

PREDETERMINATION IN RAILWAY WORK.

BY F. W. CARTER.

A presentation, with examples, of an analytical method of treatment of train-movement problems based upon the assumption that, within the working range of ordinary railway motors, the relations between tractive effort and current may be represented by a straight line, and between speed and current by a hyperbola.

I.—INTRODUCTORY.

To the steam engineer it must occasion some surprise to observe how closely the performance of an electric train can be predicted from the knowledge gained by a few stand tests on its motors. When the particulars of the problem are fully known, it is possible to determine how the motor will behave dynamically, when put to a particular service, with almost as great a degree of accuracy as its conduct can be observed. This possibility arises from the simplicity and definiteness of the circumstances on which the performance of the motor depend. At any particular voltage the torque and speed corresponding to a particular current are definite and can be determined from stand tests, whilst the change in these quantities with variation of voltage can be computed. The relation between torque and speed is all that one need know about the motor in order to determine its dynamical effect on the train. Knowing the frictional resistance to motion at any particular speed, we can, from the torque and mass moved, determine the rate of change of speed, and hence the interval of time required to produce a given small change in speed and so, by a process of point to point construction, obtain the speed-time curve, from which to determine the times, distances etc., which are the ultimate ends of the calculation. Though somewhat long and tedious, this is the natural and sagacious method of attacking the problem. It is the method of the pioneer, which is applicable to any speed-torque curve whatever.

The method of the present paper applies only to the ordinary continuous current railway motor, or to motors having similar characteristics. It shows how some of the machine tools of analysis can be employed in the treatment of the problem, for the purpose of obtaining the required results without having to build all the intermediate bridges afresh for every new case that arises.

The most general problems in the subject, present no great difficulties to the method, which nevertheless takes account of all the circumstances of the case. Given the profile of a road, with the stops, curves, etc., and the particulars of the train and its equivalent, also the mean voltage and train friction, we can lay out a time table for the train, find the energy it requires, or deter-

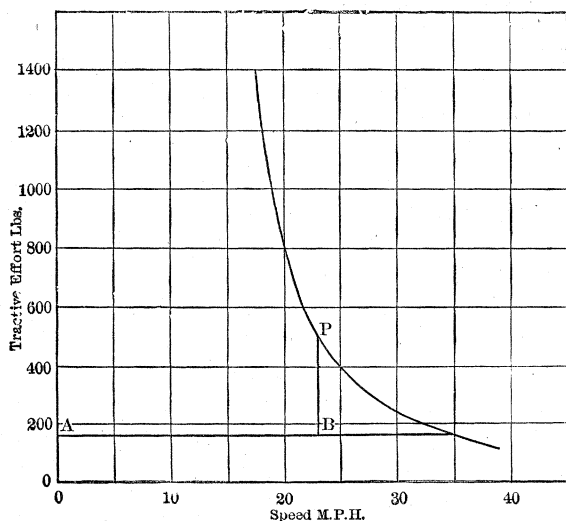


FIG. 1.

mine the motor losses. Such a problem treated by the usual method would be very long and consume a great deal of curve paper, whilst the method herein described obtains the solution easily and completely without it being necessary to plot a single curve.

The paper first discusses the kinematics of the subject, afterwards it is shown how to determine the energy used and the motor losses. Illustrative examples follow the general treatment.

The information necessary in employing the method is given in the form of curves and simple equations, which can be used without reference to the underlying analysis.

II.—PRINCIPLES OF METHOD.

We have stated that when the speed-time curve for any equipment and schedule has been found, we can easily determine all the circumstances of the run, and have indicated how the speed-time curve for the train can be computed from the speed-tractive-effort curve of the motors. Let Fig. 1 represent the speed-tractive-effort curve for a motor. Let oA be the tractive force required to overcome train resistance, so that, at the speed represented by the point P , the accelerating force per motor is PB . Let Fig. 2 be the speed-time curve deduced from Fig. 1 for a particular weight of train, when the train resistance per motor is oA . If now we draw a number of such speed-time curves for different values of train resistance oA , we shall have provided ourselves with the means of solving any problem whatever involving this train and equipment, having simply to copy the required portion of the appropriate speed-time curve. If we wish to handle a

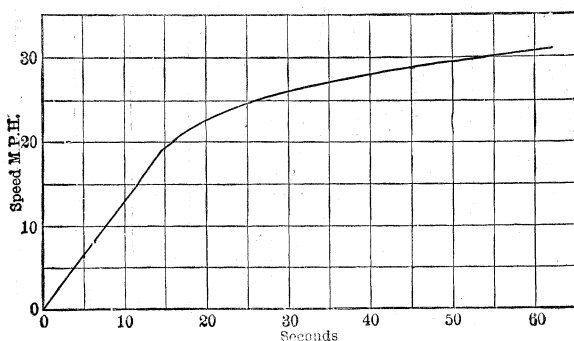


FIG. 2.

problem involving several grades or varying train resistance, we can do so; we continue for the requisite distance on the curve corresponding to the first grade, then, on change of grade step to the appropriate curve at the same speed, and copy the requisite portion of this curve, and so on.

Suppose now we halve the gear reduction ratio of the motors. This will double the speed for any current and halve the tractive effort, giving us Fig. 3, which is exactly the same as Fig. 1, except that the scales are altered. Suppose there is only half the weight of train and half the train resistance per motor; then at any point P we have half the accelerating force acting on half the mass, thus producing the same rate of acceleration as in the problem of Figs. 1 and 2. Since, however, the velocity is doubled everywhere, it will require double the time to run between a pair of points in Fig. 3, as compared with the corresponding points of

Fig. 1. Thus the speed-time curve for this case, shown in Fig. 4, is exactly the same as Fig. 2, except that the scales of time and speed are both halved. It follows then that the system of speed-time curves supposedly drawn to correspond to the equipment of rigs. 1 and 2, can be used for other equipments, by proper change in the scales of time and speed. As a matter of fact, as we shall show, such a system, drawn for a particular equipment, will apply to any equipment whatever; the only difficulty in the use of such a system occurs in determining which curve, and what scales of ordinate and abscissa are to be employed in a particular case.

The general shape of the motor characteristic is reflected in the general shape of the speed-time curves, whilst the fact that there are three quantities to be determined in any particular problem

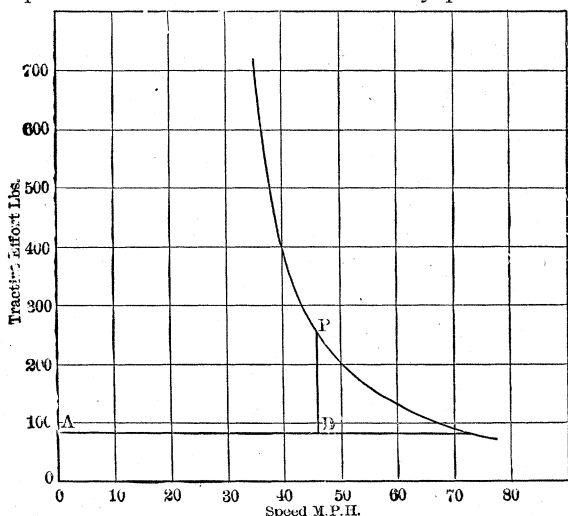


FIG. 3.

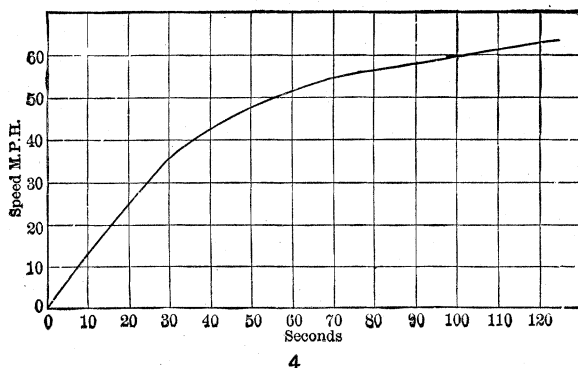
implies that the affecting circumstances are dynamically equivalent to three only. These may be taken as: (1) The motor equipment, represented by the speed-tractive-effort curve, and including the gear and size of wheels, besides the true characteristic of the motor at the voltage used. (2) The train driven by the motors, represented by its effective weight, and including the apparent additional weight due to the effect of rotary inertia. (3) The train resistance, including that due to grade as well as frictional and wind resistances. From one point of view we may describe the object of the present paper as seeking to determine the most convenient means of combining these three circumstances in order to obtain a workable solution of the dynamical problem of train motion on the lines indicated above.

