The Transformation of Linear Partial Differential Operators by Extended Linear Continuous Groups. By E. B. ELLIOTT. Received and read March 10, 1898.

1. On a number of previous occasions, and more particularly in a paper "On the Interchange of the Variables in certain Linear Differential Operators" [*Phil. Trans.*, Vol. CLXXXI. (1890), A, pp. 19–51], I have applied a simple but artificial method to the transformation of linear differential operators from forms in which their arguments are the successive derivatives of a dependent with regard to one or more independent variables to the forms which they assume when dependent and independent variables are transposed. Again, in a paper "On the Reversion of Partial Differential Expressions with Two Independent and Two Dependent Variables" (*Proc. Lond. Math. Soc.*, Vol. XXII., pp. 79–104), I have somewhat extended the method, by application to a case in which there are more independent variables than one. Up to the present time I have not, nor, so far as I know, has any one else, applied the method to the transformation of operators consequent on continuous transformations of, or substitutions for, dependent and independent variables in terms of new dependent and independent variables; but it is one which admits of wide applications in this direction. The following paper deals with a class of such applications.

I.

2. There is no new difficulty of principle when the formulæ of transformation are linear, and the original variables are supposed connected by one relation only, so that a single one of them is dependent and the rest independent. Let us first consider the case of two variables x, y, supposed connected by one relation of quite arbitrary form and not necessarily known, and consider them, and the successive derivatives of the latter with regard to the former, to be expressed in terms of x', y', and the successive derivatives of y' with regard to x' by the transformations of the general linear group

$$\left.\begin{aligned} x &= a_1x'+b_1y'+c_1 \\ y &= a_2x'+b_2y'+c_2 \end{aligned}\right\} \qquad (1)$$

and its extensions.

Let y_r and y'_r denote respectively $\frac{1}{r!}\frac{d^r y}{dx^r}$ and $\frac{1}{r!}\frac{d^r y'}{dx'^r}$ for all positive integral values of r. Corresponding finite increments ξ, η of x and y are connected by the relation

$$\eta = y_1\xi + y_2\xi^2 + y_3\xi^3 + \dots, \tag{2}$$

and corresponding increments ξ', η' of x' and y' by the relation

$$\eta' = y'_1\xi' + y'_2\xi'^2 + y'_3\xi'^3 + \dots. \tag{3}$$

Moreover, if the increments ξ', η' are those of x', y' necessitated by the increments ξ, η, of x, y,

$$\left.\begin{aligned} \xi &= a_1\xi' + b_1\eta' \\ \eta &= a_2\xi' + b_2\eta' \end{aligned}\right\}, \tag{4A}$$

which we may also write

$$\left.\begin{aligned} (a_1b_2 - a_2b_1)\,\xi' &= b_2\xi - b_1\eta \\ (a_1b_2 - a_2b_1)\,\eta' &= a_1\eta - a_2\xi \end{aligned}\right\}. \tag{4B}$$

We do not need for present purposes the expressions for either of the sets $y_1, y_2, y_3, \dots$ and $y'_1, y'_2, y'_3, \dots$ in terms of the other, but it is important to notice that the expressions in question do not involve explicitly x', y' or x, y, but that

$$y_1 = \frac{a_2 + b_2 y'_1}{a_1 + b_1 y'_1},$$

$$y_2 = \frac{(a_1b_2 - a_2b_1)\,y'_2}{(a_1 + b_1y'_1)^3},$$

and generally that we pass from y_r to y_{r+1} by a total differentiation with regard to x', *i.e.*, by operation with

$$2y'_2\frac{\partial}{\partial y'_1} + 3y'_3\frac{\partial}{\partial y'_2} + 4y'_4\frac{\partial}{\partial y'_3} + \dots,$$

and division by $(r+1)$ times $a_1 + b_1y'$. Herein lies the special virtue of an extended (*erweiterte*) *linear* group that its extensions alone, apart from the original linear equations, form a group—in the present case a group of the fourth order or four-parameter group, the parameters being a_1, b_1, a_2, b_2, while the complete extended group is of the sixth order, having the six parameters a_1, b_1, c_1, a_2, b_2, c_2.

