CONJUGATE SYSTEMS WITH EQUAL TANGENTIAL INVARIANTS AND THE TRANSFORMATION OF MOUTARD.

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In recent years one of the most interesting fields of geometrical investigations has been that of transformations of geometrical entities of a particular type into entities of the same type. As examples, we mention the well-known Bäcklund transformations of pseudospherical surfaces, the BIANCHI transformations B_k of surfaces applicable to quadrics [1]), and the transformation of the surfaces of Voss [2]). In the first two cases a surface S and a transform S_1 are the focal surfaces of a W-congruence. BIANCHI [3]) has shown the analytic relation between W-congruences and the transformation of MOUTARD [4]) and consequently we have general methods of procedure in the determination of any transformation for which the two surfaces are the focal surfaces of a W-congruence. Moreover, it should be said that there are other known transformations of this type than those mentioned above. In the third example referred to a surface S and a transform S_1 are so related that the lines joining corresponding points on their surfaces form a congruence whose developables meet S and S_1 in a conjugate system with equal tangential invariants; namely the geodesic conjugate system. It is the purpose of this paper to establish for any conjugate system with equal tangential invariants a transformation possessing the property just mentioned and determine other properties of this transformation.

When a surface S is referred to a conjugate system with equal tangential invariants, one can find by quadratures an associate surface Σ referred to its asymptotic lines. For the transformation under discussion the associate surfaces Σ and Σ_1 of S and S_1

1) L. BIANCHI, *Lezioni di Geometria differenziale,* Vol. III: *Teoria delle trasformazioni delle superficie applicabili sulle quadriche* (Pisa, Spoerri, 1909).

2) L. P. EISENHART, *Transformations of Surfaces of* Voss [Transactions of the American Mathematical Society, Vol. XV (1914), pp. 245-265].

3) BIANCHI, *Lezioni di Geometria differenziale,* Vol. II (Pisa, Spoerri, 1903), pp. 51-56.

4) TH. F. MOUTARD, *Sur la construction des équations de la forme* $\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda(x, y)$ *qui admettent une intégrale générale explicite* [Journal de l'École Polytechnique, Series I, Cahier XLV (1878), pp. 1-11]; also: L. BIANCHI, *Lezioni di Geometria differenziale,* Vol. II (Pisa, Spoerri, 1903), pp. 47, 48.

respectively are the focal sheets of a W-congruence. Accordingly we devote §§ 1, 2 to the determination in suitable form of the equations of a W-congruence. With the aid of these results we establish in § 3 the equations of the desired *transformations* Ω of the given surface S.

In § 4 we show that these general transformations possess a theorem of permutability, that is, if S_1 and S_2 are transforms of S, there exist ∞^1 surfaces S' each of which is obtained from S_1 and S_2 by transformations Ω. Moreover, if M, M_1, M_2 are corresponding points of S, S_1, S_2 the corresponding points M' of the surfaces S' lie on a conic in the plane of M, M_1, M_2; and the tangent planes to the surfaces S' pass through the point of intersection of the corresponding tangent planes to S, S_1 and S_2.

In § 5 we consider the congruence of lines of intersection of the tangent planes to a surface S and a transform S_1, and in § 6 discuss the case for which this congruence is normal. To this class belong the transformations of surfaces of Voss previously referred to [5]).

In the last section we determine under what conditions S and a transform S_1 envelope a two parameter family of spheres and find that the corresponding conjugate system of S and S_1 are lines of curvature with isothermal spherical representation. This is the transformation which we established sometime since proceeding from purely analytical considerations.

§ 1.

Equations of a non-normal W-congruence.

1. When a surface Σ is referred to its asymptotic lines and the linear element of the spherical representation of these lines is written

$$(1) \qquad d\sigma^2 = E\,du^2 + 2\sqrt{EG}\cos 2\omega\, du\, dv + G\,dv^2,$$

2ω is the angle between the parametric lines on the sphere.

If $\begin{Bmatrix}12\\1\end{Bmatrix}'$ and $\begin{Bmatrix}12\\2\end{Bmatrix}'$ are Christoffel symbols formed with respect to this linear element, and the total curvature of Σ be denoted by $-\frac{1}{\rho^2}$, we have

$$(2) \qquad \begin{Bmatrix}12\\1\end{Bmatrix}' = -\frac{1}{2}\frac{\partial \log \rho}{\partial v}, \qquad \begin{Bmatrix}12\\2\end{Bmatrix}' = -\frac{1}{2}\frac{\partial \log \rho}{\partial u} \text{ [6])}.$$

Moreover, from the definition of the symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}'$ and equations (2) follow the iden-

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5) Loc. cit. [2]).

6) L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces* (London and Boston, Ginn and Co., 1909), p. 192.

tities

$$(3)\quad \begin{cases} \dfrac{\partial \sqrt{E}}{\partial v} = -\sqrt{E}\,\dfrac{\partial \log \sqrt{\rho}}{\partial v} - \sqrt{G}\cos 2\omega\,\dfrac{\partial \log \sqrt{\rho}}{\partial u}, \\[2mm] \dfrac{\partial \sqrt{G}}{\partial u} = -\sqrt{G}\,\dfrac{\partial \log \sqrt{\rho}}{\partial u} - \sqrt{E}\cos 2\omega\,\dfrac{\partial \log \sqrt{\rho}}{\partial v}. \end{cases}$$

If X, Y, Z denote the direction-cosines of the normal to Σ and X_1, Y_1, Z_1; X_2, Y_2, Z_2 the direction-cosines of the bisectors of the angles between the parametric lines on the spherical representation of Σ, we have

$$(4)\quad \begin{cases} \dfrac{\partial X}{\partial u} = \sqrt{E}(\sin\omega\, X_1 + \cos\omega\, X_2), & \dfrac{\partial X}{\partial v} = \sqrt{G}(-\sin\omega\, X_1 + \cos\omega\, X_2), \\[2mm] \dfrac{\partial X_1}{\partial u} = -A X_2 - \sqrt{E}\sin\omega\, X, & \dfrac{\partial X_1}{\partial v} = B X_2 + \sqrt{G}\sin\omega\, X, \\[2mm] \dfrac{\partial X_2}{\partial u} = A X_1 - \sqrt{E}\cos\omega\, X, & \dfrac{\partial X_2}{\partial v} = -B X_1 - \sqrt{G}\cos\omega\, X, \end{cases}$$

where

$$(5)\quad A = \frac{\partial \omega}{\partial u} - \sqrt{\frac{E}{G}}\sin 2\omega\,\frac{\partial \log \sqrt{\rho}}{\partial v}, \qquad B = \frac{\partial \omega}{\partial v} - \sqrt{\frac{G}{E}}\sin 2\omega\,\frac{\partial \log \sqrt{\rho}}{\partial u}.$$

If we make use of the LIOUVILLE form of the GAUSSian curvature [7], we can show that

$$(6)\quad \frac{\partial A}{\partial v} + \frac{\partial B}{\partial u} + \sqrt{EG}\sin 2\omega = 0.$$

In accordance with the general theory we know that X, Y, Z are solutions of

$$(7)\quad \frac{\partial^2 \theta}{\partial u\,\partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \theta}{\partial v} + \sqrt{EG}\cos 2\omega . \theta = 0.$$

If Σ_1 is a surface which together with Σ constitutes the focal sheets of a W-congruence, the direction-cosines of the join of corresponding points may be written in the form

$$(8)\quad \alpha_1 = \cos\theta\, X_1 + \sin\theta\, X_2, \quad \beta_1 = \cos\theta\, Y_1 + \sin\theta\, Y_2, \quad \gamma_1 = \cos\theta\, Z_1 + \sin\theta\, Z_2.$$

If σ denotes the angle between the tangent planes to Σ and Σ_1, and X', Y', Z', the direction-cosines of the normal to Σ_1, we have

$$(9)\quad X' = \cos\sigma\, X + \sin\sigma(\sin\theta\, X_1 - \cos\theta\, X_2).$$

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[7] Loc. cit. [6], p. 187.

From this we derive, making use of (4),

$$(10)\quad \left\{\begin{aligned}\frac{\partial X'}{\partial u} &= -\sin\sigma\left[\frac{\partial\sigma}{\partial u} - \sqrt{E}\cos(\theta+\omega)\right]X\\ &+\left[\cos\sigma\left(\sqrt{E}\sin\omega + \sin\theta\frac{\partial\sigma}{\partial u}\right) + \sin\sigma\cos\theta\left(\frac{\partial\theta}{\partial u} - A\right)\right]X_1\\ &+\left[\cos\sigma\left(\sqrt{E}\cos\omega - \cos\theta\frac{\partial\sigma}{\partial u}\right) + \sin\sigma\sin\theta\left(\frac{\partial\theta}{\partial u} - A\right)\right]X_2,\\ \frac{\partial X'}{\partial v} &= -\sin\sigma\left[\frac{\partial\sigma}{\partial v} - \sqrt{G}\cos(\theta-\omega)\right]X\\ &+\left[\cos\sigma\left(-\sqrt{G}\sin\omega + \sin\theta\frac{\partial\sigma}{\partial v}\right) + \sin\sigma\cos\theta\left(\frac{\partial\theta}{\partial v} + B\right)\right]X_1\\ &+\left[\cos\sigma\left(\sqrt{G}\cos\omega - \cos\theta\frac{\partial\sigma}{\partial v}\right) + \sin\sigma\sin\theta\left(\frac{\partial\theta}{\partial v} + B\right)\right]X_2.\end{aligned}\right.$$

The direction-cosines X', Y', Z' are given by the quadratures

$$(11)\quad \frac{\partial}{\partial u}(\sqrt{\rho\rho_1}\,w_1 X') = -\rho w_1^2\frac{\partial}{\partial u}\left(\frac{X}{w_1}\right),\qquad \frac{\partial}{\partial v}(\sqrt{\rho\rho_1}\,w_1 X') = \rho w_1^2\frac{\partial}{\partial v}\left(\frac{X}{w_1}\right).$$

where now w_1 is a solution of the equation (7) [8].