III.—NOTATION.

The suffix 1 is used to distinguish the value a quantity has during the time of acceleration on resistance. The suffix 2 distinguishes a quantity during speed-curve running. The suffixes 3 and 4 refer similarly to coasting and braking respectively. The suffix 0 marks a constant of the same dimensions as the quantity represented by the letter to which it is attached. A dashed letter signifies the value that a quantity would take if free running speed were attained under the conditions supposed. A letter without distinguishing mark is used for the current value of a quantity. The signification of the several letters will be explained as they are met with.

IV.—GENERAL DYNAMICS.

Let w be the effective weight of the train per motor, including the apparent addition due to inertia of rotating parts. Let s be



4

the speed at time t , F the tractive force of the motor and F^1 the resistance to traction, composed of frictional resistance and grade resistance. Then, the forces being in gravitational units and g being the value of the acceleration due to gravity, the general equation of motion for the train is

$$w \frac{ds}{dt} = (F - F^1) g \quad (1)$$

During the time of acceleration on resistance, we may take F and F^1 as uniform, giving ds/dt constant, $= a_1$ where

$$w a_1 = (F_1 - F^1) g \quad (2)$$

During coasting $F = 0$, and if we assume the resistance to motion uniform, ($= F_c$) we get

$$w a_3 = -F_c g \quad (3)$$

Similarly during braking (with F_b as resistance),

$$w a_4 = -F_b g \quad (4)$$

It is not essential to the method of the paper that straight line coasting and braking curves be assumed (*i.e.*, that F_c and F_b be assumed constant), but as this is likely to be as close to the average actual case as any other specific assumptions, and as furthermore this is not a discussion of train resistance or of braking curves, we shall make the above assumptions where we deal with coasting or braking.

V.—THE SPEED CURVE.

The chief feature of this system is the treatment of speed-curve running, for the dynamical equations present no difficulty during the periods of accelerating, coasting and braking. In order to

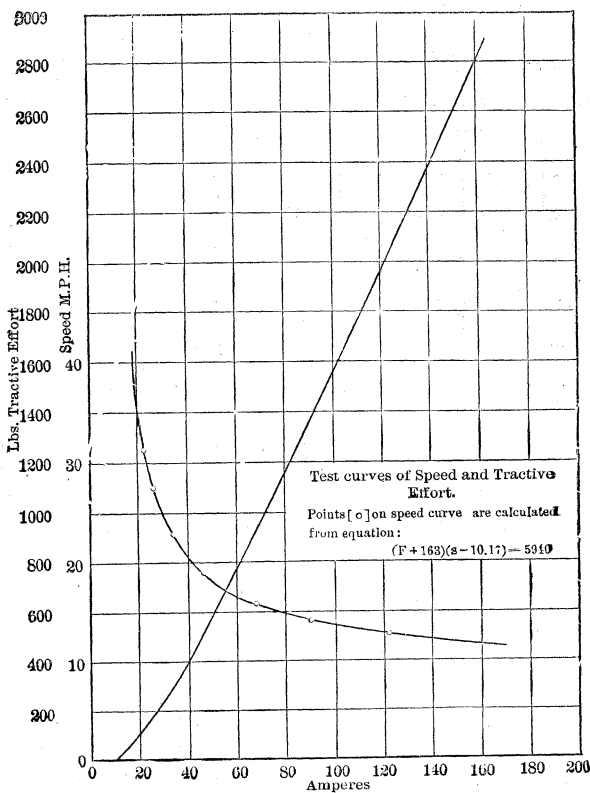


FIG. 5.

make analytical use of equation 1 during the period in question, it is necessary to construct an empirical formula connecting tractive force and speed for the working portion of the speed-curve. Now, the speed-curve has all the appearance of a rectangular hyperbola, and as a matter of fact the hyperbola,

$$(F + F_0)(s - s_0) = K F_0 s_0 \quad (5)$$

can, by proper adjustment of the constants $K F_0$ and s_0 , be made to lie extremely close to the speed curve over the whole useful range from acceleration to free running. Figs. 5 and 6 show two typical speed-curves with the hyperbolas proposed to represent them, and it will be noticed that the hyperbola lies as close to the actual speed curves—the working part being always understood—as would a speed-curve on another motor of the same type. In fact, it is as close as the test can be relied upon, and it is useless to attempt greater accuracy than this.

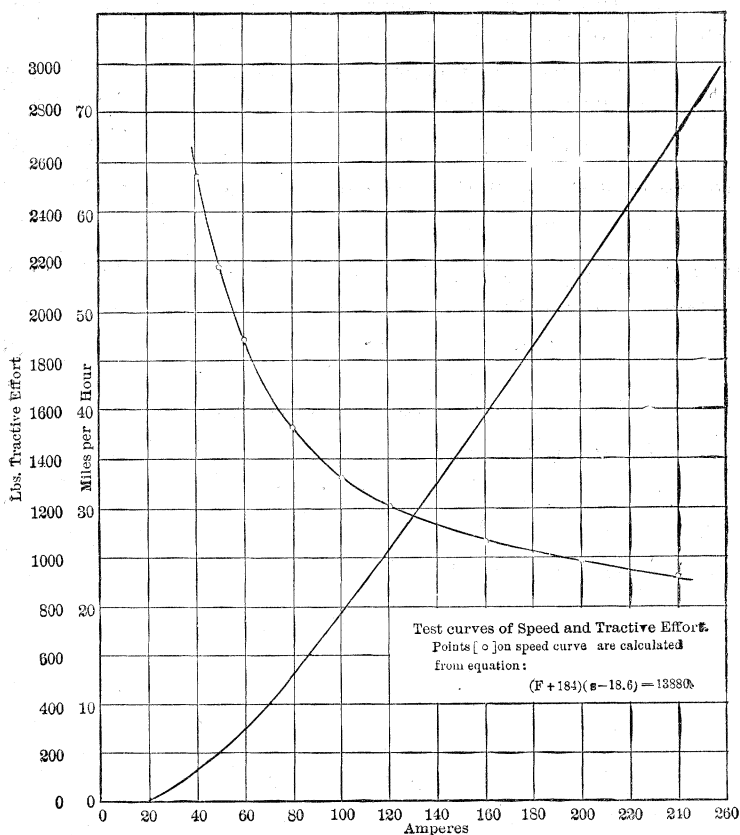


FIG. 6.

The curves of Figs. 5 and 6 are in no way exceptional, but represent what in the author's experience are average cases, so that the assumption that the speed-tractive effort curve can be represented by such an equation as 5, appears fully justified. I have dwelt upon this point as it is the fundamental assumption of this method of treatment, and anyone

doubting the accuracy of the results has practically to show where the assumption fails, for the rest is purely mathematics.

Concerning the constants.— K depends only on the *shape* of the curve, and for an average well-shaped characteristic lies in the neighborhood of $3\frac{1}{2}$ or 4. F_0 is of the dimensions of force. It varies as the gear reduction ratio and inversely as the diameter of the wheels. s_0 is a speed, varying inversely as the gear reduction ratio and directly as the diameter of the wheels. It also practically varies as the voltage, though a large change in voltage will change the shape of the speed curve slightly, and so change all the constants a little. (See also Sec. VIII.)

We shall usually express speed as a multiple of s_0 —making s_0 the unit of speed. Thus writing $s/s_0 = q$, equation 5 becomes

$$(F + F_0) (q - 1) = K F_0 \quad (6)$$

In terms of this unit the free running speed is given by

$$(F^1 + F_0) (q^1 - 1) = K F_0 \quad (7)$$

whilst the speed of striking the speed-curve after acceleration on resistance, is given by:

$$(F_1 + F_0) (q_1 - 1) = K F_0 \quad (8)$$

VI.—DYNAMICS CONTINUED.