The problem before us is the following. If

$$f(x, y, y_1, y_2, y_3, \dots) \equiv F(x', y', y'_1, y'_2, y'_3, \dots)$$

is any equivalence which holds, whatever be the relation connecting

x, y, in virtue of the transformation (1) and its extensions, it is required to express

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial y_3}, \ldots$$

as linear functions, with constant or variable coefficients, of

$$\frac{\partial F}{\partial x'}, \frac{\partial F}{\partial y'}, \frac{\partial F}{\partial y_1'}, \frac{\partial F}{\partial y_2'}, \frac{\partial F}{\partial y_3'}, \ldots.$$

The coefficients will, of course, be independent of the forms of f and F.

We have seen that x', y' only enter explicitly in F, if at all, in virtue of the explicit occurrence of x, y in f, and conversely. In fact,

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y'} &= b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} \end{aligned}\right\}, \tag{5A}$$

or, otherwise written,

$$\left.\begin{aligned} (a_1 b_2 - a_2 b_1) \frac{\partial}{\partial x} &= b_2 \frac{\partial}{\partial x'} - a_2 \frac{\partial}{\partial y'} \\ (a_1 b_2 - a_2 b_1) \frac{\partial}{\partial y} &= a_1 \frac{\partial}{\partial y'} - b_1 \frac{\partial}{\partial x'} \end{aligned}\right\}, \tag{5B}$$

where, for shortness, f is omitted as the subject on which $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ act, and the equivalent F as that on which $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}$ act.

In order to transform $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \ldots$, let us regard the relations (2), (3), (4), which connect finite increments of x, y, x', y'. Eliminating η and η', we have

$$\left.\begin{aligned} \xi &= a_1 \xi' + b_1 (y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \ldots) \\ y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots &= a_2 \xi' + b_2 (y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \ldots) \end{aligned}\right\}, \tag{6A}$$

which do not involve x, y, x', y' explicitly. These two equalities cannot be independent, but are identical in meaning. For substitution from (4B) for ξ', η' in (3) gives a relation in ξ, η, and y_1', y_2', y_3', ..., which, when these last are replaced by their expressions in terms of y_1, y_2, y_3, ..., must be identical with (2), as otherwise corresponding increments ξ, η of two variables connected by a single relation would be connected by two relations, as is not the case. Let us consider the equivalent relations (6A) in the slightly more convenient forms

$$\left.\begin{aligned} \xi &= a_1\xi' + b_1(y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \dots) \\ b_2\xi - b_1(y_1\xi + y_2\xi^2 + y_3\xi^3 + \dots) &= (a_1b_2 - a_2b_1)\,\xi' \end{aligned}\right\}. \qquad (6\text{B})$$

Here, since y is a perfectly arbitrary function of x, and ξ is also perfectly arbitrary, we may, for a given x, regard $y_1, y_2, y_3, \dots$ and ξ as quite independent variables. We must then consider $y_1', y_2', y_3', \dots$ and ξ' as dependent variables, all but the last of them as dependent on $y_1, y_2, y_3, \dots$ in virtue of the extensions of (1), and ξ' as dependent on these and also on ξ in virtue of either of the equivalent relations (6B) at present before us. Now, of the independent variables $y_1, y_2, y_3, \dots, \xi$, let y_r alone receive an infinitesimal increment δy_r, and let $\delta\xi'$ be the consequent increment of ξ'. The equalities (6B) yield us the two equivalent results

$$0 = \{a_1 + b_1(y_1' + 2y_2'\xi' + 3y_3'\xi'^2 + \dots)\}\,\delta\xi' + b_1\left\{\xi'\frac{\partial y_1'}{\partial y_r} + \xi'^2\frac{\partial y_2'}{\partial y_r} + \xi'^3\frac{\partial y_3'}{\partial y_r} + \dots\right\}\delta y_r,$$

$$-b_1\xi^r\delta y_r = (a_1b_2 - a_2b_1)\,\delta\xi'.$$

Accordingly,

$$\xi'\frac{\partial y_1'}{\partial y_r} + \xi'^2\frac{\partial y_2'}{\partial y_r} + \xi'^3\frac{\partial y_3'}{\partial y_r} + \dots$$

$$\equiv (a_1b_2 - a_2b_1)^{-1}\xi^r\{a_1 + b_1(y_1' + 2y_2'\xi' + 3y_3'\xi'^2 + \dots)\}. \quad (7)$$