The cartesian coordinates ξ_1, η_1, ζ_1, of Σ_1 are given by equations of the form

$$(12)\quad \xi_1 - \xi = \sqrt{\rho\rho_1}(YZ' - ZY') = \sqrt{\rho\rho_1}\sin\sigma(\cos\theta X_1 + \sin\theta X_2)\ ^{9)}.$$

Also we have

$$(13)\quad \frac{\partial\xi}{\partial u} = \sqrt{E}\rho(\cos\omega X_1 - \sin\omega X_2),\qquad \frac{\partial\xi}{\partial v} = -\sqrt{G}\rho(\cos\omega X_1 + \sin\omega X_2),$$

and

$$(14)\quad \mathfrak{E} = E\rho^2,\qquad \mathfrak{F} = -\rho^2\sqrt{EG}\cos 2\omega,\qquad \mathfrak{G} = G\rho^2,$$

where E, F and G denote the first fundamental coefficients of Σ.

If in (11) we replace $\frac{\partial X}{\partial u}$ and $\frac{\partial X}{\partial v}$ by their values from (4) we have

$$(15)\quad \left\{\begin{aligned}\frac{\partial X'}{\partial u} &= -\frac{\partial}{\partial u}\log(\sqrt{\rho\rho_1}\,w_1)X' - \sqrt{\frac{\rho}{\rho_1}}\left[\sqrt{E}(\sin\omega X_1 + \cos\omega X_2) - X\frac{\partial\log w_1}{\partial u}\right],\\ \frac{\partial X'}{\partial v} &= -\frac{\partial}{\partial v}\log(\sqrt{\rho\rho_1}\,w_1)X' - \sqrt{\frac{\rho}{\rho_1}}\left[\sqrt{G}(\sin\omega X_1 - \cos\omega X_2) + X\frac{\partial\log w_1}{\partial v}\right].\end{aligned}\right.$$

Equating these expressions to the values given by (10), we obtain the following

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[8]) This follows from equations (74) [loc. cit. [6]), p. 419] where we put

$\nu_1 = \sqrt{\rho}X,\quad \nu_2 = \sqrt{\rho}Y,\quad \nu_3 = \sqrt{\rho}Z;\qquad \bar\nu_1 = \sqrt{\rho_1}X',\quad \bar\nu_2 = \sqrt{\rho_1}Y',\quad \bar\nu_3 = \sqrt{\rho_1}Z';\qquad \theta_1 = \sqrt{\rho}\,\omega_1.$

[9]) Loc. cit. [6]), p. 418.

equations:

$$
(16)\quad
\left\{
\begin{aligned}
&\sin\sigma\left[\frac{\partial\sigma}{\partial u}-\sqrt{E}\cos(\theta+\omega)\right]=\cos\sigma\frac{\partial\log\sqrt{\rho\rho_1}}{\partial u}+\left(\cos\sigma-\sqrt{\frac{\rho}{\rho_1}}\right)\frac{\partial\log w_1}{\partial u},\\
&\sin\sigma\left[\frac{\partial\sigma}{\partial v}-\sqrt{G}\cos(\theta-\omega)\right]=\cos\sigma\frac{\partial\log\sqrt{\rho\rho_1}}{\partial v}+\left(\cos\sigma+\sqrt{\frac{\rho}{\rho_1}}\right)\frac{\partial\log w_1}{\partial v},\\
&\cos\sigma\frac{\partial\sigma}{\partial u}+\frac{\partial}{\partial v}\log\left(\sqrt{\rho\rho_1}\,w_1\right).\sin\sigma-\sqrt{E}\cos(\theta+\omega)\left(\cos\sigma+\sqrt{\frac{\rho}{\rho_1}}\right)=0,\\
&\cos\sigma\frac{\partial\sigma}{\partial v}+\frac{\partial}{\partial v}\log\left(\sqrt{\rho\rho_1}\,w_1\right).\sin\sigma-\sqrt{G}\cos(\theta-\omega)\left(\cos\sigma-\sqrt{\frac{\rho}{\rho_1}}\right)=0,\\
&\sin\sigma\left(\frac{\partial\theta}{\partial u}-A\right)+\sqrt{E}\left(\cos\sigma+\sqrt{\frac{\rho}{\rho_1}}\right)\sin(\theta+\omega)=0,\\
&\sin\sigma\left(\frac{\partial\theta}{\partial v}+B\right)+\sqrt{G}\left(\cos\sigma-\sqrt{\frac{\rho}{\rho_1}}\right)\sin(\theta-\omega)=0.
\end{aligned}
\right.
$$

For the present we exclude the case where $\sigma=\frac{\pi}{2}$ and thus equations (16) are equivalent to

$$
(17)\quad
\left\{
\begin{aligned}
&\frac{\partial\sigma}{\partial u}=\sqrt{E}\cos(\theta+\omega)\left(1+\cos\sigma\sqrt{\frac{\rho}{\rho_1}}\right)-\sqrt{\frac{\rho}{\rho_1}}\sin\sigma\frac{\partial\log w_1}{\partial u},\\
&\frac{\partial\sigma}{\partial v}=\sqrt{G}\cos(\theta-\omega)\left(1-\cos\sigma\sqrt{\frac{\rho}{\rho_1}}\right)+\sqrt{\frac{\rho}{\rho_1}}\sin\sigma\frac{\partial\log w_1}{\partial v},
\end{aligned}
\right.
$$

$$
(18)\quad
\left\{
\begin{aligned}
&\frac{\partial\log\sqrt{\rho\rho_1}}{\partial u}+\frac{\partial\log w_1}{\partial u}\left(1-\sqrt{\frac{\rho}{\rho_1}}\cos\sigma\right)-\sqrt{E}\sin\sigma\sqrt{\frac{\rho}{\rho_1}}\cos(\theta+\omega)=0,\\
&\frac{\partial\log\sqrt{\rho\rho_1}}{\partial v}+\frac{\partial\log w_1}{\partial v}\left(1+\sqrt{\frac{\rho}{\rho_1}}\cos\sigma\right)+\sqrt{G}\sin\sigma\sqrt{\frac{\rho}{\rho_1}}\cos(\theta-\omega)=0,
\end{aligned}
\right.
$$

$$
(19)\quad
\left\{
\begin{aligned}
&\sin\sigma\left(\frac{\partial\theta}{\partial u}-A\right)+\sqrt{E}\left(\cos\sigma+\sqrt{\frac{\rho}{\rho_1}}\right)\sin(\theta+\omega)=0,\\
&\sin\sigma\left(\frac{\partial\theta}{\partial v}+B\right)+\sqrt{G}\left(\cos\sigma-\sqrt{\frac{\rho}{\rho_1}}\right)\sin(\theta-\omega)=0.
\end{aligned}
\right.
$$

It can readily be shown that the conditions of integrability of these equations are satisfied [10]).

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[10]) Equations (17), (18), (19) are essentially the same as those found by M. PICONE, *Sulle congruence rettilinee W* [Rendiconti del Circolo Matematico di Palermo, t. XXXVII (1° semestre 1914), pp. 212-244], p. 220. We noticed this work of PICONE as we were preparing our results for publication. Our method makes use of the transformation of MOUTARD and thereby is much shorter. These equations and the following ones of this section are extremely useful in discussing particular types of *W*-congruences.

2. With the aid of equations (17), (18), (19) equations (10) are reducible to

$$(20)\begin{cases}\dfrac{\partial X'}{\partial u}=\sqrt{\dfrac{\rho}{\rho_1}}\Big\{[X\sin\sigma-\cos\sigma(\sin\theta X_1-\cos\theta X_2)]\left[\dfrac{\partial \log w_1}{\partial u}\sin\sigma-\sqrt{E}\cos\sigma\cos(\theta+\omega)\right]\\ \qquad -\sqrt{E}\sin(\theta+\omega)(X_1\cos\theta+X_2\sin\theta)\Big\},\\ \dfrac{\partial X'}{\partial v}=\sqrt{\dfrac{\rho}{\rho_1}}\Big\{[X\sin\sigma-\cos\sigma(\sin\theta X_1-\cos\theta X_2)]\left[-\dfrac{\partial \log w_1}{\partial v}\sin\sigma+\sqrt{G}\cos\sigma\cos(\theta-\omega)\right]\\ \qquad +\sqrt{G}\sin(\theta-\omega)(X_1\cos\theta+X_2\sin\theta)\Big\}.\end{cases}$$

We introduce functions θ_1; X'_1, Y'_1, Z'_1; X'_2, Y'_2, Z'_2 defined by

$$(21)\quad\begin{cases}\cos\theta_1 X'_1+\sin\theta_1 X'_2=\cos\theta X_1+\sin\theta X_2,\\ -\sin\theta_1 X'_1+\cos\theta_1 X'_2=X\sin\sigma-\cos\sigma(\sin\theta X_1-\cos\theta X_2),\end{cases}$$

and analogous equations in the $Y's$ and $Z's$.

It follows from (9) and (21) that

$$(22)\qquad\begin{vmatrix}X' & Y' & Z'\\ X'_1 & Y'_1 & Z'_1\\ X'_2 & Y'_2 & Z'_2\end{vmatrix}=+1.$$

From (21) follow also

$$(23)\quad\begin{cases}X'_1=\cos\theta_1[\cos\theta X_1+\sin\theta X_2]-\sin\theta_1[X\sin\sigma-\cos\sigma(\sin\theta X_1-\cos\theta X_2)],\\ X'_2=\sin\theta_1[\cos\theta X_1+\sin\theta X_2]+\cos\theta_1[X\sin\sigma-\cos\sigma(\sin\theta X_1-\cos\theta X_2)].\end{cases}$$

We denote by ω_1, E_1, G_1 functions for Σ_1 analogous to ω, E, G for Σ. We imagine θ_1 determined so that we hawe

$$(24)\quad \frac{\partial X'}{\partial u}=\sqrt{E_1}(\sin\omega_1 X'_1+\cos\omega_1 X'_2),\qquad \frac{\partial X'}{\partial v}=\sqrt{G_1}(-\sin\omega_1 X'_1+\cos\omega_1 X'_2).$$

Comparing these with (20), we have

$$(25)\begin{cases}\sqrt{E_1}\sin\omega_1=-\sqrt{\dfrac{\rho}{\rho_1}}\Big\{\sin\theta_1\left[\dfrac{\partial\log w_1}{\partial u}\sin\sigma-\sqrt{E}\cos\sigma\cos(\theta+\omega)\right]+\sqrt{E}\sin(\theta+\omega)\cos\theta_1\Big\},\\ \sqrt{E_1}\cos\omega_1=\sqrt{\dfrac{\rho}{\rho_1}}\Big\{\cos\theta_1\left[\dfrac{\partial\log w_1}{\partial u}\sin\sigma-\sqrt{E}\cos\sigma\cos(\theta+\omega)\right]-\sqrt{E}\sin(\theta+\omega)\sin\theta_1\Big\},\\ \sqrt{G_1}\sin\omega_1=-\sqrt{\dfrac{\rho}{\rho_1}}\Big\{\sin\theta_1\left[\dfrac{\partial\log w_1}{\partial v}\sin\sigma-\sqrt{G}\cos\sigma\cos(\theta-\omega)\right]+\sqrt{G}\sin(\theta-\omega)\cos\theta_1\Big\},\\ \sqrt{G_1}\cos\omega_1=\sqrt{\dfrac{\rho}{\rho_1}}\Big\{-\cos\theta_1\left[\dfrac{\partial\log w_1}{\partial v}\sin\sigma-\sqrt{G}\cos\sigma\cos(\theta-\omega)\right]+\sqrt{G}\sin(\theta-\omega)\sin\theta_1\Big\}.\end{cases}$$