From equations 6 and 7,

$$F - F^1 = K F_0 \left(\frac{1}{q-1} - \frac{1}{q^1-1} \right) = \frac{K F_0}{q^1-1} \cdot \frac{q^1 - q}{q - 1} \quad (9)$$

From equations 7 and 8

$$F_1 - F^1 = \frac{K F_0}{q^1-1} \cdot \frac{q^1 - q_1}{q_1 - 1}$$

whence, from equation 2,

$$a_1 = \frac{K F_0 g}{w} \cdot \frac{1}{q^1-1} \cdot \frac{q^1 - q_1}{q_1 - 1}$$

Now,

$$\begin{aligned} t_1 &= \frac{s_1}{a_1} = s_0 \frac{q_1}{a_1} \\ &= \frac{w s_0}{K F_0 g} \cdot (q^1 - 1) q_1 \frac{q_1 - 1}{q^1 - q_1} \end{aligned} \quad (10)$$

The distance $d_1 = \frac{1}{2} s_1 t_1 = \frac{1}{2} s_0 q_1 t_1$

$$= \frac{w s_0^2}{K F_0 g} \cdot (q^1 - 1) \frac{q_1^2}{2} \cdot \frac{q_1 - 1}{q^1 - q_1} \quad (11)$$

From equation 1,

$$d t = \frac{w}{g} \cdot \frac{d s}{F - F^1} = \frac{w s_0}{g} \cdot \frac{d q}{F - F^1}$$

whence, from equation 9,

$$d t = \frac{w s_0}{K F_0 g} \cdot (q^1 - 1) \frac{q - 1}{q^1 - q} d q \quad (12)$$

$$\begin{aligned} \therefore t_2 &= \frac{w s_0}{K F_0 g} (q^1 - 1) \int_{q^1}^{q_2} \frac{q - 1}{q^1 - q} d q \\ &= \frac{w s_0}{K F_0 g} (q^1 - 1) \left[- (q - 1) - \log_e \frac{q^1 - q}{q^1 - 1} \right]_1^{q_2} \end{aligned} \quad (13)$$

$$\begin{aligned} d_2 &= \int s d t = s_0 \int q d t \\ &= \frac{w s_0^2}{K F_0 g} (q^1 - 1) \int_{q^1}^{q_2} \frac{q (q - 1)}{q^1 - q} d q \\ &= q^1 s_0 t_1 - \frac{w s_0^2}{K F_0 g} \frac{q^1 - 1}{2} \left[(q - 1)^2 \right]_1^{q_2} \end{aligned} \quad (14)$$

Equations 10 and 13 give respectively the times t_1 and t_2 each as the product of two factors—one, $\frac{w s_0}{K F_0 g}$ depending on the equipment only, and the other depending on q^1 and therefore on the resistance to traction, *i.e.*, on the grade. Similarly the distances given by equations 11 and 14 are each the product of $\frac{w s_0^2}{K F_0 g}$ and a function of q^1 . Now, just as we have taken s_0 as our unit of speed, we will take $T = \frac{w s_0}{K F_0 g}$ as our unit of time, and $D = \frac{w s_0^2}{K F_0 g}$ as our unit of distance, writing T and D for shortness, as these quantities occur frequently. This is an obvious piece of manipulation, but at the same time is most important. It enables all the infinite possibilities of the subject to be expressed in one system of curves, which cover all equipments and all

grades, and which, having been once computed, reduce all problems to mere reference to them. These curves are placed on record in various forms in plates I., II. and III..

With T and D as units of time and distance, equation 10 reduces to

$$t_1 = (q^1 - 1) q_1 \frac{q_1 - 1}{q^1 - q_1} \quad (15)$$

Equation 11, to

$$d_1 = (q^1 - 1) \frac{q_1^2}{2} \frac{q_1 - 1}{q^1 - q_1} \quad (16)$$

Equation 13, to

$$t_2 = (q^1 - 1) \left\{ \left[- (q_2 - 1) - \log_e \frac{q^1 - q_2}{q^1 - 1} \right] - \left[- (q_1 - 1) - \log_e \frac{q^1 - q_1}{q^1 - 1} \right] \right\} \quad (17)$$

Equation 14, to

$$d_2 = q^1 t_2 - \frac{q^1 - 1}{2} \left\{ (q_2 - 1)^2 - (q_1 - 1)^2 \right\} \quad (18)$$

If we adopt these same units of time and distance whilst coasting and braking, we get from equation 3

$$t_3 = \frac{K F_0}{F_c} \cdot (q_2 - q_3) \quad (19)$$

$$d_3 = \frac{K F_0}{F_c} \frac{q_2^2 - q_3^2}{2} \quad (20)$$

where q_3 is the speed at the end of coasting.

Similarly, from equation 4,

$$t_4 = \frac{K F_0}{F_b} (q_3 - q_4) \quad (21)$$

$$d_4 = \frac{K F_0}{F_b} \frac{q_3^2 - q_4^2}{2} \quad (22)$$

where q_4 is the speed after braking, which of course equals zero if the train is stopped.

For the usual stop, coasting followed by braking, we get for the total time with power off,

$$t_3 + t_4 = \frac{K F_0}{F_c} q_2 - \left(\frac{K F_0}{F_c} - \frac{K F_0}{F_b} \right) q_3 \quad (23)$$

and for the total distance,

$$d_3 + d = \frac{K F_0}{F_c} \frac{q_2^2}{2} - \left(\frac{K F_0}{F_c} - \frac{K F_0}{F_b} \right) \frac{q_3^2}{2} \quad (24)$$

whilst, if there is no coasting, $q_2 = q_3$.

The kinematics of the subject, as treated by this method, is entirely included in equations 15 to 24.

VII.—UNIVERSAL SPEED TIME AND DISTANCE CURVES.

Plate I. gives curves between q and t , where

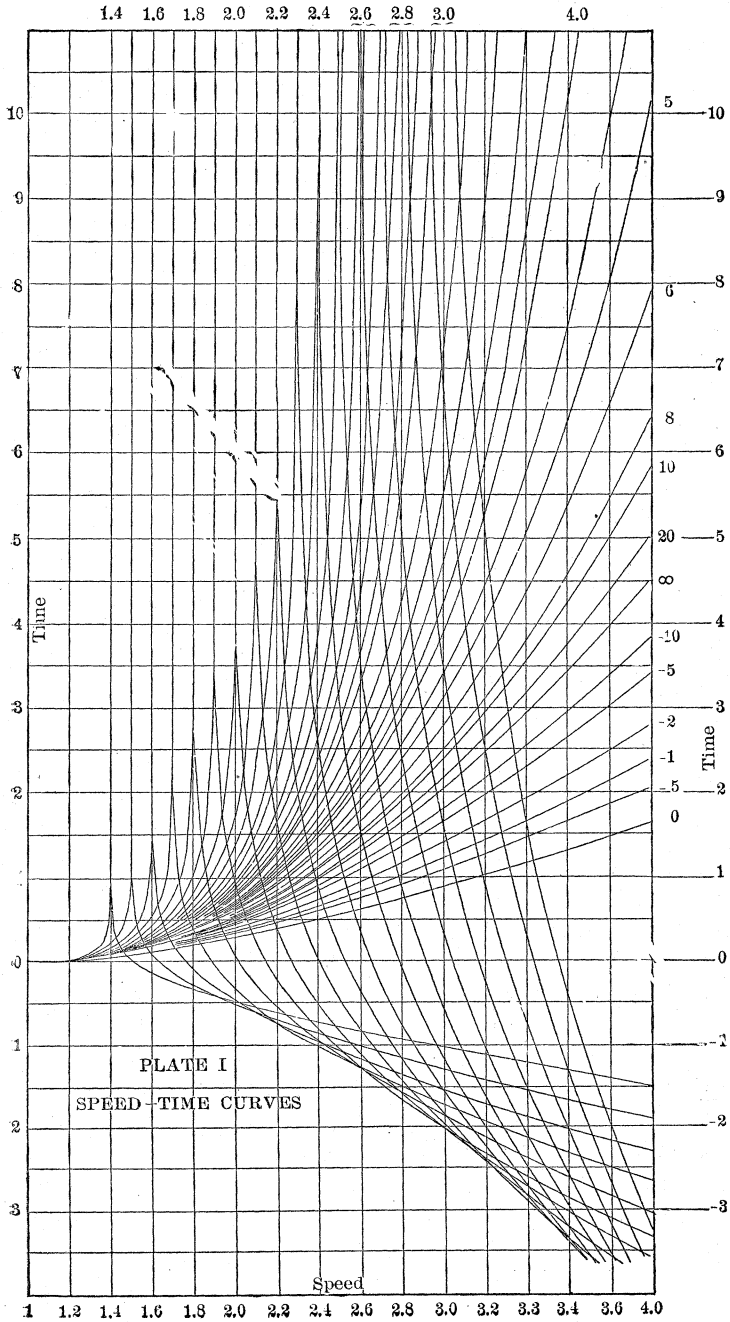
$$t = (q^1 - 1) \left\{ - (q - 1) - \log_e \frac{q^1 - q}{q^1 - 1} \right\} \quad (\text{see equation 17})$$

