infty$ and 0 for the limits of each integration, is the same multiple of $\{\phi(0)-\phi(-\infty)\}^2$ under the same conditions, which may be written as seven, thus

$$ab = cd = ef = a'b' = c'd' = e'f',$$

$$\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}+\log\frac{e}{e}\log\frac{e}{f'} = 0,$$

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}$$
$$+\log\frac{a}{f'}\log\frac{b}{f'}\log\frac{c}{f'}\log\frac{d}{f'}\log\frac{e}{f'}\log\frac{f}{f'} = 0.$$

9. Take, lastly, the simplest example of another class,

$$\int_0^\infty\int_0^\infty\ldots\int_0^\infty\frac{\phi(ax+by+cz)\phi(du+ev+fw)-\phi(a'x+b'y+c'z)\phi(d'u+e'v+f'w)}{xyzuvw}$$
$$\times\, dx\, dy\, dz\, du\, dv\, dw.$$

By two applications of (5), this may be written

$$=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\phi\,(ax+by+cz)}{xyz}\,dx\,dy\,dz\,.\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'}\{\phi(\infty)-\phi(0)\}$$
$$+\int_0^\infty\int_0^\infty\int_0^\infty\frac{\phi(d'u+e'v+f'w)}{uvw}\,du\,dv\,dw\,.\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(\infty)-\phi(0)\},$$

subject to the four conditions

$$abc = a'b'c',\quad def = d'e'f',\quad \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'} = \log\frac{a}{a'}\log\frac{b}{a'}\log\frac{c}{a'},$$
$$\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'} = \log\frac{d}{d'}\log\frac{e}{d'}\log\frac{f}{d'};$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(\infty)-\phi(0)\}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty\frac{\phi\,(ax+by+cz)-\phi\,(d'x+e'y+f'z)}{xyz}\,dx\,dy\,dz,$$

if in addition

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}+\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'} = 0;$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}\{\phi(\infty)-\phi(0)\}^2 \;\ldots\; (18),$$

if the two other conditions

$$abc = d'e'f',\quad \log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'} = \log\frac{a}{d'}\log\frac{b}{d'}\log\frac{c}{d'}\text{ hold.}$$

Similarly, under the same seven conditions, the like integral, with all the lower limits $-\infty$ and all the upper ones 0, is the same multiple of $\{\phi(0)-\phi(-\infty)\}^2$.

10. The above theorems may be multiplied by easy transformations. Thus, in (3), the substitution of $\log x$, $\log y$ for x, y respectively gives that, if $ab = a'b'$,

$$\int_1^\infty\int_1^\infty \frac{\phi[\log(x^a y^b)]-\phi[\log(x^{a'}y^{b'})]}{xy\log x\log y}\,dx\,dy = \log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\},$$

which may be written

$$\int_1^\infty\int_1^\infty \frac{\psi(x^a y^b)-\psi(x^{a'}y^{b'})}{xy\log x\log y}\,dx\,dy = \log\frac{a}{a'}\log\frac{a}{b'}\{\psi(\infty)-\psi(1)\},$$

the function ψ being such as not to become infinite for any value of its argument equal to or greater than unity.

So (4) gives that, under the same conditions $ab = a'b'$, if ψ be a function not made infinite by any value of its argument between 0 and 1 inclusively,

$$\int_0^1\int_0^1 \frac{\psi(x^a y^b)-\psi(x^{a'}y^{b'})}{xy\log x\log y}\,dx\,dy = \log\frac{a}{a'}\log\frac{a}{b'}\{\psi(1)-\psi(0)\}.$$

The same substitutions applied to (9) and (10) give

$$\int_1^\infty\int_1^\infty \frac{\psi(x^a)\psi(y^b)-\psi(x^{a'})\psi(y^{b'})}{xy\log x\log y}\,dx\,dy = -\log\frac{a}{a'}\log\frac{a}{b'}\{\psi(\infty)-\psi(1)\}^2,$$

and

$$\int_0^1\int_0^1 \frac{\psi(x^a)\psi(y^b)-\psi(x^{a'})\psi(y^{b'})}{xy\log x\log y}\,dx\,dy = -\log\frac{a}{a'}\log\frac{a}{b'}\{\psi(1)-\psi(0)\}^2,$$

under the same condition $ab = a'b'$.

And so like transformations may be applied to all the other above results (5) to (8) and (11) to (18), giving a new series of theorems true under the same conditions as before.

All these results may however be proved independently, using instead of (1) and (2) the equivalent formulæ

$$\int_1^\infty \frac{\psi(x^a)-\psi(x^{a'})}{x\log x}\,dx = \log\frac{a}{a'}\{\psi(\infty)-\psi(1)\},$$

$$\int_0^1 \frac{\psi(x^a)-\psi(x^{a'})}{x\log x}\,dx = \log\frac{a}{a'}\{\psi(1)-\psi(0)\},$$

which may be seen at once just as (1) and (2) were.