These equations are equivalent to

$$(26)\quad\begin{cases}\dfrac{\partial\log w_1}{\partial u}\sin\sigma=\sqrt{E}[\cos\sigma\cos(\theta+\omega)-\sin(\theta+\omega)\cot(\theta_1+\omega_1)],\\ \dfrac{\partial\log w_1}{\partial v}\sin\sigma=\sqrt{G}[\cos\sigma\cos(\theta-\omega)-\sin(\theta-\omega)\cot(\theta_1-\omega_1)],\end{cases}$$

and

$$(27)\quad \sqrt{E_1} = -\sqrt{\frac{\rho}{\rho_1}}\sqrt{E}\,\frac{\sin(\theta+\omega)}{\sin(\theta_1+\omega_1)}, \qquad \sqrt{G_1} = \sqrt{\frac{\rho}{\rho_1}}\sqrt{G}\,\frac{\sin(\theta-\omega)}{\sin(\theta_1-\omega_1)}.$$

If one calculates E_1 and G_1 from (20), it is seen that equations (26) and (27) are consistent as determining $(\theta_1+\omega_1)$ and $(\theta_1-\omega_1)$. Hence equations (26) may be solved for θ_1 as desired.

Now equations (17) and (18) are reducible by (26) to

$$(28)\quad \begin{cases} \dfrac{\partial\sigma}{\partial u} = \sqrt{E}\left[\cos(\theta+\omega) + \sqrt{\dfrac{\rho}{\rho_1}}\sin(\theta+\omega)\cot(\theta_1+\omega_1)\right], \\ \dfrac{\partial\sigma}{\partial v} = \sqrt{G}\left[\cos(\theta-\omega) - \sqrt{\dfrac{\rho}{\rho_1}}\sin(\theta-\omega)\cot(\theta_1-\omega_1)\right], \end{cases}$$

and

$$(29)\quad \begin{cases} \dfrac{\partial}{\partial u}\log\sqrt{\rho\rho_1} = \dfrac{-\sqrt{E}}{\sin\sigma}\left[\left(\cos\sigma - \sqrt{\dfrac{\rho}{\rho_1}}\right)\cos(\theta+\omega) \right. \\ \qquad\qquad \left. + \left(\sqrt{\dfrac{\rho}{\rho_1}}\cos\sigma - 1\right)\sin(\theta+\omega)\cot(\theta_1+\omega_1)\right], \\ \dfrac{\partial}{\partial v}\log\sqrt{\rho\rho_1} = \dfrac{-\sqrt{G}}{\sin\sigma}\left[\left(\cos\sigma + \sqrt{\dfrac{\rho}{\rho_1}}\right)\cos(\theta-\omega) \right. \\ \qquad\qquad \left. - \left(\sqrt{\dfrac{\rho}{\rho_1}}\cos\sigma + 1\right)\sin(\theta-\omega)\cot(\theta_1-\omega_1)\right]. \end{cases}$$

From these we derive also

$$(30)\quad \begin{cases} \dfrac{\partial}{\partial u}\log(\sqrt{\rho\rho_1}\,w_1) = \dfrac{\sqrt{E}}{\sin\sigma}\sqrt{\dfrac{\rho}{\rho_1}}[\cos(\theta+\omega) - \cos\sigma\sin(\theta+\omega)\cot(\theta_1+\omega_1)], \\ \dfrac{\partial}{\partial v}\log(\sqrt{\rho\rho_1}\,w_1) = \dfrac{\sqrt{G}}{\sin\sigma}\sqrt{\dfrac{\rho}{\rho_1}}[-\cos(\theta-\omega) + \cos\sigma\sin(\theta-\omega)\cot(\theta_1-\omega_1)]. \end{cases}$$

In view of the fact that θ_1 has been chosen so that (24) hold, we must have also, analogous to (4)

$$(31)\quad \begin{cases} \dfrac{\partial X_1'}{\partial u} = -A_1X_2' - \sqrt{E_1}\sin\omega_1 X', & \dfrac{\partial X_1'}{\partial v} = B_1X_2' + \sqrt{G_1}\sin\omega_1 X', \\ \dfrac{\partial X_2'}{\partial u} = A_1X_1' - \sqrt{E_1}\cos\omega_1 X', & \dfrac{\partial X_2'}{\partial v} = -B_1X_1' - \sqrt{G_1}\cos\omega_1 X', \end{cases}$$

where

$$(32)\quad A_1 = \frac{\partial\omega_1}{\partial u} - \sqrt{\frac{E_1}{G_1}}\sin 2\omega_1\frac{\partial\log\sqrt{\rho_1}}{\partial v}, \qquad B_1 = \frac{\partial\omega_1}{\partial v} - \sqrt{\frac{G_1}{E_1}}\sin 2\omega_1\frac{\partial\log\sqrt{\rho_1}}{\partial u}.$$

In order to find the conditions imposed by this set of equations, we calculate first the derivatives of α_1 defined by (8) and also of α_2, where

$$(33)\quad \alpha_2 = -\sin\theta X_1 + \cos\theta X_2.$$

One finds readily

$$(34)\quad \begin{cases} \dfrac{\partial \alpha_1}{\partial u} = -\sqrt{E}\dfrac{\sin(\theta+\omega)}{\sin\sigma}\left[\left(\cos\sigma + \sqrt{\dfrac{\rho}{\rho_1}}\right)\alpha_2 + X\sin\sigma\right], \\ \dfrac{\partial \alpha_1}{\partial v} = -\sqrt{G}\dfrac{\sin(\theta-\omega)}{\sin\sigma}\left[\left(\cos\sigma - \sqrt{\dfrac{\rho}{\rho_1}}\right)\alpha_2 + X\sin\sigma\right], \end{cases}$$

$$(35)\quad \begin{cases} \dfrac{\partial \alpha_2}{\partial u} = \dfrac{\sqrt{E}}{\sin\sigma}\left[\sin(\theta+\omega)\left(\cos\sigma + \sqrt{\dfrac{\rho}{\rho_1}}\right)\alpha_1 - X\sin\sigma\cos(\theta+\omega)\right], \\ \dfrac{\partial \alpha_2}{\partial v} = \dfrac{\sqrt{G}}{\sin\sigma}\left[\sin(\theta-\omega)\left(\cos\sigma - \sqrt{\dfrac{\rho}{\rho_1}}\right)\alpha_1 - X\sin\sigma\cos(\theta-\omega)\right]. \end{cases}$$

With the aid of these formulas we show that when the expressions (23) for X_1' and X_2' are substituted in (31) equations (25) must be satisfied and also

$$(36)\quad \begin{cases} \dfrac{\partial \theta_1}{\partial u} - A_1 + \sqrt{E}\dfrac{\sin(\theta+\omega)}{\sin\sigma}\left(\cos\sigma\sqrt{\dfrac{\rho}{\rho_1}} + 1\right) = 0, \\ \dfrac{\partial \theta_1}{\partial v} + B_1 + \sqrt{G}\dfrac{\sin(\theta-\omega)}{\sin\sigma}\left(\cos\sigma\sqrt{\dfrac{\rho}{\rho_1}} - 1\right) = 0. \end{cases}$$

§ 2.

Normal W-congruences.

3. We consider now the case $\sigma = \dfrac{\pi}{2}$. Equations (16) reduce in this case to

$$(37)\quad \begin{cases} \dfrac{\partial \log w_1}{\partial u} = \sqrt{\dfrac{\rho_1}{\rho}}\sqrt{E}\cos(\theta+\omega), \\ \dfrac{\partial \log w_1}{\partial v} = -\sqrt{\dfrac{\rho_1}{\rho}}\sqrt{G}\cos(\theta-\omega), \end{cases}$$

$$(38)\quad \begin{cases} \dfrac{\partial \log\sqrt{\rho\rho_1}}{\partial u} = \sqrt{E}\cos(\theta+\omega)\left(\sqrt{\dfrac{\rho}{\rho_1}} - \sqrt{\dfrac{\rho_1}{\rho}}\right), \\ \dfrac{\partial \log\sqrt{\rho\rho_1}}{\partial v} = -\sqrt{G}\cos(\theta-\omega)\left(\sqrt{\dfrac{\rho}{\rho_1}} - \sqrt{\dfrac{\rho_1}{\rho}}\right), \end{cases}$$

$$(39)\quad \begin{cases} \dfrac{\partial \theta}{\partial u} - A = -\sqrt{E}\sqrt{\dfrac{\rho}{\rho_1}}\sin(\theta+\omega), \\ \dfrac{\partial \theta}{\partial v} + B = \sqrt{G}\sqrt{\dfrac{\rho}{\rho_1}}\sin(\theta-\omega). \end{cases}$$

The condition of integribility of the last equations is satisfied identically, whereas the conditions of integribility of (37) and (38) require that

$$(40)\qquad \cos(\theta+\omega)\sqrt{E}\,\frac{\partial\log\sqrt{\rho}}{\partial v}+\cos(\theta-\omega)\sqrt{G}\,\frac{\partial\log\sqrt{\rho}}{\partial u}=0.$$

In order to determine normal W-congruences one must express the condition that θ given by (40) satisfies the preceding equations. It is important to observe, however, that from the form of (40) it follows that a surface satisfying the conditions can be the focal surface of *only one* normal W-congruence.