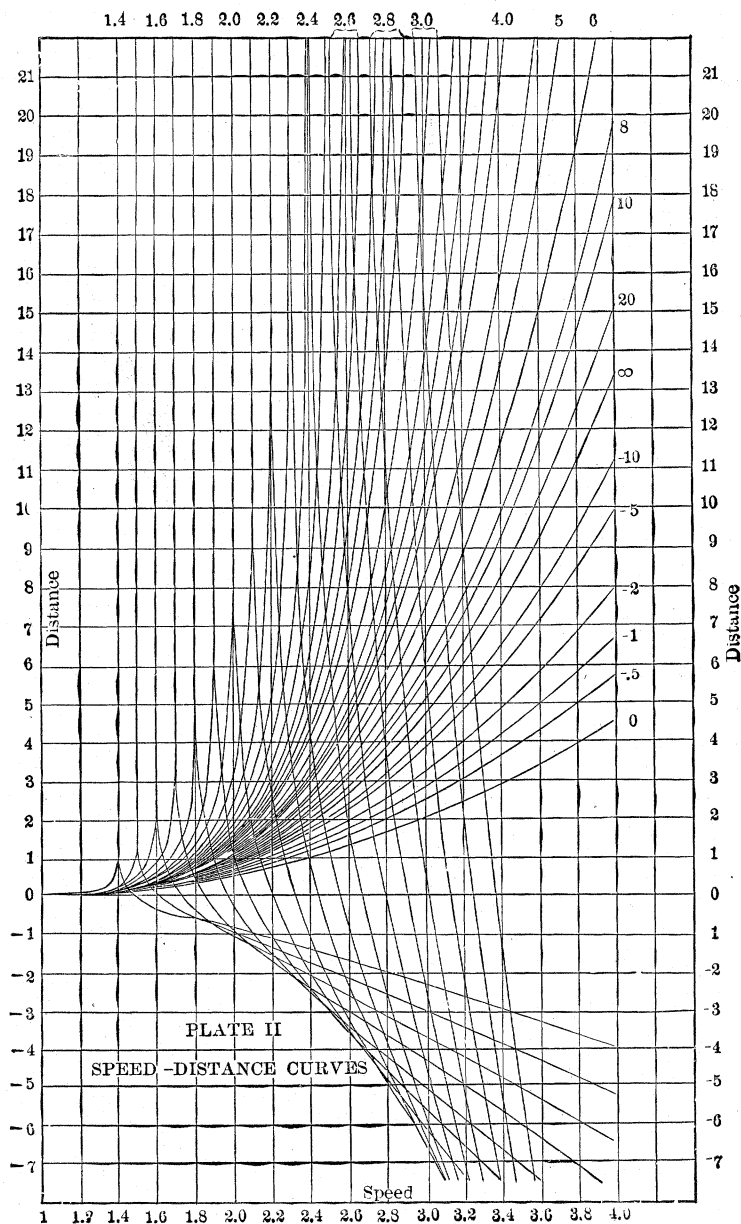
for different values of q^1 , that is, with a particular equipment, of train resistance; (the figures attached to the several curves are the values of q^1 for which they are plotted.) These are universal speed-time curves corresponding to that portion of the run which is made on the speed-curve. The time spent on the speed-curve is the difference between the ordinates at q_2 and q_1 of the particular curve for which the constant q^1 corresponds to the equipment and resistance in question. If, having reached the speed q_2 , we strike another grade, we simply determine the value of q^1 for this grade, and stepping to the corresponding curve at the same abscissa q_2 , proceed the requisite time on this latter curve, and so on.

Curves corresponding to the lower free-running speeds have a second branch in which the speed approaches free-running from above. These are for use when we happen to strike a grade at a speed higher than the free-running speed corresponding to that grade, and require no special treatment.

Plate II. gives universal speed-distance curves, from the equation,

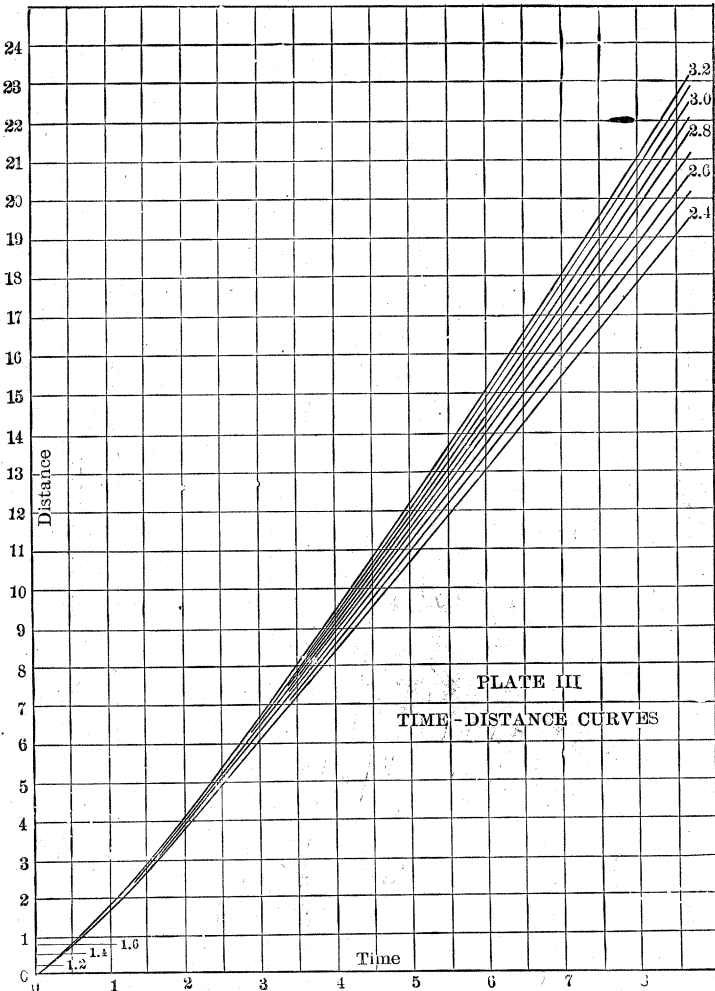
$$d = q^1 t - \frac{q^1 - 1}{2} (q - 1)^2 \quad (\text{see equation 18})$$





These are used in the same manner as the speed-time curves, and are perhaps more generally useful, since in such work as laying out a time table it is the distance on each grade that is known.

Plate III. gives universal time-distance curves corresponding to speed-curve running for such values of q as occur on approxi-



mately level track. They will be found useful in certain preliminary calculations. [See Section XIV. c.]

The speed-time curve corresponding to any run can easily be plotted. The portions of it which represent acceleration, coasting, and braking are straight lines, and present no difficulty

whilst the portion representing speed-curve running is copied directly from the appropriate curve in Plate I.—the time co-ordinate being multiplied by T and the speed co-ordinate by s_0 . Similar remarks apply to the speed-distance, and distance-time curves. [See Section XIV. e.]

VIII.—SERIES RUNNING.

If any part of the run is to be made with motors in series, we can negotiate it by means of our speed-time and speed-distance curves, by changing the scale of ordinate and abscissa, practically by halving s_0 , better, if there is much series running, by recalculating the constants of the speed-curve so as to give the correct curve corresponding to half potential. From these new constants we obtain new values of T , D and s_0 , which determine the scales in the speed-time and speed-distance curves.

IX.—THE CURRENT EQUATION.

An inspection of the curve between tractive force and current for a motor will show that it is never far removed from a straight line—remembering always that we need only concern ourselves with the portion between accelerating and free-running—and a straight line judiciously chosen is perhaps as nearly accurate as the nature of the subject will permit. However, it does not introduce any very great complication to proceed to greater accuracy, and we shall perform the analysis in such manner as to include the more general case, whilst being immediately reducible to the form corresponding to a straight line tractive effort curve.