In this, on the right, substitute for ξ its equivalent

$$\xi = a_1\xi' + b_1(y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \dots) \quad (8)$$

from (4A). We then have on the right a rational integral expression in ξ', which we can expand and arrange by powers of ξ'. We are thus given in (7) the identical equality of two expansions in powers of ξ', the coefficients on the two sides being all in no way dependent on ξ'. Corresponding coefficients on left and right must then be equal. In other words, the coefficients of successive powers of ξ' on the right are, in succession, the expressions for

$$\frac{\partial y_1'}{\partial y_r}, \ \frac{\partial y_2'}{\partial y_r}, \ \frac{\partial y_3'}{\partial y_r}, \ \dots$$

in terms of $y_1', y_2', y_3', \dots$. It appears at once that the lowest power of ξ' which occurs on the right is ξ'^r, so that $\dfrac{\partial y_r'}{\partial y_r}$ is the first of these derivatives which does not vanish. In other words, as is otherwise obvious, y_r' is the lowest of the derivatives of y' with regard to x' whose expression in terms of $y_1, y_2, y_3, \dots$ involves y_r.

3. But in the identity (7), when the right-hand member is, as above described, expressed in terms of ξ' and expanded, we may put for ξ'^s on the two sides any quantity or operator we please, and so for every other power of ξ'. Let us then put the corresponding $\frac{\partial}{\partial y'_s}$ for each ξ'^s ($s = 1, 2, 3, \ldots, \infty$), the subject of operation being any function $F(x', y', y'_1, y'_2, y'_3, \ldots)$. We thus get on the left

$$\frac{\partial y'_1}{\partial y_r}\frac{\partial}{\partial y'_1}+\frac{\partial y'_2}{\partial y_r}\frac{\partial}{\partial y'_2}+\frac{\partial y'_3}{\partial y_r}\frac{\partial}{\partial y'_3}+\ldots,$$

i.e.,

$$\frac{\partial}{\partial y_r},$$

the subject of operation being the function $f(x, y, y_1, y_2, y_3, \ldots)$ of unaccented letters which is equivalent to F. Consequently, to obtain the equivalent of $\frac{\partial}{\partial y_r}$ in the form

$$A'_1\frac{\partial}{\partial y'_1}+A'_2\frac{\partial}{\partial y'_2}+A'_3\frac{\partial}{\partial y'_3}+\ldots,$$

which is its proper one when the subject of operation is expressed in terms of $x', y', y'_1, y'_2, y'_3, \ldots$ instead of in terms of $x, y, y_1, y_2, y_3, \ldots$, we have the following rule :—Expand

$$(a_1b_2-a_2b_1)^{-1}\{a_1\xi'+b_1(y'_1\xi'+y'_2\xi'^2+y'_3\xi'^3+\ldots)\}^r$$
$$\times\{a_1+b_1(y'_1+2y'_2\xi'+3y'_3\xi'^2+\ldots)\}$$

in ascending powers of ξ', writing the power of ξ' last in each term, and then put in the expanded result, for every power ξ'^s which occurs, the corresponding $\frac{\partial}{\partial y'_s}$.

Referring to (6A), this product to be expanded is seen to be briefly expressible as

$$(a_1b_2-a_2b_1)^{-1}\xi^r\left\{\frac{\partial\xi}{\partial\xi'}+\frac{\partial\xi}{\partial\eta'}\frac{\partial\eta'}{\partial\xi'}\right\},$$

or, more shortly still, as

$$(a_1b_2-a_2b_1)\,\xi^{-1}\frac{d\xi}{d\xi'},$$

where ξ is supposed to be expressed, as if an actual finite increment of x, in terms of ξ' and the derivatives y'_s, which latter are not functions of ξ'.