In consequence of (39) equations (10) become reducible by (23) to the form

$$(41)\qquad \begin{cases} \dfrac{\partial X'}{\partial u}=\sqrt{E}\left[\cos(\theta+\omega)(-\sin\theta_1 X_1'+\cos\theta_1 X_2')-\sqrt{\dfrac{\rho}{\rho_1}}\sin(\theta+\omega)(\cos\theta_1 X_1'+\sin\theta_1 X_2')\right], \\ \dfrac{\partial X'}{\partial v}=\sqrt{G}\left[\cos(\theta-\omega)(-\sin\theta_1 X_1'+\cos\theta_1 X_2')+\sqrt{\dfrac{\rho}{\rho_1}}\sin(\theta-\omega)(\cos\theta_1 X_1'+\sin\theta_1 X_2')\right]. \end{cases}$$

Comparing these with (24), we are brought to the equations

$$(42)\qquad \begin{cases} \sin(\theta_1+\omega_1)\cos(\theta+\omega)+\sqrt{\dfrac{\rho}{\rho_1}}\cos(\theta_1+\omega_1)\sin(\theta+\omega)=0, \\ \sin(\theta_1-\omega_1)\cos(\theta-\omega)-\sqrt{\dfrac{\rho}{\rho_1}}\cos(\theta_1-\omega_1)\sin(\theta-\omega)=0. \end{cases}$$

$$(43)\qquad \sqrt{E_1}=-\sqrt{\frac{\rho}{\rho_1}}\sqrt{E}\,\frac{\sin(\theta+\omega)}{\sin(\theta_1+\omega_1)},\qquad \sqrt{G_1}=\sqrt{\frac{\rho}{\rho_1}}\sqrt{G}\,\frac{\sin(\theta-\omega)}{\sin(\theta_1-\omega_1)}.$$

In consequence of (42) equations (37) are equivalent to (26) when in the latter we put $\sigma=\frac{\pi}{2}$. Hence all of the formulas (19), (26), (29), (36) are equally true for $\sigma=\frac{\pi}{2}$.

The following known theorems concerning normal W-congruences are immediate consequences of equations (38) and (40):

The total curvature of the focal surfaces at corresponding points is equal only when these surfaces are pseudospherical.

There is a functional relation between the total curvatures of the focal sheets of any W-congruence.

§ 3.

Transformations Ω of conjugate systems with equal tangential invariants.

4. Let S be a surface with the spherical representation of a conjugate parametric system given by (1) satisfying the conditions involved in (2). The surface Σ with the spherical representation (1) of its asymptotic lines is the associate of S for the given

parametric conjugate system. In this sense we shall refer to it as *the associate of S*. The tangential coordinates X, Y, Z, W of S are solutions of (7). The point coordinates (x, y, z) are given by expressions of the form [11])

$$(44)\quad x = WX + \frac{1}{EG\sin^2 2\omega}\left[\frac{\partial W}{\partial u}\left(G\frac{\partial X}{\partial u} - F\frac{\partial X}{\partial v}\right) + \frac{\partial W}{\partial v}\left(E\frac{\partial X}{\partial v} - F\frac{\partial X}{\partial u}\right)\right],$$

which in consequence of equations (4) may be written

$$(45)\quad x=WX+\frac{1}{\sqrt{EG}\sin 2\omega}\left[\sqrt{G}\frac{\partial W}{\partial u}(X_1\cos\omega+X_2\sin\omega)+\sqrt{E}\frac{\partial W}{\partial v}(-X_1\cos\omega+X_2\sin\omega)\right].$$

If D, $D' = 0$, D'' denote the second fundamental coefficients of S, we have

$$(46)\quad \begin{cases} \dfrac{\partial x}{\partial u} = \dfrac{-D}{\sqrt{E}\sin 2\omega}(\cos\omega X_1 + \sin\omega X_2), \\[2ex] \dfrac{\partial x}{\partial v} = \dfrac{D''}{\sqrt{G}\sin 2\omega}(\cos\omega X_1 - \sin\omega X_2) \text{ [12]}. \end{cases}$$

Moreover, we have

$$(47)\quad \begin{cases} D = -\left[\dfrac{\partial^2 W}{\partial u^2} - \begin{Bmatrix}11\\1\end{Bmatrix}'\dfrac{\partial W}{\partial u} - \begin{Bmatrix}11\\2\end{Bmatrix}'\dfrac{\partial W}{\partial v} + EW\right], \\[2ex] D'' = -\left[\dfrac{\partial^2 W}{\partial v^2} - \begin{Bmatrix}22\\1\end{Bmatrix}'\dfrac{\partial W}{\partial u} - \begin{Bmatrix}22\\2\end{Bmatrix}'\dfrac{\partial W}{\partial v} + GW\right] \text{ [13]}, \end{cases}$$

where in consequence of (3) the expressions for CHRISTOFFEL symbols may be put in the form

$$(48)\quad \begin{cases} \begin{Bmatrix}11\\1\end{Bmatrix}' = \dfrac{\partial\log\sqrt{E}}{\partial u} + 2\cot 2\omega\dfrac{\partial\omega}{\partial u} - \sqrt{\dfrac{E}{G}}\cos 2\omega\dfrac{\partial\log\sqrt{\rho}}{\partial v}, \\[2ex] \begin{Bmatrix}11\\2\end{Bmatrix}' = \dfrac{E}{G}\dfrac{\partial\log\sqrt{\rho}}{\partial v} - 2\sqrt{\dfrac{E}{G}}\dfrac{1}{\sin 2\omega}\dfrac{\partial\omega}{\partial u}, \\[2ex] \begin{Bmatrix}22\\1\end{Bmatrix}' = \dfrac{G}{E}\dfrac{\partial\log\sqrt{\rho}}{\partial u} - 2\sqrt{\dfrac{G}{E}}\dfrac{1}{\sin 2\omega}\dfrac{\partial\omega}{\partial v}, \\[2ex] \begin{Bmatrix}22\\2\end{Bmatrix}' = \dfrac{\partial\log\sqrt{G}}{\partial v} + 2\cot 2\omega\dfrac{\partial\omega}{\partial v} - \sqrt{\dfrac{G}{E}}\cos 2\omega\dfrac{\partial\log\sqrt{\rho}}{\partial u}. \end{cases}$$

The function W_1 determined by the equations

$$(49)\quad \begin{cases} \dfrac{\partial}{\partial u}(\sqrt{\rho\rho_1}\,w_1 W_1) = -\rho\left(w_1\dfrac{\partial W}{\partial u} - W\dfrac{\partial w_1}{\partial u}\right), \\[2ex] \dfrac{\partial}{\partial v}(\sqrt{\rho\rho_1}\,w_1 W_1) = \rho\left(w_1\dfrac{\partial W}{\partial v} - W\dfrac{\partial w_1}{\partial v}\right), \end{cases}$$

and X', Y', Z' are the tangential coordinates of a surface S_1, of which Σ_1 given by (12) is the associate.

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[11]) Loc. cit. [6]), p. 163.
[12]) Loc. cit. [6]), p. 200.
[13]) Loc. cit. [6]), p. 164.

The coordinates (x_1, y_1, z_1) of S_1 are of the form

$$(50)\quad \begin{cases} x_1 = W_1 X' + \dfrac{1}{\sqrt{E_1 G_1}\sin 2\omega_1}\Big[\sqrt{G_1}\dfrac{\partial W_1}{\partial v}(X_1'\cos\omega_1 + X_2'\sin\omega_1) \\ \qquad\qquad + \sqrt{E_1}\dfrac{\partial W_1}{\partial v}(-X_1'\cos\omega_1 + X_2'\sin\omega_1)\Big]. \end{cases}$$

We substitute in (50) the expressions for $\frac{\partial W_1}{\partial u}$ and $\frac{\partial W_1}{\partial v}$ given by (49); also the expressions for X', X_1', X_2' given by (9) and (23). If we put

$$(51)\quad R = \frac{\rho}{\rho_1}\frac{\sqrt{EG}}{\sqrt{E_1 G_1}\sin 2\omega_1}\Big[\frac{W\cos\sigma - W_1}{\sin\sigma} + \frac{1}{\sqrt{E}}\frac{\sin(\theta-\omega)}{\sin 2\omega}\frac{\partial W}{\partial u} - \frac{1}{\sqrt{G}}\frac{\sin(\theta+\omega)}{\sin 2\omega}\frac{\partial W}{\partial v}\Big],$$

the result of the above substitution becomes

$$(52)\quad \begin{cases} \dfrac{x_1 - x}{R} = X_1\{\cos\omega[\cot(\theta_1-\omega_1)\sin(\theta-\omega) - \cot(\theta_1+\omega_1)\sin(\theta+\omega)] - \cos\sigma\sin\theta\sin 2\omega\} \\ \quad + X_2\{-\sin\omega[\cot(\theta_1-\omega_1)\sin(\theta-\omega) + \cot(\theta_1+\omega_1)\sin(\theta+\omega)] + \cos\sigma\cos\theta\sin 2\omega\} \\ \quad + X\sin\sigma\sin 2\omega. \end{cases}$$

Since W_1 enters only in the factor R and not in the right-hand member of the equation, and since W_1 is determined only to within the additive function $\frac{c}{\sqrt{\rho\rho_1}\, w_1}$, where c is an arbitrary constant, we see that on the line joining corresponding points of S and S_1 there is an infinity of points describing surfaces with the same spherical representation of a conjugate system as S_1. In like manner in equations (49) W may be replaced by $W + c w_1$ without changing the right-hand member. Hence on the line referred to there are an infinity of points describing conjugate systems parallel to the conjugate system on S. One knows that if two conjugate systems are parallel the lines joining corresponding points form a congruence whose developables cut the two surfaces in these conjugate lines. In view of these results we have the theorem:

If S is a surface referred to a conjugate system with equal tangential invariants and Σ its associate surface; if Σ_1 is a surface which with Σ constitute the focal sheets of a W-congruence, determined by a set of functions θ, θ_1, w_1, σ; when one draws through points of S lines having the direction determined by the right-hand member of (52), these lines generate a congruence whose developables meet S in the parametric conjugate system, and one can determine upon these lines by one quadrature an infinity of points each generating a surface S_1, cut in a conjugate system by the developlables of the congruence, and Σ_1 is the associate surface of S_1.

We say that S and S_1 are in the relation of a *transformation* Ω.