Let the straight line between current (I) and tractive force be,

$$I = \frac{I_0}{K F_0} (F + F_i) \quad (25)$$

where F_i is a constant. Substituting for F from equation, 6 we get,

$$I = \frac{I_0}{K F_0} \left\{ \frac{K F_0}{q - 1} - F_0 + F_i \right\}$$

Now writing, $F_i - F_0 = K F_0 b$, we get

$$I = I_0 \left\{ \frac{1}{q - 1} + b \right\} \quad (26)$$

This is a hyperbola connecting current and speed, which assumes a straight line law connecting current and tractive force. If now we put,

$$I = I_0 \left\{ \frac{1}{q - q_i} + b \right\} \quad (27)$$

we can adjust the three constants I_0 , q_i and b , so as to make the hyperbola of equation 27 cross the speed-current curve at three points within the useful range, and practically coincide with it. By putting $q_i = 1$ in any result deduced from equation 27 we reduce it to the corresponding result for equation 26.

The free running current is,

$$I^1 = I_0 \left(\frac{1}{q^1 - q_i} + b \right) \quad (28)$$

and the accelerating current,

$$I_1 = I_0 \left(\frac{1}{q_1 - q_i} + b \right) \quad (29)$$

The ampere-time curve, for that portion of the run which is made on the speed-curve, can be deduced from the speed-time curve and equation 26 or 27. [See Section XIV. e.]

X.—ENERGY REQUIRED.

Assuming series-parallel control, the energy taken per motor during acceleration in series, is,

$$\frac{1}{2} V \times \frac{1}{2} t_1 T I_1$$

during acceleration in multiple twice this amount is used, so that the total energy required during acceleration is,

$$E_1 = \frac{3}{4} V I_1 t_1 T \text{ watt-seconds} \quad (30)$$

The energy required during speed curve running is,

$$\begin{aligned} E_2 &= V \int I dt \\ &= V T \int_{q_i}^{q_2} \left(\frac{1}{q - q_i} + b \right) (q^1 - 1) \frac{q - 1}{q^1 - q} dq \quad (\text{See equation 19}) \\ &= V T \left\{ I^1 t_2 + I_0 \frac{q^1 - 1}{q^1 - q_i} (q_2 - q_1) + I_0 \frac{(q^1 - 1)(q_i - 1)}{q^1 - q_i} \right. \\ &\quad \left. \log_e \frac{q_2 - q_i}{q_1 - q_i} \right\} \quad (31) \end{aligned}$$

When $q_i = 1$

$$E_2 = V T \{ I^1 t_2 + I_0 (q_2 - q_1) \} \quad (32)$$

For an extended run on several grades the quantity within the brackets becomes,—“ the sum of the value of $I^1 t_2$ for the several grades, $+I_0$ times the total increase of speed between striking the speed curve and cutting off power ”—intermediate speeds on the grades cancelling. With a little trouble equation 31 can be reduced to a form almost as simple for calculation. [See Section XII Equation 38]

XI.—COPPER LOSSES.

If R be the resistance of the motor (or of armature or field), the $I^2 R$ loss in this resistance is, during acceleration,

$$C_1 = R I_1^2 t_1 T \text{ watt-seconds} \quad (33)$$

During speed-curve running,

$$\begin{aligned} C_2 &= R \int I^2 dt \\ &= R T \int_{q_1}^{q_2} I_0^2 \left(\frac{1}{q - q_i} + b \right)^2 (q^1 - 1) \frac{q - 1}{q^1 - q} dq \\ &= R T \left\{ I^2 t_2 + I_0 (I^1 + b I_0) \frac{q^1 - 1}{q^1 - q_i} (q_2 - q_1) \right. \\ &\quad \left. + I_0^2 \left[\frac{(q^1 - 1)^2}{(q^1 - q_i)^2} + 2b \frac{(q_i - 1)(q^1 - 1)}{q^1 - q_i} \right] \log_e \frac{q_2 - q_i}{q_1 - q_i} \right. \\ &\quad \left. + I_0^2 \frac{(q_i - 1)(q^1 - 1)}{q^1 - q_i} \left(\frac{1}{q_1 - q_i} - \frac{1}{q_2 - q_i} \right) \right\} \quad (34) \end{aligned}$$

If we put $q_i = 1$, this reduces to

$$C_2 = R T \left\{ I^2 t_2 + I_0 (I^1 + b I_0) (q_2 - q_1) + I_0^2 \log_e \frac{q_2 - 1}{q_1 - 1} \right\} \quad (35)$$

For a run in which power is kept on over a number of different grades the terms $I^2 t_2$ and $I_0 I^1 (q_2 - q_1)$ have to be computed for each grade, whilst in the other terms intermediate speeds cut out and the sum depends only on the speed of striking and leaving the speed-curve. Although it is not usual to compute the losses for such extended runs, it is nevertheless advisable to place the method of calculation on record.

XII.—IRON LOSSES.

The core loss of a motor cannot be calculated with any accuracy, but when it has been determined by test, we can compute the iron losses corresponding to any run, although of course the reliability of the result is not greater than that of the core loss, so that it is useless to aspire to too great accuracy.

During acceleration on resistance the mean core loss is rather less than a half of the maximum and may with sufficient accuracy be taken as $0.4 W_1$ where W_1 is the core loss on reaching the motor curve at the accelerating current. Thus we may write

$$L_1 = .4 W_1 t_1 T \quad (36)$$

The core loss at constant voltage has usually to be drawn through a very few test points, and if we can find an empirical formula which represents a smooth continuous curve through these test points—at any rate such of them as lie within the useful range—this formula will represent the core loss quite as well as the usual arbitrary curve through the points. Whilst I am not prepared to propose a formula that will suit all core loss curves, I have found that the commonest shape—that which rises continually with the current, if plotted against speed, can be closely represented by a hyperbola of the form,

$$(W - W_0) (q - q_0) = P \quad (37)$$

where the constants P , w_0 and q_0 are chosen to make the curve pass through three test points within the useful range.

The iron loss during speed-curve running is obtained from equation 37 by the same integration as the energy is obtained in equation 31. Thus,

$$\begin{aligned} L_2 &= \int W dt \\ &= T \left\{ W^1 t_2 + P \frac{q^1 - 1}{q^1 - q_0} (q_2 - q_1) - P \frac{(1 - q_0) (q^1 - 1)}{q_1 - q_0} \log_e \frac{q_2 - q_0}{q^1 - q_0} \right\} \\ &= T \left\{ W^1 t_2 - P \frac{1 - q_0}{q^1 - q_0} (q_2 - q_1) + P \frac{(1 - q_0)^2}{q_1 - q_0} \log_e \frac{q_2 - q_0}{q^1 - q_0} \right. \\ &\quad \left. + P (q_2 - q_1) - (1 - q_0) \log_e \frac{q_2 - q_0}{q_1 - q_0} \right\} \quad (38) \end{aligned}$$

The last form of the above equation is for use on a run over several grades. In it the first term is the important one, the second and third are small, whilst the last two depend only on the speeds of reaching and leaving the motor curve.

XIII.—REMARKS.

The foregoing sections provide us with all the tools required in the subject, in the form of a score or so of equations and a few systems of curves. The equations, being in general form, are independent of any particular system of units. In the examples that follow, intended merely to show the use and application of the method—we shall adopt the units usual in this country—miles per hour, watt hours per ton mile, etc.

XIV.—EXAMPLES.