11. Again, transform (9) by substituting e^x, e^y for x, y respectively;

and for convenience write also α for $\log a$, β for $\log b$, &c. Then it becomes that, if $\alpha+\beta = \alpha'+\beta'$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\phi(\epsilon^{x+\alpha})\,\phi(\epsilon^{y+\beta})-\phi(\epsilon^{x+\alpha'})\,\phi(\epsilon^{y+\beta'})\}\,dx\,dy$$
$$=-(\alpha-\alpha')\,(\alpha-\beta')\,\{\phi(\infty)-\phi(0)\}^2,$$

which may be written

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\mathrm{F}(x+\alpha)\,\mathrm{F}(y+\beta)-\mathrm{F}(x+\alpha')\,\mathrm{F}(y+\beta')\}\,dx\,dy$$
$$=-(\alpha-\alpha')(\alpha-\beta')\{\mathrm{F}(\infty)-\mathrm{F}(-\infty)\}^2,$$

F being a function which no value of its argument makes infinite.

A like transformation applied to (11) gives that, under the conditions

$$\alpha+\beta+\gamma=\alpha'+\beta'+\gamma',$$
$$(\alpha-\alpha')\,(\alpha-\beta')\,(\alpha-\gamma') = (\alpha-\gamma')\,(\beta-\gamma')\,(\gamma-\gamma'),$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\mathrm{F}(x+\alpha)\,\mathrm{F}(y+\beta)\,\mathrm{F}(z+\gamma)$$
$$-\mathrm{F}(x+\alpha')\,\mathrm{F}(y+\beta')\,\mathrm{F}(z+\gamma')\}\,dx\,dy\,dz$$
$$=(\alpha-\alpha')(\beta-\beta')(\gamma-\gamma')\,\{\mathrm{F}(\infty)-\mathrm{F}(-\infty)\}^3.$$

In like manner, from (13), if $\alpha+\beta=\gamma+\delta=\alpha'+\beta'=\gamma'+\delta'$, and $(\alpha-\alpha')\,(\alpha-\beta')+(\gamma-\gamma')\,(\gamma-\delta')=0$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\mathrm{F}(x+\alpha)\,\mathrm{F}(y+\beta)\,\mathrm{F}(z+\gamma)\,\mathrm{F}(u+\delta)$$
$$-\mathrm{F}(x+\alpha')\,\mathrm{F}(y+\beta')\,\mathrm{F}(z+\gamma')\,\mathrm{F}(u+\delta')\}\,dx\,dy\,dz\,du$$
$$=-(\alpha-\alpha')\,(\alpha-\beta')\,(\alpha-\gamma')\,(\alpha-\delta')\,\{\mathrm{F}(\infty)-\mathrm{F}(-\infty)\}^4.$$

Other though less simply expressed theorems are obtained by similar transformations from (3), (5), (7), (15), (17), (18) above.

All these, however, may be proved easily without transformation. For they can be deduced, in the same way as their corresponding theorems have been from (1), by means of

$$\int_{-\infty}^{\infty}\{\mathrm{F}(x+\alpha)-\mathrm{F}(x+\beta)\}\,dx=(\alpha-\beta)\,\{\mathrm{F}(\infty)-\mathrm{F}(-\infty)\},$$

a transformation of (1) proved readily in the same way.

12. If it be allowable to substitute x^m, y^n, z^p, &c., for x, y, z, &c., in results (3) to (18), much more general results are obtained under the same conditions as before. I have not, however, been able to prove directly any of these theorems except in the special case when $m=n=p=$ &c., and have considerable doubt as to the lawfulness in general of the transformation. The class, however, for which

$m = n =$ &c., m being any positive quantity, of which the first, the tranformation of (3), is

$$\int_0^\infty \int_0^\infty \frac{\phi(ax^m+by^m)-\phi(a'x^m+b'y^m)}{xy}\, dx\, dy$$
$$= \frac{1}{m^2}\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\},$$

under the condition $ab = a'b'$, are readily proved, using instead of (1)

$$\int_0^\infty \frac{\phi(ax^m)-\phi(bx^m)}{x}\, dx = \frac{1}{m}\{\phi(\infty)-\phi(0)\}\log\frac{a}{b},$$

and the corresponding formula instead of (2).

NOTE.*—In formula (3), $ab = a'b'$, and factor in the result is $\log\frac{a}{a'}\log\frac{a}{b'}$.