We shall show that for one of the surfaces associate to a given Σ and for a given set of functions θ, θ_1, w_1, σ all the lines of the congruence pass through the origin. The analytical condition for this is that there exist a function t such that

$$x_1 - x = t x, \qquad y_1 - y = t y, \qquad z_1 - z = t z.$$

From (44) and (52) it follows that we must have

$$
(53)\quad \begin{cases} R \sin\sigma \sin 2\omega = t\,W, \\ R\{\cot(\theta_1 - \omega_1)\sin(\theta - \omega) - \cot(\theta_1 + \omega_1)\sin(\theta + \omega) - 2\cos\sigma\sin\theta\sin\omega\} \\ \qquad = \dfrac{t}{\sqrt{EG}\sin 2\omega}\left(\sqrt{G}\dfrac{\partial W}{\partial u} - \sqrt{E}\dfrac{\partial W}{\partial v}\right), \\ R\{\cot(\theta_1 - \omega_1)\sin(\theta - \omega) + \cot(\theta_1 + \omega_1)\sin(\theta + \omega) - 2\cos\sigma\cos\theta\cos\omega\} \\ \qquad = \dfrac{-t}{\sqrt{EG}\sin 2\omega}\left(\sqrt{G}\dfrac{\partial W}{\partial u} + \sqrt{E}\dfrac{\partial W}{\partial v}\right). \end{cases}
$$

One finds readily that these equations are consistent when $W = w_1$, given by (26), and only in this case.

Referring to (49), we see that the functions W_1 are given by

$$
(54)\qquad W_1 = \frac{c}{\sqrt{\rho\rho_1}\,w_1}.
$$

We denote by $\bar{S}$ and $\bar{S}_1$ the surfaces associate to Σ and Σ_1 respectively, for which $W = w_1$ and equation (54) holds.

The surface $\bar{S}$ determines an infinitesimal deformation of Σ for which the coordinates of the surface Σ' corresponding to Σ with orthogonality of linear elements are equal to

$$
\sqrt{\rho\rho_1}\,w_1 X', \qquad \sqrt{\rho\rho_1}\,w_1 Y', \qquad \sqrt{\rho\rho_1}\,w_1 Z' \quad [14]).
$$

Owing to the reciprocal relation of Σ and Σ' the latter admits an infinitesimal deformation for which Σ is the surface corresponding with orthogonality of linear elements. The coordinates of the corresponding associate surface Σ_1' are $\frac{X}{w_1}, \frac{Y}{w_1}, \frac{Z}{w_1}$. If we consider Σ' as the associate surface in an infinitesimal deformation of Σ_1' the surface corresponding to the latter with orthogonality of linear elements is Σ_1 given by (12) [15]). Moreover, $\bar{S}_1$ is the associate of Σ_1 corresponding to Σ_1'. Hence Σ, Σ', Σ_1', Σ_1, $\bar{S}$ and $\bar{S}_1$ form part of a group of the so-called *twelve surfaces of* DARBOUX [16]).

§ 4.

Theorem of permutability for the transformations Ω.

5. Suppose that we have two solutions w_1 and w_2 of equation (7). The function w_2 leads to a W-congruence of which Σ and a surface Σ_2 are the focal sheets; we denote by ρ_2 the function for Σ_2 analogous to ρ for Σ.

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14) Loc. cit. 6), p. 420.

15) Loc cit. 6), p. 419.

16) Cfr. G. DARBOUX, *Leçons sur la théorie générale des surfaces et les applications géométriques du Calcul infinitésimal* (Paris, Gauthier-Villars), Vol. IV (1896), pp. 48-72; also: L. P. EISENHART, *The Twelve Surfaces of* DARBOUX *and the Transformation of* MOUTARD [American Journal of Mathematics, Vol. XXXII (1910), pp. 17-36].

By the transformation of MOUTARD two functions w'_1 and w'_2 are given by the quadratures

$$(55)\qquad \begin{cases} \dfrac{\partial}{\partial u}(\sqrt{\rho\rho_i}\,w_i w'_i) = -\rho\left(w_i\dfrac{\partial w_j}{\partial u} - w_j\dfrac{\partial w_i}{\partial u}\right), \\ \dfrac{\partial}{\partial v}(\sqrt{\rho\rho_i}\,w_i w'_i) = \rho\left(w_i\dfrac{\partial w_j}{\partial v} - w_j\dfrac{\partial w_i}{\partial v}\right) \end{cases} \qquad \left(\begin{matrix} i, j = 1, 2 \\ i \neq j \end{matrix}\right).$$

[17]

The functions w'_i are determined only to within the additive quantity $\dfrac{c}{\sqrt{\rho\rho_i}\,w_i}$, where c is an arbitrary constant. These constants can be chosen so that pairs of functions satisfy the relation

$$(56)\qquad \sqrt{\rho\rho_1}\,w_1 w'_1 + \sqrt{\rho\rho_2}\,w_2 w'_2 = 0.$$

With this pairing of the functions w'_1 and w'_2 we have

$$(57)\qquad \sqrt{\rho_1}\,w'_1\frac{\partial^2}{\partial u\partial v}\left(\frac{1}{\sqrt{\rho_1}\,w'_1}\right) = \sqrt{\rho_2}\,w'_2\frac{\partial^2}{\partial u\partial v}\left(\frac{1}{\sqrt{\rho_2}\,w'_2}\right).$$

Also one has that w'_1 and w'_2 are solutions of

$$(58)\qquad \frac{\partial^2}{\partial u\partial v}(\sqrt{\rho_i}\,\theta_i) = \sqrt{\rho}\,w_i\frac{\partial^2}{\partial u\partial v}\left(\frac{1}{\sqrt{\rho}\,w_i}\right)\cdot\sqrt{\rho_i}\,\theta_i \qquad (i = 1, 2).$$

If we compare (55) with (49) we note that w'_1 is obtained from w_2 just as W_1 is obtained from W. Hence w'_1 and W_1 are solutions of the same equation, namely (58) for $i = 1$. Accordingly w'_1 can be used to transform W_1 into a function W'_1. If we denote by W_2 the function obtained from W with the aid of w_2 by means of equations similar to (49), it follows that w'_2 and W_2 satisfy the same equation, namely (58), for $i = 2$. Hence w'_2 enables us to obtain from W_2 a function W'_2 by equations of the form (49). It is our purpose to show that in consequence of (56) the functions W'_1 and W'_2 are equal to one another; we denote them by W'.

In this case we must have

$$(59)\qquad \begin{cases} \dfrac{\partial}{\partial u}(\sqrt{\rho'\rho_i}\,w'_i W') = -\rho_i\left(w'_i\dfrac{\partial W_i}{\partial u} - W_i\dfrac{\partial w'_i}{\partial u}\right), \\ \dfrac{\partial}{\partial v}(\sqrt{\rho'\rho_i}\,w'_i W') = \rho_i\left(w'_i\dfrac{\partial W_i}{\partial v} - W_i\dfrac{\partial w'_i}{\partial v}\right), \end{cases} \qquad (i = 1, 2)$$

where ρ' is a function analogous to ρ, and also

$$(60)\qquad \begin{cases} \dfrac{\partial}{\partial u}(\sqrt{\rho\rho_i}\,w_i W_i) = -\rho\left(w_i\dfrac{\partial W}{\partial u} - W\dfrac{\partial w_i}{\partial u}\right), \\ \dfrac{\partial}{\partial v}(\sqrt{\rho\rho_i}\,w_i W_i) = \rho\left(w_i\dfrac{\partial W}{\partial v} - W\dfrac{\partial w_i}{\partial v}\right), \end{cases} \qquad (i = 1, 2)$$

these equations being similar to (49).

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[17] L. BIANCHI, loc. cit. [3]), p. 70.

If we take the first of equations (59) for $i = 1$ and 2 and substitute for $\frac{\partial W_i}{\partial u}$ from (60), and then eliminate $\frac{\partial W'}{\partial u}$, we obtain

$$(61) \qquad \sqrt{\rho'}\, W' = \sqrt{\rho}\, W + t(W_1\sqrt{\rho_1} - W_2\sqrt{\rho_2}),$$

where

$$(62) \qquad t = \frac{w_1\sqrt{\rho}}{w_2'\sqrt{\rho_2}} = -\frac{w_2\sqrt{\rho}}{w_1'\sqrt{\rho_1}}.$$

If we proceed in like manner with the second of equations (59) for $i = 1$ and 2, we are brought to the same result.

Furthermore, one shows readily that the expression (61) satisfies equations (59), so that the assumption that W_1' and W_2' are equal is correct.

The direction-cosines of the normal to S_1 are obtained from those of S by equations (11) which are of the same form as (60). In like manner the direction-cosines X'', Y'', Z'' of the normal to S_2 are obtained from those of S by (60). Evidently the direction-cosines $\bar{X}$, $\bar{Y}$, $\bar{Z}$ of the normal to S' for which W' given by (61) is a tangential coordinate are given by expressions of the form

$$(63) \qquad \sqrt{\rho'}\,\bar{X} = \sqrt{\rho}\, X + t(X'\sqrt{\rho_1} - X''\sqrt{\rho_2}).$$

From (61) and (63) it follows that *the tangent planes of* S, S_1, S_2, S' *at corresponding points meet in a point.*

If Σ, Σ_1, Σ_2 are the surfaces associate to S, S_1, S_2, so that Σ and Σ_1 are the focal sheets of a W-congruence, and also Σ and Σ_2, the associate Σ' of S' can be so placed in space that Σ_1 and Σ' are the focal sheets of a W-congruence and likewise Σ_2 and Σ' as Bianchi has shown. In fact, the cartesian coordinates of Σ' are of the form

$$(64) \qquad \xi' = \xi + t\sqrt{\rho_1\rho_2}(Y'Z'' - Z'Y'').$$

It was this result of Bianchi which gave a geometrical interpretation of the theorem of Moutard. Our results have given another interpretation. There are other properties of the theorem of permutability which we shall develop now.