Now, just as we have found it convenient to take ξ'^s as a symbolic representative of $\frac{\partial}{\partial y'_s}$ for every positive integral s, so it occurs to take ξ^r as a symbolic representative of $\frac{\partial}{\partial y_r}$. Our result may then be stated as follows:—If

$$f(x, y, y_1, y_2, y_3, \ldots) \equiv F(x', y', y'_1, y'_2, y'_3, \ldots)$$

be any identity consequent on the general linear transformation (1) and its extensions, then

$$\frac{\partial}{\partial y_r} f(x, y, y_1, y_2, y_3, \ldots) \quad (r = 1, 2, 3, \ldots, \infty),$$

whose symbolical form is $\xi^r f$,

has for its equivalent

$$(a_1 b_2 - a_2 b_1)^{-1} \xi^r \frac{d\xi}{d\xi'} F(x', y', y'_1, y'_2, y'_3, \ldots),$$

where
$$a_1 \xi' + b_1 (y'_1 \xi' + y'_2 \xi'^2 + y'_3 \xi'^3 + \ldots)$$

has to be substituted for ξ, as it would have were ξ, ξ' corresponding increments of x, x' instead of mere symbols, after which the result obtained has to be expanded in powers of ξ', and then in the expansion every power ξ'^s to be replaced by the corresponding differential operator $\frac{\partial}{\partial y'_s}$.

The rule for transforming any linear operator

$$P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + A_1\frac{\partial}{\partial y_1} + A_2\frac{\partial}{\partial y_2} + A_3\frac{\partial}{\partial y_3} + \ldots$$

is immediately deduced. As to the first two terms they become at once

$$(a_1 b_2 - a_2 b_1)^{-1} \left\{ (Pb_2 - Qb_1)\frac{\partial}{\partial x'} + (Qa_1 - Pa_2)\frac{\partial}{\partial y'} \right\},$$

by (5B). As to the rest of the operator, write it symbolically

$$A_1 \xi + A_2 \xi^2 + A_3 \xi^3 + \ldots,$$

multiply it by
$$(a_1 b_2 - a_2 b_1)^{-1} \frac{d\xi}{d\xi'},$$

then substitute for ξ its expansion in terms of ξ' as above, expand the result and arrange it as a series in powers of ξ', writing each such power of ξ' as the last factor in its term, and, finally, for every power ξ'^{s} in the expansion, write the corresponding $\dfrac{\partial}{\partial y'_s}$.

If P, Q, A_1, A_2, A_3, ... are constants, the transformation is thus completed. If they are functions of x, y, y_1, y_2, y_3, ..., they need for useful purposes to be expressed in terms of x', y', y'_1, y'_2, y'_3, ... by means of (1) and its extensions. But, as we shall see, this expression is automatically effected in a wide and important class of cases.

Since $$\frac{d\xi}{d\xi'}\,\frac{d\xi'}{d\xi}=1,$$

and since the determinant of the coefficients of ξ, η in the expressions for ξ', η' is the reciprocal of the determinant of the coefficients of ξ', η' in the expressions for ξ, η, we at once see that, as should be the case, the rule obtained for the expression of an unaccented operator as an accented one affords the exactly corresponding rule for the expression of an accented operator as unaccented.

4. Let us apply these conclusions to the transformation of what I have called in the *Phil. Trans.* (*loc. cit.* in § 1) MacMahon operators. Such operators (of four elements) are those included in the definition

$$\{\mu,\nu\,;\,m,n\}_\nu=\frac{1}{m}\,\Sigma\left\{(\mu+\nu s)\,Y_s^{(m)}\frac{\partial}{\partial y_{n+s}}\right\},$$

the summation being with regard to s, which assumes in turn all positive integral values not less than the greater of m and $-n+1$, m and n being integral or zero, and $Y_s^{(m)}$ denoting the coefficient of ξ^s in the expansion of

$$(y_1\xi+y_2\xi^2+y_3\xi^3+\dots)^m.$$

This operator is the result of replacing in the $(\mu,\nu\,;\,m,n)$ of Major MacMahon's remarkable first paper on multilinear operators*

$$a,\ b,\ c,\ d,\ \dots\quad\text{by}\quad 0,\ y_1,\ y_2,\ y_3,\ \dots,$$

* "The Theory of a Multilinear Partial Differential Operator, with Applications to the Theories of Invariants and Reciprocants" (*Proc. Lond. Math. Soc.*, Vol. XVIII., pp. 61–88).