Write $\log a$, $\log b$, $\log a'$, $\log b' = \alpha, \beta, \alpha', \beta'$; then $\alpha+\beta = \alpha'+\beta'$, and $\log\frac{a}{a'}\log\frac{a}{b'} = (\alpha-\alpha')(\alpha-\beta') = \alpha^2-\alpha(\alpha'+\beta')+\alpha'\beta', = \alpha^2-\alpha(\alpha+\beta)$ $+\alpha'\beta', = \alpha'\beta'-\alpha\beta$, viz., the factor is $= -\{\log a \log b - \log a' \log b'\}$.

So, in formula (5),

$$abc = a'b'c' \text{ and } \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'} = \log\frac{a}{c'}\log\frac{b}{c'}\log\frac{c}{c'}.$$

Factor in result is $\qquad \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}$.

Write $\log a$, &c. $= \alpha, \beta, \gamma, \alpha', \beta', \gamma'$; then $\alpha+\beta+\gamma = \alpha'+\beta'+\gamma'$; and it is clear *à priori*, that the factor should be expressible in any one of the six forms

$$\begin{aligned} \text{factor} &= \alpha-\alpha'.\,\alpha-\beta'.\,\alpha-\gamma' = \alpha-\alpha'.\,\beta-\alpha'.\,\gamma-\alpha' \\ &= \beta-\alpha'.\,\beta-\beta'.\,\beta-\gamma' = \alpha-\beta'.\,\beta-\beta'.\,\gamma-\beta' \\ &= \gamma-\alpha'.\,\gamma-\beta'.\,\gamma-\gamma' = \alpha-\gamma'.\,\beta-\gamma'.\,\gamma-\gamma'. \end{aligned}$$

It at once appears that, besides $\alpha+\beta+\gamma = \alpha'+\beta'+\gamma'$, we must have $\beta\gamma+\gamma\alpha+\alpha\beta = \beta'\gamma'+\gamma'\alpha'+\alpha'\beta'$; viz., these two relations existing, each form of the factor reduces itself to $\alpha\beta\gamma-\alpha'\beta'\gamma'$, thus

$$\alpha-\alpha'.\,\alpha-\beta'.\,\alpha-\gamma' = \alpha^3-\alpha^2.\,(\alpha+\beta+\gamma)+\alpha(\beta\gamma+\gamma\alpha+\alpha\beta)-\alpha'\beta'\gamma'$$
$$= \alpha\beta\gamma-\alpha'\beta'\gamma'.$$

Thus relations between the constants are

$$abc = a'b'c', \ \textit{i.e.}, \ \log a+\log b+\log c = \log a'+\log b'+\log c',$$

* For this note I have to thank Professor Cayley.

and $\quad \log b \log c + \log c \log a + \log a \log b$

$$= \log b' \log c' + \log c' \log a' + \log a' \log b',$$

and $\quad$ factor $= \log a \log b \log c - \log a' \log b' \log c'$.

It is presumed that, in the other cases considered, the forms of the factors might be modified in like manner, but this has not been examined.

On certain Identical Differential Relations.

By J. W. L. GLAISHER, M.A., F.R.S.

[*Read November 9th*, 1876.]

1. In the "Nouvelle Correspondance Mathématique," Tom. ii., pp. 240—243 (August, 1876), I have shown that the function $e^{\sqrt{x}}$ possesses the curious property that its $(n+1)^{\text{th}}$ differential coefficient is equal to its n^{th} integral, to a power of $4x$ *près*; that is to say, that

$$\left(\int^x dx\right)^n e^{\sqrt{x}} = (4x)^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} e^{\sqrt{x}};$$

or, otherwise, that

$$2^{2n+1} \left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} e^{\sqrt{x}} = e^{\sqrt{x}} \quad \text{..................(1)}.$$

This result is there obtained in the way in which I was led to it, viz., by means of the integral

$$\int_0^\infty e^{-ax^2-\frac{b}{x^2}}\, dx = \frac{\sqrt{\pi}}{2} \frac{e^{-2\sqrt{(ab)}}}{\sqrt{a}};$$

but it can be proved more easily by expanding $e^{\sqrt{x}}$ in ascending powers of $\sqrt{x}$, that is, by replacing $e^{\sqrt{x}}$ by $1 + x^{\frac{1}{2}} + \frac{x}{1.2} + \frac{x^{\frac{3}{2}}}{1.2.3} +$ &c., when it is readily seen that this series reproduces itself. It is rather interesting to note how this reproduction is brought about by the differentiations.

2. If in (1) we write $a\sqrt{x}$ for $\sqrt{x}$, we have

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} e^{a\sqrt{x}} = \frac{a^{2n+1}}{2^{2n+1}} e^{a\sqrt{x}}$$

$$= \frac{1}{2^{2n+1}} \left(\frac{d}{d\,.\,\sqrt{x}}\right)^{2n+1} e^{a\sqrt{x}};$$