6. We denote by s_1 and s_2 the surfaces whose parametric conjugate system has the spherical representation (1), and which are determined by the respective functions w_1 and w_2. We denote by x_{01}, y_{01}, z_{01}; x_{02}, y_{02}, z_{02} the coordinates of s_1 and s_2. Then for $i = 1$ and 2

$$(65) \quad x_{0i} = w_i X + \frac{1}{\sqrt{EG}\sin 2\omega}\left[\left(\sqrt{G}\frac{\partial w_i}{\partial u} - \sqrt{E}\frac{\partial w_i}{\partial v}\right)\cos\omega X_1 + \left(\sqrt{G}\frac{\partial w_i}{\partial u} + \sqrt{E}\frac{\partial w_i}{\partial v}\right)\sin\omega X_2\right].$$

It is readily shown that equation (52) can be written

$$(66) \qquad x_i - x = \frac{R_i \sin\sigma_i \sin 2\omega}{w_i} x_{0i}. \qquad (i = 1, 2).$$

If we compare equations (55) with (49), we observe that the surface s_1' determined by w'_1 is a transform of s_2. Hence its coordinates $x'_{01}, y'_{01}, z'_{01}$ are given by expressions of the form

$$(67) \qquad x'_{01} - x_{02} = \frac{R'_{01} \sin\sigma_1 \sin 2\omega}{w_1} x_{01},$$

where

$$(68) \quad R'_{01} = \frac{\rho}{\rho_1} \frac{\sqrt{EG}}{\sqrt{E_1 G_1} \sin 2\omega_1}\left[\frac{w_2 \cos\sigma_1 - w'_1}{\sin\sigma_1} + \frac{1}{\sqrt{E}} \frac{\sin(\theta-\omega)}{\sin 2\omega} \frac{\partial w_2}{\partial u} - \frac{1}{\sqrt{G}} \frac{\sin(\theta+\omega)}{\sin 2\omega} \frac{\partial \omega_2}{\partial v}\right].$$

Since w'_1 is the transforming function of S_1, it follows from equations analogous to (66) that

$$(69) \qquad x' - x_1 = \frac{R'_1 \sin\sigma'_1 \sin 2\omega_1}{w'_1} x'_{01},$$

where, analogously to (68),

$$(70) \quad \left\{ \begin{aligned} R'_1 = \frac{\rho_1}{\rho'} \frac{\sqrt{E_1 G_1}}{\sqrt{E' G'} \sin 2\omega'} &\left[\frac{W_1 \cos\sigma'_1 - W'}{\sin\sigma'_1}\right. \\ &\left. + \frac{1}{\sqrt{E_1}} \frac{\sin(\theta'-\omega_1)}{\sin 2\omega_1} \frac{\partial W_1}{\partial u} - \frac{1}{\sqrt{G_1}} \frac{\sin(\theta'+\omega_1)}{\sin 2\omega_1} \frac{\partial W_1}{\partial v}\right]. \end{aligned} \right.$$

From (66), (67) and (69) it follows that the point M' of S' with coordinates x', y', z' lies in the plane determined by M, M_1, M_2. In fact M' is the point of intersection of the lines through M_1 and M_2 whose direction parameters are proportional to $x'_{01}, y'_{01}, z'_{01}$; $x'_{02}, y'_{02}, z'_{02}$ respectively.

We have observed that w'_1 is determined only to within the additive function $\frac{c}{\sqrt{\rho\rho_1}\, w_1}$. Accordingly we put

$$(71) \qquad w'_1 = \bar{w}'_1 + \frac{c}{\sqrt{\rho\rho_1}\, w_1},$$

where $\bar{w}'_1$ is independent of c. In order that (56) may hold w'_2 must be of the form

$$(72) \qquad w'_2 = \bar{w}'_2 - \frac{c}{\sqrt{\rho\rho_2}\, w_2},$$

it being understood that $\bar{w}'_1$ and $\bar{w}'_2$ satisfy (56). From (67) we have accordingly

$$(73) \qquad x'_{01} = x_{02} + (P_1 + c\,Q_1) x_{01}, \qquad x'_{02} = x_{01} + (P_2 + c\,Q_2) x_{02},$$

where P_1, Q_1, P_2, Q_2 are independent of c.

Evidently the two pencils of lines through M_1 and M_2 whose direction-parameters have the values (73) respectively are projective. Hence the points M' lie on a conic. From (71) it follows that as c becomes very large the term $\frac{c}{\sqrt{\rho\rho_1}\, w_1}$ is the controlling term. Hence for $c = \infty$ we have the point M.

The question arises as to whether the conic is degenerate so that in fact the pencils are perspective and all the points lie on a line. In order to settle this question we determine the coordinates of M'.

From (61) and (63) we have that S' is the envelope of the plane

$$(74) \quad \sum x'[(T+c)X+w_1w_2(X'\sqrt{\rho\rho_1}-X''\sqrt{\rho\rho_2})]=(T+c)W+w_1w_2(W_1\sqrt{\rho\rho_1}-W_2\sqrt{\rho\rho_2}),$$

where

$$T = w_2 w_2' \sqrt{\rho\rho_2}.$$

If we differentiate this equation with respect to u and v respectively, and in the reduction make use of the preceding results, we have

$$(75) \quad \left\{\begin{aligned} &\sum x'\left[(T+c)\frac{\partial X}{\partial u} + w_1\frac{\partial w_2}{\partial u}X'\sqrt{\rho\rho_1} - w_2\frac{\partial w_1}{\partial u}X''\sqrt{\rho\rho_2}\right] \\ &= (T+c)\frac{\partial W}{\partial u} + w_1\frac{\partial w_2}{\partial u}W_1\sqrt{\rho\rho_1} - w_2\frac{\partial w_1}{\partial u}W_2\sqrt{\rho\rho_2}, \\ &\sum x'\left[(T+c)\frac{\partial X}{\partial v} + w_1\frac{\partial w_2}{\partial v}X'\sqrt{\rho\rho_1} - w_2\frac{\partial w_1}{\partial v}X''\sqrt{\rho\rho_2}\right] \\ &= (T+c)\frac{\partial W}{\partial v} + w_1\frac{\partial w_2}{\partial v}W_1\sqrt{\rho\rho_1} - w_2\frac{\partial w_1}{\partial v}W_2\sqrt{\rho\rho_2}. \end{aligned}\right.$$

If for the sake of brevity we put

$$(\theta, \varphi, \psi) = \begin{vmatrix} \theta & \varphi & \psi \\ \frac{\partial\theta}{\partial u} & \frac{\partial\varphi}{\partial u} & \frac{\partial\psi}{\partial u} \\ \frac{\partial\theta}{\partial v} & \frac{\partial\varphi}{\partial v} & \frac{\partial\psi}{\partial v} \end{vmatrix}, \qquad H = \sqrt{EG}\sin 2\omega,$$

the determinant Δ of equations (74) and (75) is reducible to

$$(76) \quad \begin{aligned} \Delta = (T+c)^3H + (T+c)^2H[w_1\sqrt{\rho\rho_1}\textstyle\sum X'x_{02} - w_2\sqrt{\rho\rho_2}\sum X''x_{01}] \\ + (T+c)\rho\sqrt{\rho_1\rho_2}\,w_1w_2\textstyle\sum(X'Y''-X''Y').(w_1, w_2, Z). \end{aligned}$$

Solving equations (74) and (75) for x', we obtain

$$(76') \quad \left\{\begin{aligned} &\Delta x' = (T+c)^3Hx + (T+c)^2\{(W, Y, w_2)w_1\sqrt{\rho\rho_1}Z' - (W, Y, w_1)w_2\sqrt{\rho\rho_2}Z'' \\ &-(W, Z, w_2)w_1\sqrt{\rho\rho_1}Y' + (W, Z, w_1)w_2\sqrt{\rho\rho_1}Y'' + W_1w_1\sqrt{\rho_1\rho}(w_2, Y, Z) \\ &- W_2w_2\sqrt{\rho\rho_2}(w_1, Y, Z)\} + (T+c)w_1w_2\rho\sqrt{\rho_1\rho_2}\{(W, w_1, w_2)(Y'Z''-Y''Z') \\ &\quad + (Y, w_1, w_2)(Z'W_2 - Z''W_1) + (Z, w_1, w_2)(W_1Y'' - W_2Y')\}. \end{aligned}\right.$$

From expressions (76) and (76′) it follows that

$$(x'-x)\Delta = a_1(T+c)^2 + a_0(T+c),$$
$$(y'-y)\Delta = b_1(T+c)^2 + b_0(T+c),$$
$$(z'-z)\Delta = c_1(T+c)^2 + c_0(T+c),$$

where a_1, b_1, c_1; a_0, b_0, c_0 are determinate functions of u and v. If the locus of M' is to be a straight line, the ratios

$$\frac{x'-x}{y'-y}, \qquad \frac{x'-x}{z'-z},$$

must be independent of c. The necessary and sufficient condition for this is that

$$\frac{a_1}{a_0} = \frac{b_1}{b_0} = \frac{c_1}{c_0}.$$

This gives two equations linear in W_1 and W_2. The coefficient of W_1 in these two equations involve X'', Y'', Z'', w_1, w_2. Evidently they cannot be zero for X'' is independent of w_1. Hence W_1 and W_2 could be found by solving two linear equations. But they are determined by (60) only to within additive functions $\frac{\alpha}{\sqrt{\rho\rho_1} w_1}$ and $\frac{\beta}{\sqrt{\rho\rho_2} w_2}$ where α and β are constants. Hence in general the above conditions are not satisfied.

Accordingly the theorem of permutability may be stated as follows:

Let s_1 and s_2 be two surfaces corresponding to the given surface S with parallelism of tangent planes, and let the tangential coordinates of s_1 and s_2 be X, Y, Z, w_1 and X, Y, Z, w_2; by means of w_1 and w_2 we transform S into two surfaces S_1 and S_2; also by means of w_1 we transform s_2 into s'_1 and by w_2 we transform s_1 into s'_2, the corresponding tangential coordinates of s'_1 and s'_2 being denoted by w'_1 and w'_2.

If through a point M_1 of S_1 we draw a line parallel to the radius vector of s'_1 at the corresponding point, and in like manner through the corresponding point M_2 of S_2 a line parallel to the radius vector of s'_2 the lines meet in a point M' of the plane M, M_1, M_2. The locus of M', a surface S', is the transform of S_1 by w'_1 and of S_2 by w'_2. When the parameter in w'_1 varies we get ∞' of points M' which lie on a conic passing through M, M_1, M_2.

§ 5.

The Adjoint Congruence.

7. Since the joins of corresponding points of S and S_1 form a congruence whose developables cut these surfaces in a conjugate system, the developables of the congruence of lines of intersection of the tangent planes to S and S_1 correspond to the developables of the former congruence. We say that the new congruence is *adjoint* to the other. We know that its focal points are the points of intersection of the tangents to the parametric curves of S and S_1 at corresponding points. It is our purpose now to find the coordinates a_1, b_1, c_1; a_2, b_2, c_2 of these focal points.

In consequence of (46) and analogous equations for S_1 it follows that the expressions for a_1 and a_2 may be given the form

$$(77)\quad \begin{cases} a_1 = x + \lambda_1(\cos\omega\, X_1 + \sin\omega\, X_2) = x_1 + \mu_1(\cos\omega_1 X'_1 + \sin\omega_1 X'_2), \\ a_2 = x + \lambda_2(\cos\omega\, X_1 + \sin\omega\, X_2) = x_1 + \mu_2(\cos\omega_1 X'_1 - \sin\omega_1 X'_2), \end{cases}$$

where λ_1, μ_1, λ_2, μ_2 are to be determined.