or, again, that of replacing in $(\mu+\nu m, \nu;\ m, n+m-1)$,

$$a, b, c, d, \dots \text{ by } y_1, y_2, y_3, y_4, \dots .$$

We shall confine attention to cases in which m is not negative, and $m+n$ not less than unity. The summation for s in $\{\mu, \nu;\ m, n\}_y$ is then $\sum_{s=m}^{s=\infty}$, *i.e.*, no coefficient $Y_s^{(m)}$ which actually occurs in the expansion of the multinomial is absent from the summation. The elaboration of results for the excepted cases of $m+n<1$ could be added, much as in my paper to which reference has been made, but would unduly lengthen the present communication. The case $m=0$ has some speciality, and will be only partially included.

5. Symbolically expressed, as in § 3, it is clear that

$$\{\mu, \nu;\ m, n\}_y = \frac{\mu}{m}\xi^n\eta^m + \nu\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi},$$

where η stands for $y_1\xi+y_2\xi^2+y_3\xi^3+\dots$; and that in particular

$$\{1, 0;\ m, n\}_y = \frac{1}{m}\xi^n\eta^m,$$

and $$\{0, 1;\ m, n\}_y = \xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}.$$

We can at once then apply the rule arrived at in § 3; and obtain that the results of transforming these two last operators by the extended linear scheme (1) are respectively in symbolic form

$$\frac{1}{m}(a_1b_2-a_2b_1)^{-1}\xi^n\eta^m\frac{d\xi}{d\xi'},$$

i.e., $$\frac{1}{m}(a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m\left(a_1+b_1\frac{d\eta'}{d\xi'}\right),$$

and $$(a_1b_2-a_2b_1)^{-1}\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}\frac{d\xi}{d\xi'},$$

i.e., $$(a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^{n+1}(a_2\xi'+b_2\eta')^{m-1}\left(a_2+b_2\frac{d\eta'}{d\xi'}\right).$$

Each of these expressions expanded is a sum of multiples of terms of the two types $\xi'^p\eta'^{m+n-p}$, $\xi'^p\eta'^{m+n-p}\frac{d\eta'}{d\xi'}$, *i.e.*, a sum of multiples of MacMahon y'-operators with $m'+n'=m+n$. In fact, if, as usual, $\binom{n}{r}$ denote the number of combinations of n things r together, the results of the transformation are, when n as well as m is not negative,

$$m\,(a_1 b_2 - a_2 b_1)\,\{1, 0;\ m, n\}_y$$

$$\equiv \sum_{r=0,\,s=0}^{r=n,\,s=m} \binom{n}{r}\binom{m}{s} a_1^{r+1} b_1^{n-r} a_2^s b_2^{m-s} \xi'^{r+s} \eta'^{m+n-r-s}$$

$$+ \sum_{r=0,\,s=0}^{r=n,\,s=m} \binom{n}{r}\binom{m}{s} a^r b^{n-r+1} a_2^s b_2^{m-s} \xi'^{r+s} \eta'^{m+n-r-s} \frac{d\eta'}{d\xi'}$$

$$\equiv \sum_{r=0,\,s=0}^{r=n,\,s=m} \binom{n}{r}\binom{m}{s} a_1^{r+1} b_1^{n-r} a_2^s b_2^{m-s} (m+n-r-s) \times \{1, 0;\ m+n-r-s, r+s\}_{y'}$$

$$+ \sum_{r=0,\,s=0}^{r=n,\,s=m} \binom{n}{r}\binom{m}{s} a_1^r b_1^{n-r+1} a_2^s b_2^{m-s} \{0, 1;\ m+n-r-s+1, r+s-1\}_{y'},$$

and

$$(a_1 b_2 - a_2 b_1)\,\{0, 1;\ m, n\}_y$$

$$\equiv \sum_{r=0,\,s=0}^{r=n+1,\,s=m-1} \binom{n+1}{r}\binom{m-1}{s} a_1^r b_1^{n+1-r} a_2^{s+1} b_2^{m-1-s} (m+n-r-s) \times \{1, 0;\ m+n-r-s, r+s\}_{y'}$$

$$+ \sum_{r=0,\,s=0}^{r=n+1,\,s=m-1} \binom{n+1}{r}\binom{m-1}{s} a_1^r b_1^{n+1-r} a_2^s b_2^{m-s} \times \{0, 1;\ m+n-r-s+1, r+s-1\}_{y'}.$$

The transformation of $\{\mu, \nu;\ m, n\}_y$ is immediately deduced by adding $\frac{\mu}{m}$ times the former of these results to ν times the latter.