(a) Equation of speed-tractive-effort curve:

In order to determine the constants K , F_0 and s_0 of equation 5 we select three points on the motor curve—one near free running, one near accelerating, and the third about half-way between these two—and compute the values of the above constants, which will make the hyperbola pass through these points. Thus from Fig. 5 we get,

Speed (m.p.h.)	Tractive Effort (lbs.)
27.5	180
19.0	510
14.1	1350

Now, equation 5 may be written,

$$F s + F_0 s - F s_0 = (K + 1) F_0 s_0 \quad (39)$$

Thus,

$$27.5 \times 180 + 27.5 F_0 - 180 s_0 =$$

$$19.0 \times 510 + 19.0 F_0 - 510 s_0 =$$

$$14.1 \times 1350 + 14.1 F_0 - 1350 s_0 = (K + 1) F_0 s_0$$

or $4950 + 27.5 F_0 - 180 s_0 =$

$$9690 + 19 F_0 - 510 s_0 =$$

$$19035 + 14.1 F_0 - 1350 s_0 = (K + 1) F_0 s_0 \quad (40)$$

$$\therefore 4740 = 8.5 F_0 + 330 s_0 \quad (41)$$

$$9345 = 4.9 F_0 + 840 s_0 \quad (42)$$

Multiplying equation 41 by 4.9/8.5, we get,

$$2735 = 4.9 F_0 + 190 s_0$$

subtracting this from 42 gives

$$6610 = 650 s_0$$

$$\therefore s_0 = 10.17 \text{ m.p.h.} = 14.92 \text{ f.p.s.}$$

From equation 41 we get

$$8.5 F_0 = 4740 - 3356 = 1384$$

$$\therefore F_0 = 163 \text{ lbs.}$$

whence from equation 40,

$$(K+1) \times 1658 = 4950 + 4480 - 1830 = 7600$$

or

$$K = 3.585$$

Thus, the equation of the speed curve is

$$\begin{aligned} (F+163)(s-10.17) &= 3.585 \times 10.17 \times 163 \\ &= 5940 \end{aligned} \quad (43)$$

This gives the constants for this motor, running at 500 volts with gear reduction ratio 4.31. Fig. 5 shows some points calculated from the above equation.

(b) Equation of current curve:

If we wish to determine the current accurately, as given by equation 27, the problem is the same as the last. We select three speeds, and read the corresponding currents from the test curve, then proceed to find the constants of equation 27 just as we did those of equation 5 in the last problem. Thus, from the curve of Fig. 5 we get,

$$I = 40.6 \left\{ \frac{1}{q - .918} + .081 \right\} \quad (44)$$

If the less exact form of equation 26 is sufficient, we select two points and read current and tractive effort, then determine the equation of the straight line joining the points. Thus, from Fig. 5 we get,

Amperes.	Tractive-effort (lbs.)
34	300
80	1155

whilst

$$K F_0 = 584$$

Hence, from equation 25, we get,

$$584 \times 34 = I_0 (300 + F_i)$$

$$584 \times 80 = I_0 (1155 + F_i)$$

$$\therefore 584 \times 46 = 855 I_0$$

$$\therefore I_0 = 31.4$$

$$F_i = 332$$

giving,

$$b = (332 - 163)/584 \\ = .29$$

Thus,

$$I = 31.4 \left\{ \frac{1}{q-1} + .29 \right\} \quad (45)$$

The current equation in this form (or that or equation 44), is independent of gear reduction ratio and practically of the voltage.

(c) To determine the characteristics of a motor equipment required to handle a given service:

Let it be required to make a schedule speed of 17 m.p.h., allowing two stops per mile of 10 seconds each, and assuming that 10 per cent. of the total time will be consumed in unavoidable delays. Let the estimated effective weight of the train be eight tons per motor, and the friction 20 lbs. per ton of effective weight, or 160 lbs. per motor in all. Let acceleration and braking retardation be both at the rate of 1.32 m.p.h. per sec., requiring 120 lbs. per ton, so that the total force required per motor during acceleration is $8 \times 140 = 1120$ lbs. Let us allow moreover for 10 per cent. of the distance being coasted. Thus we must calculate on making 17 miles in 54 minutes, and the typical run is one of one-half mile in 95.3 seconds, including a stop, or of 2,640 feet in 85.3 seconds actual running time.

We first make a calculation of the distance and time power is off—estimating the maximum speed reached, say 31 m.p.h in this case. Then,

Coasting distance, 264 feet. Coasting time, 5.9 seconds.

Speed at end of coasting, 29.7 m.p.h.

Braking distance, 490 feet. Braking time, 22.5 seconds.

Distance power off, 754 feet. Time power off, 28.4 seconds.

Distance power on, 1886 feet. Time power on, 56.9 seconds.

Assume for the speed curve,

$$(F + F_0) (q - 1) = 4 F_0$$

which is about an average shape; also assume that free running on the level is reached when $q = 2.5$, (this is about the usual thing in street railway work), then,

$$(160 + F_0) (2.5 - 1) = 4 F_0 \quad (\text{See equation 7})$$

$$\therefore F_0 = 96 \text{ lbs.}$$

$$T = \frac{16000}{4 \times 96 \times 32.2} s_0 = 1.3 s_0$$

$$D = 1.3 s_0^2$$

Thus, if t and d are respectively the total time and distance that power is on, expressed in terms of our usual units:

$$1.3 s_0 t = 56.9 \text{ seconds.}$$

$$1.3 s_0^2 d = 1886 \text{ feet}$$

$$\therefore \frac{\sqrt{d}}{t} = \frac{\sqrt{1886 \times 1.3}}{56.9} = .87$$

Now, the speed of striking the speed-curve is given by,

$$(1120 + 96) (q_1 - 1) = 384$$

(See equation 8)

$$\therefore q_1 = 1.316$$

$$t_1 = .525 \quad (\text{See equation 15})$$

$$d_1 = .35 \quad (\text{See equation 16})$$

Referring to Plate III. (or Plates I. and II.), we see that the reading on the curve, "2.5" corresponding to speed 1.316 is—time .065 distance, .06. Now, if τ be the reading of time, and δ that of distance, when the power is cut off,

$$t = \tau + .525 - .065 = \tau + .46$$

$$d = \delta + .35 - .06 = \delta + .29$$

Thus, we have to determine the point on curve "2.5," Plate III., which satisfies,

$$\frac{\sqrt{\delta + .29}}{\tau + .46} = .87$$

This we find by trial to be,

$$\tau = 1.68$$

Hence

$$\delta = 3.18$$

$$s_0 = \frac{56.9}{1.3 (1.68 + .46)} = 20.5 \text{ f.p.s.} = 13.95 \text{ m.p.h.}$$

The reading of speed at time 1.68 on curve "2.5," plate I., is 2.18, which makes the speed just before cutting off power,

$2.18 \times 13.95 = 30.4$ m.p.h. For the purpose of a preliminary calculation this will probably be sufficiently near to the 31 m.p.h. assumed.

Thus we have found the constants of the speed-tractive-effort curve suitable for this service. Fig. 7 shows the curve so determined.

The curves of Plate III. can be used in a very similar manner to determine the gear reduction required with a given motor in order to handle a given service.

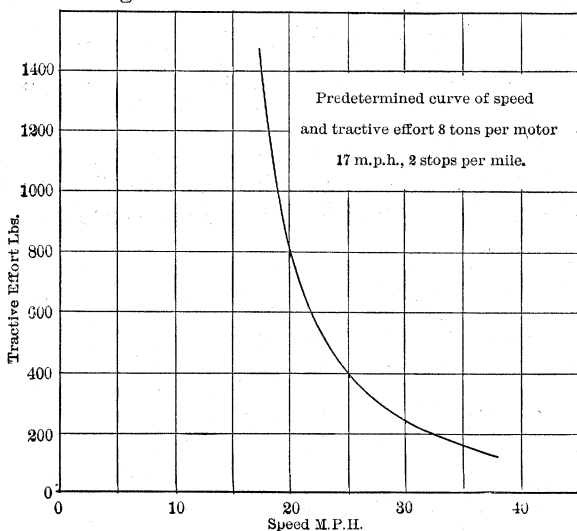


FIG. 7.

(d) Given the train and equipment, to determine how to make the schedule.

Let the train and schedule be the same as in the last example (c), and the motor that whose characteristic is given in Fig. 5 (see example a), but with gear reduction 3.05

Then from equation 43 we get,

$$F_0 = \frac{3.05}{4.31} \times 163 = 115 \text{ lbs.}$$

$$s_0 = \frac{4.31}{3.05} \times 10.17 = 14.4 \text{ m.p.h.} = 21.1 \text{ f.p.s.}$$

$$K = 3.585$$

$$w = 16000$$

$$T = \frac{16000 \times 21.1}{3.585 \times 115 \times 32.2} = 25.45 \text{ seconds.}$$

$$D = 25.45 \times 21.1 = 537 \text{ feet.}$$

As in the last example, we have to make 2640 feet in 85.3 seconds, or in our units, distance 4.92 in time 3.355. Let acceleration and braking be as in the last example; thus the speed q_1 of striking the speed curve is given by

$$(1120 + 115) (q_1 - 1) = 3.585 \times 115 \text{ (see equation 8)}$$

or $q_1 = 1.335$

(Equation 44 shows this corresponds to acceleration at about 100 amperes.)