The equality of the two expressions for a_1 necessitates the equations

$$R \sin 2\omega - \mu_1 \sin(\theta_1 - \omega_1) = 0,$$

$$\begin{aligned} &R\{\cos\omega[\cot(\theta_1 - \omega_1)\sin(\theta - \omega) - \cot(\theta_1 + \omega_1)\sin(\theta + \omega)] - \cos\sigma \sin\theta \sin 2\omega\} \\ &\quad + \mu_1\{\cos(\theta_1 - \omega_1)\cos\theta + \sin(\theta_1 - \omega_1)\cos\sigma\sin\theta\} - \lambda_1 \cos\omega = 0, \end{aligned}$$

$$\begin{aligned} &R\{-\sin\omega[\cot(\theta_1 - \omega_1)\sin(\theta - \omega) + \cot(\theta_1 + \omega_1)\sin(\theta + \omega)] + \cos\sigma\cos\theta\sin 2\omega\} \\ &\quad + \mu_1\{\cos(\theta_1 - \omega_1)\sin\theta - \sin(\theta_1 - \omega_1)\cos\sigma\cos\theta\} - \lambda_1 \sin\omega = 0. \end{aligned}$$

From these we have

$$(78) \qquad \lambda_1 = \frac{R\sin(\theta + \omega)\sin 2\omega_1}{\sin(\theta_1 - \omega_1)\sin(\theta_1 + \omega_1)}, \qquad \mu_1 = \frac{R\sin 2\omega}{\sin(\theta_1 - \omega_1)}.$$

In like manner we find

$$(79) \qquad \lambda_2 = \frac{R\sin(\theta - \omega)\sin 2\omega_1}{\sin(\theta_1 - \omega_1)\sin(\theta_1 + \omega_1)}, \qquad \mu_2 = \frac{R\sin 2\omega}{\sin(\theta_1 + \omega_1)}.$$

If we put

$$(80) \quad T = \frac{W_1 - W\cos\sigma}{\sin\sigma} - \frac{1}{\sqrt{E}}\frac{\sin(\theta - \omega)}{\sin 2\omega}\frac{\partial W}{\partial u} + \frac{1}{\sqrt{G}}\frac{\sin(\theta + \omega)}{\sin 2\omega}\frac{\partial W}{\partial v},$$

it follows from (78) and (79), when the expression (51) for R is substituted, that

$$(81) \qquad \lambda_1 = \frac{T}{\sin(\theta - \omega)}, \qquad \lambda_2 = \frac{T}{\sin(\theta + \omega)}.$$

From (80) we have

$$\frac{\partial T}{\partial u} = -\frac{\cos(\theta + \omega)\left(\cos\sigma + \sqrt{\frac{\rho}{\rho_1}}\right)\sqrt{E}}{\sin\sigma} T + \frac{D}{\sqrt{E}}\frac{\sin(\theta - \omega)}{\sin 2\omega},$$

$$\frac{\partial T}{\partial v} = -\frac{\cos(\theta - \omega)\left(\cos\sigma - \sqrt{\frac{\rho}{\rho_1}}\right)\sqrt{G}}{\sin\sigma} T - \frac{D''}{\sqrt{G}}\frac{\sin(\theta + \omega)}{\sin 2\omega}.$$

Making use of these results, we obtain from (77) by differentiation

$$(82) \quad \left\{ \begin{aligned} \frac{\partial a_1}{\partial u} &= \frac{\lambda_1\sqrt{E}\sin 2\omega}{\sin(\theta - \omega)}\left[-\sin(\theta - \omega)X + \frac{\cos\sigma + \sqrt{\frac{\rho}{\rho_1}}}{\sin\sigma}(\cos\omega X_1 + \sin\omega X_2) \right. \\ &\qquad \left. + \frac{1}{\sqrt{G}}\frac{\partial \log\sqrt{\rho}}{\partial v}(\cos\theta X_1 + \sin\theta X_2)\right], \\ \frac{\partial a_1}{\partial v} &= \left[\lambda_1\left(\frac{\partial\omega}{\partial v} + B\right) - \frac{D''}{\sqrt{G}}\right]\frac{\cos\theta X_1 + \sin\theta X_2}{\sin(\theta - \omega)}, \end{aligned} \right.$$

$$
(83)\left\{\begin{aligned}
\frac{\partial a_2}{\partial u} &= -\left[\lambda_2\left(\frac{\partial\omega}{\partial u}+A\right)+\frac{D}{\sqrt{E}}\right]\frac{\cos\theta X_1+\sin\theta X_2}{\sin(\theta+\omega)},\\
\frac{\partial a_2}{\partial v} &= \frac{\lambda_2\sqrt{G}\sin 2\omega}{\sin(\theta+\omega)}\left[\sin(\theta+\omega)X-\frac{\cos\sigma-\sqrt{\frac{\rho}{\rho_1}}}{\sin\sigma}(\cos\omega X_1-\sin\omega X_2)\right.\\
&\qquad\left.-\frac{1}{\sqrt{E}}\frac{\partial\log\sqrt{\rho}}{\partial u}(\cos\theta X_1-\sin\theta X_2)\right].
\end{aligned}\right.
$$

From these expressions we find that the direction-cosines A_1, B_1, C_1; A_2, B_2, C_2 of the normals to the focal surfaces are given by

$$
(84)\left\{\begin{aligned}
A_1 &= \frac{1}{\sqrt{1+\frac{\rho}{\rho_1}+2\cos\sigma\sqrt{\frac{\rho}{\rho_1}}}}\left[\left(\cos\sigma+\sqrt{\frac{\rho}{\rho_1}}\right)X+\sin\sigma(\sin\theta X_1-\cos\theta X_2)\right],\\
A_2 &= \frac{1}{\sqrt{1+\frac{\rho}{\rho_1}-2\cos\sigma\sqrt{\frac{\rho}{\rho_1}}}}\left[\left(\cos\sigma-\sqrt{\frac{\rho}{\rho_1}}\right)X+\sin\sigma(\sin\theta X_1-\cos\theta X_2)\right].
\end{aligned}\right.
$$

If x_0, y_0, z_0 are the coordinates of any point on the adjoint congruence, and we put

$$P=\sum X(x-x_0),\qquad Q=\sum(\sin\theta X_1-\cos\theta X_2)(x-x_0),$$

where x, y, z are current coordinates, the equations of planes tangent to S and S_1 and of the focal planes are of the form

$$P=0,\qquad P\cos\sigma-Q\sin\sigma=0,$$

$$\left(\cos\sigma+\sqrt{\frac{\rho}{\rho_1}}\right)P+\sin\sigma\, Q=0,\qquad \left(\cos\sigma-\sqrt{\frac{\rho}{\rho_1}}\right)P+\sin\sigma\, Q=0.$$

Evidently the latter planes are harmonic conjugate with respect to the first two.

Consequently we have the theorem:

When S and S_1 are two surfaces in the relation of a transformation Ω their tangent planes are harmonic conjugate to the focal planes of the congruence generated by the line of intersection of these tangent planes.

From (84) it is seen that the necessary and sufficient condition that the focal planes be perpendicular is that $\rho_1=\rho$. Hence we have the theorem:

The necessary and sufficient condition that the lines of intersection of the tangent planes to the two surfaces S and S_1 in the relation of a transformation Ω form a normal congruence is that the associate surfaces Σ and Σ_1 have equal total curvature at corresponding points; in this case the focal planes bisect the angles between the tangent planes to S and S_1.

BIANCHI has considered W-congruences of this kind [18]); we shall apply the preceding formulas to this case.

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[18]) L. BIANCHI, *Sopra alcune nuove classi di superficie e di sistemi tripli ortogonali* [Annali di Matematica pura ed applicata, Ser. II, Vol. XVIII (1890), pp. 301-358], pp. 330-334.

§ 6.

Normal adjoint congruences.

8. When $\rho_1 = \rho$, equations (17) and (18) may be replaced by

$$(85) \qquad \frac{\partial \sigma}{\partial u} = \frac{\sin\sigma}{1-\cos\sigma}\frac{\partial \log \rho}{\partial u}, \qquad \frac{\partial \sigma}{\partial v} = -\frac{\sin\sigma}{1+\cos\sigma}\frac{\partial \log \rho}{\partial v},$$

$$(86) \qquad \begin{cases} \dfrac{\partial \log w_1}{\partial u} = \sqrt{E}\dfrac{\sin\sigma}{1-\cos\sigma}\cos(\theta+\omega) - \dfrac{\partial \log\rho}{\partial u}\dfrac{1}{1-\cos\sigma}, \\ \dfrac{\partial \log w_1}{\partial v} = -\sqrt{G}\dfrac{\sin\sigma}{1+\cos\sigma}\cos(\theta-\omega) - \dfrac{\partial \log\rho}{\partial v}\dfrac{1}{1+\cos\sigma}. \end{cases}$$

Moreover, equations (19) become

$$(87) \qquad \begin{cases} \dfrac{\partial \theta}{\partial u} - A + \sqrt{E}\dfrac{\cos\sigma+1}{\sin\sigma}\sin(\theta+\omega) = 0, \\ \dfrac{\partial \theta}{\partial v} + B + \sqrt{G}\dfrac{\cos\sigma-1}{\sin\sigma}\sin(\theta-\omega) = 0. \end{cases}$$

The condition of integrability of (87) is satisfied. The condition of integrability of (85) is

$$(88) \qquad \frac{\partial^2 \log\rho}{\partial u \partial v} + \frac{\partial \log\rho}{\partial u}\frac{\partial \log\rho}{\partial v} = 0$$

or

$$(89) \qquad \rho = \varphi(u) + \psi(v).$$

In like manner the condition of integrability of (86) reduces to (88).

By means of (86) equations (26) become for the present case

$$(90) \qquad \begin{cases} \sqrt{E}\sin(\theta+\omega)[\cot(\theta_1+\omega_1)+\cot(\theta+\omega)] = \dfrac{1+\cos\sigma}{\sin\sigma}\dfrac{\partial\log\rho}{\partial u}, \\ \sqrt{G}\sin(\theta-\omega)[\cot(\theta_1-\omega_1)-\cot(\theta-\omega)] = \dfrac{1-\cos\sigma}{\sin\sigma}\dfrac{\partial\log\rho}{\partial v}, \end{cases}$$

equations which give θ_1 and ω_1.