6. Such formulæ have necessarily some cumbrousness of expression; and clearness of realization is, I am sure, gained by thinking of them as before us in their symbolical forms, namely,

$$\xi^n \eta^m \equiv (a_1 b_2 - a_2 b_1)^{-1} \xi^n \eta^m \frac{d\xi}{d\xi'}$$

$$= (a_1 b_2 - a_2 b_1)^{-1} (a_1 \xi' + b_1 \eta')^n (a_2 \xi' + b_2 \eta')^m \left(a_1 + b_1 \frac{d\eta'}{d\xi'}\right)$$

and

$$\xi^{n+1} \eta^{m-1} \frac{d\eta}{d\xi} \equiv (a_1 b_2 - a_2 b_1)^{-1} \xi^{n+1} \eta^{m-1} \frac{d\eta}{d\xi'}$$

$$= (a_1 b_2 - a_2 b_1)^{-1} (a_1 \xi' + b_1 \eta')^{n+1} (a_2 \xi' + b_2 \eta')^{m-1} \left(a_2 + b_2 \frac{d\eta'}{d\xi'}\right),$$

where before interpretation expansion on the left has to be in powers of ξ, and on the right in powers of ξ'.

The interpretation presents no difficulty when m as well as $m+n$

is a positive integer. Moreover, the value $m = 0$ has no speciality as far as the transformation of $\xi^n\eta^m$ is concerned; but this value has in connexion with the transformation of $\xi^{n+1}\eta^{m-1}\dfrac{d\eta}{d\xi}$. As it is not proposed to deal with this, the second of the two above transformations is better written

$$\xi^n\eta^m\frac{d\eta}{d\xi} \equiv (a_1b_2-a_2b_1)^{-1}\xi^n\eta^m\frac{d\eta}{d\xi}$$
$$= (a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m\left(a_2+b_2\frac{d\eta'}{d\xi'}\right),$$

in which, as in the first, m is a positive integer or zero, and $m+n$ a positive integer. We will, in fact, only attend to cases in which neither m nor n is negative in the last article, the cases to which the forms of equalities written at the end of that article apply.

7. Let us speak of an operator which is a linear function with constant coefficients of operators of the two types $\xi^n\eta^m$, $\xi^n\eta^m\dfrac{d\eta}{d\xi}$ with the same value of $m+n$ as being of the $(m+n)^{\text{th}}$ order. The results before us involve the fact that it transforms into another operator of the same order $m+n$.

Some facts with regard to the linearly independent operators

$$\xi,\ \eta,\ \xi\frac{d\eta}{d\xi},\ \eta\frac{d\eta}{d\xi},\quad i.e.,\ \tfrac{1}{2}\frac{d}{d\xi}(\eta^2),$$

of the first order will now be adduced. Written at length they are respectively

$$\xi = \frac{\partial}{\partial y_1},$$
$$\eta = y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}+y_3\frac{\partial}{\partial y_3}+\dots,$$
$$\xi\frac{d\eta}{d\xi} = y_1\frac{\partial}{\partial y_1}+2y_2\frac{\partial}{\partial y_2}+3y_3\frac{\partial}{\partial y_3}+\dots,$$
$$\eta\frac{d\eta}{d\xi} = 2.\tfrac{1}{2}y_1^2\frac{\partial}{\partial y_1}+3y_1y_2\frac{\partial}{\partial y_2}+4(y_1y_3+\tfrac{1}{2}y_2^2)\frac{\partial}{\partial y_3}+5(y_1y_4+y_2y_3)\frac{\partial}{\partial y_4}$$
$$+6(y_1y_5+y_2y_4+\tfrac{1}{2}y_3^2)\frac{\partial}{\partial y_5}+\dots$$
$$= -y_1^2\xi+y_1\left(\eta+\xi\frac{d\eta}{d\xi}\right)+V,$$

where V is Sylvester's operator

$$4 \cdot \tfrac{1}{2} y_2^2 \frac{\partial}{\partial y_3} + 5 y_2 y_3 \frac{\partial}{\partial y_4} + 6 \left(y_2 y_4 + \tfrac{1}{2} y_3^2\right) \frac{\partial}{\partial y_5} + 7 \left(y_2 y_5 + y_3 y_4\right) \frac{\partial}{\partial y_6} + \dots .$$