The equipment speed q^1 is given by,

$$(160 + 115) (q^1 - 1) = 3.585 \times 115 \text{ (see equation 7)}$$

or $q^1 = 2.50$

The coasting resistance (F_c) we will take as 160 lbs., whilst the braking resistance (F_b) is 960 lbs. Thus, the constants of equations 23 and 24 are,

$$\frac{KF_0}{F_c} = 2.58, \frac{KF_0}{F_b} = 0.43, \frac{KF_0}{F_c} - \frac{KF_0}{F_b} = 2.15$$

The time of acceleration is (equation 15).

$$t_1 = .575$$

and the distance is (equation 16),

$$d_1 = .38$$

We now proceed to determine the instant when we must cut off power in order to make the given distance in the given time. Thus try $q_2 = 2.2$, whence from Plate I.,

$$\begin{array}{rcl} t_2 & = & 1.81 - .06 = 1.75 \\ t_1 & = & .575 \end{array}$$

$$\begin{array}{rcl} \text{Time power on,} & & 2.325 \\ \text{Total time,} & & 3.355 \end{array}$$

$$\therefore \text{Time power off, } 1.03$$

whence (equation 23),

$$2.15 q_3 = 2.58 \times 2.2 - 1.03 = 4.645$$

$$q_3 = 2.16$$

Thence from equation 18 (or Plate II.),

$$\begin{array}{rcl} d_2 & = & 3.38 \\ d_1 & = & .38 \\ d_3 + d_4 & = & 1.23 \end{array} \quad (\text{see equation 24})$$

Total distance, $\underline{\hspace{1.5cm}}$ 4.99, instead of 4.92

Thus we make somewhat too great a distance in the time and must cut off power earlier. Taking $q_2 = 2.13$, we get,

$$\begin{array}{rcl} t_2 & = & 1.37 \\ t_1 & = & .575 \\ \hline t_1 + t_2 & = & 1.945 \\ \text{Total time,} & & \underline{3.355} \end{array} \quad \begin{array}{rcl} q_3 & = & 1.90 \\ d_2 & = & 2.55 \\ d_1 & = & .38 \\ d_3 + d_4 & = & 1.98 \\ \hline \text{Total distance,} & = & 4.91 \end{array}$$

This then leads to distance 4.91 in time 3.355, which is sufficiently near to the required solution.

Resuming familiar units we get,

Time of acceleration,	= 14.6 secs.
Time power on,	= 49.4 secs.
Time coasting,	= 15.1 secs.
Time braking,	= 20.8 secs.
Distance of acceleration,	= 206 feet.
Distance power on,	= 1573 feet.
Distance coasting,	= 648 feet.
Distance braking,	= 416 feet.
Speed on striking motor curve,	= 19.2 m.p.h.
Speed on cutting off power,	= 30.65 m.p.h.
Speed on beginning to brake,	= 27.35 m.p.h.

(e) To plot speed-time and current-time curves.

Having determined the duration of the several parts of the run as in the last problem, we can immediately put in the portions of the speed-time curve corresponding to accelerating, coasting and braking—which consist of straight lines. The portion that is run on the motor curve is copied from the appropriate curve ($q^1 = 2.5$) in Plate I, the speed coördinate being multiplied by $s_0 = 14.4$ m.p.h., and the time coördinate by $T = 25.45$ seconds. If we add to the reading of time in plate I. the time of acceleration (.575) and subtract the reading (.06) on first striking the speed curve, we shall arrive at the time measured from the beginning of the run.

Thus, for $q = 2.0$, we get for the reading of time, .97, whence the time from the beginning of the run is $.97 + .515 = 1.485$ or 37.8 secs., and the speed is $2.0 \times 14.4 = 28.8$ m.p.h. The current per motor is given by equation 44 or 45, according to the accuracy desired.

These curves are plotted in Fig. 8, which shows also some points calculated by the usual point-to-point method, from the curve in Fig. 5.

Speed distance curves can be plotted in a similar manner from equations 16, 20 and 22 and Plate II.

(j) Energy used.

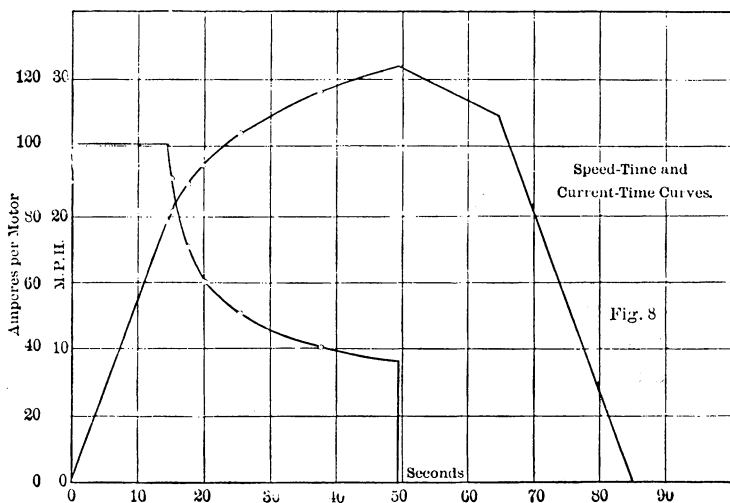


Fig. 8.

Equation 44 with equations 28 and 29, gives:

$$I^1 = 29 \text{ amps.}$$

$$I_1 = 100.7 \text{ amps.}$$

Hence, from equation 30,

$$\begin{aligned} E_1 &= \frac{3}{4} \times 500 \times 100.7 \times .575 \times 25.45 \\ &= 552000 \text{ watt-seconds} \end{aligned}$$

From equation 31,

$$\begin{aligned} E_2 &= 500 \times 25.45 \left\{ 29 \times 1.37 + 40.6 \frac{1.5}{1.582} \times .795 \right. \\ &\quad \left. - 40.6 \frac{1.5}{1.582} \times .082 \times 2.3 \times .463 \right\} \\ &= 500 \times 25.45 \{ 39.8 + 30.6 - 3.4 \} \\ &= 852000 \text{ watt-seconds} \end{aligned}$$

Total energy input = 1404000 watt-seconds

Which gives 97.5 watt-hours per ton mile.

Equations 45, 30 and 31 lead to,

$$E_1 = 564000 \text{ watt-seconds}$$

$$E_2 = 845000 \text{ watt-seconds}$$

In all 1409000 watt-seconds, differing from the more exact calculation given above, by only 0.35% in this case.

(g) Motor losses:

Calculating copper losses from the less exact current equation (No. 45), we get, from equation 33,

$$\begin{aligned} C_1 &= R \times 103^2 \times .575 \times 25.45 \\ &= R \times 155000 \text{ watt-seconds} \end{aligned}$$

whilst from equation 35 we get,

$$\begin{aligned} C_2 &= R \times 25.45 \{1236 + 975 + 1200\} \\ &= R \times 86800 \text{ watt-seconds.} \end{aligned}$$

Total copper loss = $R \times 242000$ watt-seconds.

The more exact calculation, using equations 44 and 34, gives the total copper loss as $R \times 237000$ watt-seconds.

Now, the hot resistances are,

$$\begin{array}{ll} \text{Armature,} & .108 \text{ ohms} \\ \text{Field,} & .214 \text{ ohms} \end{array}$$

Thus, the total copper losses are,

$$\begin{array}{ll} \text{Armature,} & 25600 \text{ watt-seconds.} \\ \text{Field,} & 50700 \text{ watt-seconds.} \end{array}$$

These losses correspond to $\frac{1}{2}$ mile or $1/34$ of an hour, thus the mean losses during the time of running are,

$$\begin{array}{ll} \text{Armature,} & 242 \text{ watts.} \\ \text{Field,} & 478 \text{ watts.} \end{array}$$

The core loss on this machine is closely represented by,

$$(W - 940) (q - .705) = 456 \quad (\text{See equation 37.})$$

Thus,

$$W_1 = 940 + \frac{456}{.63} = 1665 \text{ watts}$$

$$W^1 = 940 + \frac{456}{1.795} = 1195 \text{ watts}$$

Hence, from equation 36,

$$\begin{aligned} L_1 &= .4 \times 1665 \times .575 \times 25.45 \\ &= 9730 \text{ watt-seconds} \end{aligned}$$

From equation 38,

$$\begin{aligned} L_2 &= 25.45 \{1638 + 303 - 91\} \\ &= 47100 \text{ watt-seconds} \end{aligned}$$

Hence, the total core loss during the run is 56830 watt-seconds, and the mean loss is 537 watts.