Since all the conditions of integrability of the foregoing equations are satisfied, we have the following theorems of Bianchi [19]:

The necessary and sufficient condition that the curvature at corresponding points of the focal surfaces of a W-congruence be equal is that this curvature be of the form

$$(91) \qquad K = \frac{-1}{[\varphi(u)+\psi(v)]^2}.$$

When the curvature of a surface Σ referred to its asymptotic lines is of the form (91), the determination of a surface Σ_1 referred to its asymptotic lines, whose curvature

19) L. Bianchi, loc. cit. [18]).

is given by (91) *and such that* Σ *and* Σ_1 *are the focal surfaces of a W-congruence requires the solution of equations* (85), (86) *and* (87), *in which* ρ *has the value* (89).

The integral of equations (85) is given by

$$(92) \qquad \tan\frac{\sigma}{2} = \sqrt{\frac{\psi(v) - k}{\varphi(u) + k}},$$

where k is a constant.

The surfaces S whose associates possess this property are the only surfaces which can be deformed continuously with the preservation of a conjugate system. In fact, it is the conjugate system with the spherical representation determined by (89). Recalling the preceding results we have the following theorem:

If S is a surface possessing a conjugate system which is preserved in a continuous deformation of S, there exist surfaces S_1 possessing the same property, determined by a solution of equations (87), *in which σ is given by* (82), *and by a quadrature* (49), *such that the lines joining corresponding points on S and S_1 form a congruence whose developables meet S and S_1 in these conjugate systems; moreover, the lines of intersection of the tangent planes to S and S_1 form a normal congruence whose focal planes bisect the angles between the tangent planes to S and S_1.*

§ 7.

When S and S_1 envelope a two-parameter family of spheres.

9. In this section we consider the case where S and S_1 are the envelope of a two-parameter family of spheres. In order that this be the case we must have

$$(93) \quad x + rX = x_1 + rX', \qquad y + rY = y_1 + rY', \qquad z + rZ = z_1 + rZ',$$

where r denotes the radius of a sphere.

Substituting the values of x_1 and X' from (52) and (9), we are led to the three conditions

$$(94) \quad \left\{\begin{aligned} & R \sin 2\omega (1 + \cos\sigma) = r \sin\sigma, \\ & R\{\cos\omega [\cot(\theta_1 - \omega_1)\sin(\theta - \omega) - \cot(\theta_1 + \omega_1)\sin(\theta + \omega)] \\ & \qquad - \cos\sigma \sin\theta \sin 2\omega\} + r \sin\sigma \sin\theta = 0, \\ & R\{- \sin\omega [\cot(\theta_1 - \omega_1)\sin(\theta - \omega) + \cot(\theta_1 + \omega_1)\sin(\theta + \omega)] \\ & \qquad + \cos\sigma \cos\theta \sin 2\omega\} - r \sin\sigma \cos\theta = 0. \end{aligned}\right.$$

Eliminating r from the last two equations, we have

$$\cot(\theta + \omega)\cot(\theta_1 - \omega_1) - \cot(\theta - \omega)\cot(\theta_1 + \omega_1) = 0.$$

Accordingly we introduce a function λ by

$$(95) \qquad \cot(\theta_1 - \omega_1) = \lambda \cot(\theta - \omega), \qquad \cot(\theta_1 + \omega_1) = \lambda \cot(\theta + \omega).$$

Substituting these values in the last two equations (94), multiplying by $\sin\theta$ and

$-\cos\theta$ respectively and adding, we get

$$R\sin 2\omega(\lambda - \cos\sigma) + r\sin\sigma = 0.$$

From this and the first of (94), it follows that $\lambda = -1$. Hence (95) become

$$(96)\qquad \cot(\theta_1 - \omega_1) = -\cot(\theta - \omega), \qquad \cot(\theta_1 + \omega_1) = -\cot(\theta + \omega).$$

In consequence of (96) equations (26) and (28) reduce to

$$(97)\qquad \begin{cases} \dfrac{\partial \log w_1}{\partial u} = \sqrt{E}\,\dfrac{\cos\sigma + 1}{\sin\sigma}\cos(\theta + \omega), \\ \dfrac{\partial \log w_1}{\partial v} = \sqrt{G}\,\dfrac{\cos\sigma + 1}{\sin\sigma}\cos(\theta - \omega), \end{cases}$$

and

$$(98)\qquad \begin{cases} \dfrac{\partial \sigma}{\partial u} = \sqrt{E}\cos(\theta + \omega)\left(1 - \sqrt{\dfrac{\rho}{\rho_1}}\right), \\ \dfrac{\partial \sigma}{\partial v} = \sqrt{G}\cos(\theta - \omega)\left(1 + \sqrt{\dfrac{\rho}{\rho_1}}\right). \end{cases}$$

The condition of integrability of equations (97) is reducible by means of (98) to

$$\cos 2\omega = 0.$$

Hence the system on the sphere forms an orthogonal system from which it follows that Σ is a minimal surface; also S that is referred to its lines of curvature.

From (96) it follows that

$$2\omega_1 = n\pi - 2\omega,$$

where n is on integer. Consequently $\cos 2\omega_1 = 0$, which was to have been foreseen because of symmetry.

10. We shall find the equations for this case. From equations (3) follows the well-known fact that the system on the sphere is isothermic. We take

$$(99)\qquad \sqrt{E} = \sqrt{G} = e^{\alpha}.$$

From equations (2) and the expressions for $\begin{Bmatrix}12\\1\end{Bmatrix}'$ and $\begin{Bmatrix}12\\2\end{Bmatrix}'$ it follows that

$$(100)\qquad \sqrt{\rho} = e^{-\alpha}.$$

In like manner we write

$$(101)\qquad \sqrt{E_1} = \sqrt{G_1} = e^{\alpha_1}, \qquad \sqrt{\rho_1} = e^{-\alpha_1}.$$

Equations (97) and (98) become

$$(102)\qquad \frac{\partial \log w_1}{\partial u} = e^{\alpha}\cos\left(\theta + \frac{\pi}{4}\right)\frac{\cos\sigma + 1}{\sin\sigma}, \qquad \frac{\partial \log w_1}{\partial v} = e^{\alpha}\cos\left(\theta - \frac{\pi}{4}\right)\frac{\cos\sigma + 1}{\sin\sigma},$$

$$(103)\qquad \frac{\partial \sigma}{\partial u} = \cos\left(\theta + \frac{\pi}{4}\right)(e^{\alpha} - e^{\alpha_1}), \qquad \frac{\partial \sigma}{\partial v} = \cos\left(\theta - \frac{\pi}{4}\right)(e^{\alpha} + e^{\alpha_1}).$$

Also equations (29) reduce to

$$(104)\qquad \begin{cases} \dfrac{\partial}{\partial u}(\alpha_1 + \alpha) = \cos\left(\theta + \dfrac{\pi}{4}\right)(e^{\alpha} - e^{\alpha_1})\dfrac{\cos\sigma + 1}{\sin\sigma}, \\ \dfrac{\partial}{\partial v}(\alpha_1 + \alpha) = \cos\left(\theta - \dfrac{\pi}{4}\right)(e^{\alpha} + e^{\alpha_1})\dfrac{\cos\sigma + 1}{\sin\sigma}. \end{cases}$$

Equations (103) and (104) may be combined so that we have by integration

$$(105)\qquad \sin\frac{\sigma}{2} = c\, e^{\frac{\alpha+\alpha_1}{2}},$$

where c denotes a constant.

Furthermore, equations (19) assume that form

$$(106)\qquad \begin{cases} \sin\sigma\left(\dfrac{\partial\theta}{\partial u} - \dfrac{\partial\alpha}{\partial v}\right) + (e^{\alpha}\cos\sigma + e^{\alpha_1})\sin\left(\theta + \dfrac{\pi}{4}\right) = 0, \\ \sin\sigma\left(\dfrac{\partial\theta}{\partial v} + \dfrac{\partial\alpha}{\partial u}\right) + (e^{\alpha}\cos\sigma - e^{\alpha_1})\sin\left(\theta - \dfrac{\pi}{4}\right) = 0. \end{cases}$$

Equations (103), (104), (106) determine a transformation of THYBAUT [20]) of minimal surfaces.

If we put

$$(107)\qquad \varphi = e^{-(\alpha_1+\alpha)} w_1,$$

it follows from (102) and (104) that

$$(108)\qquad \frac{\partial\log\varphi}{\partial u} = \cos\left(\theta + \frac{\pi}{4}\right)e^{\alpha_1}\frac{\cos\sigma + 1}{\sin\sigma}, \quad \frac{\partial\log\varphi}{\partial v} = -\cos\left(\theta - \frac{\pi}{4}\right)e^{\alpha_1}\frac{\cos\sigma + 1}{\sin\sigma}.$$

Again from (102), (107) and (108) it follows that

$$(109)\qquad \frac{\partial\varphi}{\partial u} = e^{-2\alpha}\frac{\partial w_1}{\partial u}, \qquad \frac{\partial\varphi}{\partial v} = -e^{-2\alpha}\frac{\partial w_1}{\partial v}.$$

We have observed that in the present case the surfaces of which Σ and Σ_1 are associate have their lines of curvature represented on the sphere by an isothermic orthogonal system, and conversely all such surfaces S have a minimal surface for associate. Hence we have a transformation of such surfaces S possessing the property that S and a transform form the envelope of a two parameter family of spheres. When the functions θ, α_1 and σ are known, the further determination of S_1 requires the integration of equations (49) which in consequence of (107) may be given the form

$$\frac{\partial}{\partial u}(\varphi W_1) = -e^{-2\alpha} w_1^2 \frac{\partial}{\partial u}\left(\frac{W}{w_1}\right), \qquad \frac{\partial}{\partial v}(\varphi W_1) = e^{-2\alpha} w_1^2 \frac{\partial}{\partial v}\left(\frac{W}{w_1}\right).$$

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[20]) A. THYBAUT, *Sur la déformation du paraboloïde et sur quelques problèmes qui s'y rattachent* [Annales Scientifiques de l'École Normale supérieure (Paris), Ser. III, Vol. XIV, pp. 45-98; cfr. also: L. BIANCHI, loc. cit. [3]), pp. 334-338.

These are the same transformations which we discovered in a former paper [21]) by analytical considerations, but now their bearing upon the general problems of transformations of conjugate systems with equal tangential invariants is clearly in evidence.

Princeton University, March 7, 1914

LUTHER PFAHLER EISENHART.

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[21]) L. P. EISENHART, *Surfaces with isothermal representation of their lines of curvature and their transformations* [Transactions of the American Mathematical Society, Vol. IX (1908), pp. 149-177].