(h) Calculation of time between stops, taking account of grades, curves, etc.; also determination of energy input.

As a final example take the case of a train having actual weight per motor car of $52\frac{1}{2}$ tons, and effective weight, 58 tons, each car being equipped with four motors of the type whose speed curve is given in Fig. 6. Thus, $F_0 = 184$ lbs., $K = 4.06$, $s_0 = 18.6$ m.p.h. = 27.25 f.p.s.

The grades in a long run between two stops are given in Table I., with the results of the calculations. The first 500 feet is run in series as the speed is limited to 10 m.p.h. over the following 400 feet, on account of crossings, afterwards, the run is made in full multiple.

The current curve is approximately,

$$I = 58 \left(\frac{1}{q-1} + .39 \right)$$

Acceleration is taken at 190 amperes per motor; braking at 120 lbs. and train resistance at 15 lbs. per effective ton.

Thus,

$$w = 29000 \text{ lbs.}$$

$$T = \frac{29000 \times 27.25}{4.06 \times 184 \times 32.2} = 32.95 \text{ seconds.}$$

$$D = 32.95 \times 27.25 = 897 \text{ feet}$$

$$\text{Friction} = 217.5 \text{ lbs.}$$

q_1 is given by

$$\frac{1}{q_1 - 1} = \frac{190}{58} - .39, \text{ or } q_1 = 1.346 \quad (\text{equation 29})$$

Also,

$$\frac{K F}{F_b} = .43 \quad (\text{equations 21 and 22})$$

In the accompanying table, the first column gives the grade; the second gives the distance on that grade, in feet; the third this distance divided by 897, *i.e.*, reduced to our usual units; the fourth, the resistance per motor on the particular grade, being $217.5 + 262.5 \times \% \text{ grade}$, whilst the fifth gives the free running speed corresponding to the grade, from equation 7, with the above constants. The sixth and seventh columns give respectively the speeds of striking and leaving the speed curve on the particular grade. The eighth column gives the distance accelerating on resistance; the ninth, the distance run on the motor curve, the tenth the distance coasted, and the eleventh the distance passed over in braking. The twelfth, thirteenth, fourteenth and fifteenth columns give respectively the times corresponding to the distances in the eighth, ninth, tenth and eleventh columns. The sixteenth column gives the free running current from equation 28, with the above constants; the seventeenth is $E_1 \div VT$, from equation 30, whilst the eighteenth column is $E_2 \div VT$, from equation 32, being, after the second line, simply the value of $I^1 t_2$, supplemented by an additional term at the end, in accordance with the remark following equation 32. The figures in brackets in the first line are in units appropriate to series running [see Sec. VIII.], whilst the second line gives the same quantities in terms of the units used in the remainder of the table, for convenience of addition. ♣

In making the calculations we can conveniently start by setting down or determining the first five columns, after which we develop the table line by line. Thus, in the fourth line we determine t_1 and d_1 , by obvious modifications of equations 15 and 16, allowing for power being applied when the speed is 0.54. Thus, we arrive at $d_1 = .29$, $q_1 = 1.345$. Subtracting .29 from the total distance 1.505, we get 1.215. Adding this to the reading (.07) of distance at $q = 1.345$ on curve 4.42, Plate II., we get 1.285, which on this curve corresponds to speed $q_2 = 2.07$, whence t_2 from equation 18. Passing now to curve 2.435 at $q = 2.07$, we get distance reading 2.26, to which to add 3.345, the distance on this grade, giving 5.605, corresponding to speed reading $q_2 = 2.285$, whence t_2 from equation 18, and so on to the end of the table. The braking point in the last line has to be determined by trial so as to bring the train to rest at the required place. The columns of time, distance and energy are summed and converted into familiar units. The speed of striking or leaving a grade can be got by multiplying q_1 or q_2 by 18.6, m.p.h.

whilst at any intermediate point the corresponding value of q is got from Plate II., so that complete curves of speed-time or speed-distance could easily be plotted. Variable friction might be taken account of by dividing the variation into definite steps and treating the several steps just as different grades are treated.

XV.—CONCLUSION.

The method developed above is flexible enough to take account of all the vital circumstances of a problem, whilst it is free from the possible cumulative errors of a point-to-point method, and introduces no inaccuracy in any way comparable with that due to the uncertainty of conditions. The test curves form the basis of this, just as much as of the more obvious methods, the constants pertaining to any equipment being derived directly from these curves. We may note that the result of a calculation for an extended run, such as is given at the end of the last section, may be more accurate than the speed-time and speed-distance curves employed in deriving it. A little consideration will show that an error of interpolation on one grade will affect the time on succeeding grades in just the opposite manner to that on the grade where the error is made, and the total error from this cause is exceedingly small.

In such examples as are worked out in the last section, it is in no wise a complication to introduce actual conditions so far as they are known. In many average cases, however, actual conditions may not be determinable with any degree of accuracy, or the importance of the results may not be such as to warrant the labor of minute calculation, if it can be dispensed with. If it were not that the paper is already unduly long, we might develop a system of simple working curves, based on average conditions, which would be sufficiently accurate for the purpose in view, though they must necessarily make some sacrifices in the interests of facility.

In conclusion, I wish to express my indebtedness to the General Electric Company for its permission to make use of the curves of some of its motors in the examples worked out to illustrate the method of the paper.

APPENDIX.

On Variable Train Resistance.

So long as speeds are low, the variation in train resistance with change of speed is hardly worth considering, especially as the value of this resistance is not usually known with any degree of accuracy. But as the speed increases the variation of resistance

with speed becomes considerable, and must be taken into account. As indicated in Sec. XIV. (*h*) we can do this by dividing the variation into suitable steps, over each of which we assume a constant mean resistance, and treat the several steps as we should so many different grades. This is probably the easiest way of treating a single isolated problem. But where we have a good deal of work to do on a particular train and equipment, the better plan is to combine the train resistance, or the variable portion of it, with the motor curve, and so derive a fictitious

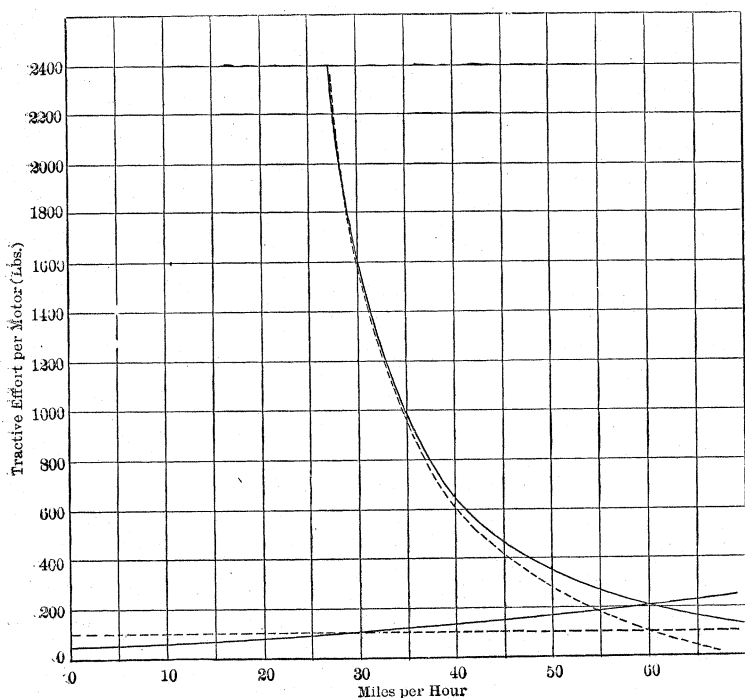


FIG. 9.

motor curve, with which we use a fictitious constant train resistance. These fictitious curves will lead to the same results as the actual motor curve and actual variable train resistance.

Fig. 9 shows, in full lines, a motor curve and resistance per motor for a particular equipment, whilst in broken lines is shown a derived fictitious motor curve, to be used with the constant train resistance also shown. The effect of grade can be added to this, just as if it were the real friction, since the resistance due to grade is independent of speed.