They are recognized as being the known operators which determine the four independent infinitesimal transformations of the group which consists of the extensions of the general linear group (1). (A more general fact including this will be proved in § 16 below.) The general infinitesimal transformation of this group of extensions is determined by

$$\lambda \xi + \mu \eta + \nu \xi \frac{d\eta}{d\xi} + \varpi \eta \frac{d\eta}{d\xi} .$$

i.e., by the general operator of the first order. The functions of the derivatives $y_1, y_2, y_3, \dots$ which are absolute differential invariants of the group of extensions are exactly those functions which have the four operators for annihilators. They are also the absolute differential invariants of the extended linear group (1) itself, as the two annihilators of such invariants given by infinitesimal variation of c_1 and c_2 in the first place are $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, which show that there is no differential invariant of the group which involves x or y.

Now, a function $f(y_1, y_2, y_3, \dots)$ annihilated by ξ or $\frac{\partial}{\partial y_1}$ is free from y_1; one annihilated by η is homogeneous of degree zero; one annihilated by $\xi \frac{d\eta}{d\xi}$ is isobaric of weight zero, weight being measured by sum of suffixes; and one annihilated by $\eta \frac{d\eta}{d\xi}$ in addition to the above, or, less exactingly, in addition to ξ and $\eta + \xi \frac{d\eta}{d\xi}$, is annihilated by V.

Consider now the transformations of these operators of the first order, as given by § 6. They are

$$\xi \equiv (a_1 b_2 - a_2 b_1)^{-1} \left\{ a_1^2 \xi' + a_1 b_1 \left(\eta' + \xi' \frac{d\eta'}{d\xi'} \right) + b_1^2 \eta' \frac{d\eta'}{d\xi'} \right\},$$

$$\eta \equiv (a_1 b_2 - a_2 b_1)^{-1} \left\{ a_1 a_2 \xi' + a_1 b_2 \eta' + a_2 b_1 \xi' \frac{d\eta'}{d\xi'} + b_1 b_2 \eta' \frac{d\eta'}{d\xi'} \right\},$$

$$\xi \frac{d\eta}{d\xi} \equiv (a_1 b_2 - a_2 b_1)^{-1} \left\{ a_1 a_2 \xi' + a_2 b_1 \eta' + a_1 b_2 \xi' \frac{d\eta'}{d\xi'} + b_1 b_2 \eta' \frac{d\eta'}{d\xi'} \right\},$$

$$\eta \frac{d\eta}{d\xi} \equiv (a_1 b_2 - a_2 b_1)^{-1} \left\{ a_2^2 \xi' + a_2 b_2 \left(\eta' + \xi' \frac{d\eta'}{d\xi'} \right) + b_2^2 \eta' \frac{d\eta'}{d\xi'} \right\}.$$

We notice here a difference of character between the middle pair and the extreme pair—the coefficients of ξ', η', $\xi'\dfrac{d\eta'}{d\xi'}$, $\eta'\dfrac{d\eta'}{d\xi'}$ are linearly independent in the one pair of cases and not so in the other pair. Look, as we of course may do, upon the equivalences as results of applying a general linear transformation to the accented variables instead of to the unaccented, *i.e.*, look upon $f(y_1, y_2, y_3, \ldots)$ as having been found as the equivalent of a function $F(y_1', y_2', y_3', \ldots)$ and not *vice versa*. We see that for η to annihilate f, whatever the linear transformation may have been, or for $\xi\dfrac{d\eta}{d\xi}$ to annihilate it in like case, it is necessary for all four of ξ', η', $\xi'\dfrac{d\eta'}{d\xi'}$, $\eta'\dfrac{d\eta'}{d\xi'}$ to annihilate F. On the other hand, for ξ or for $\eta\dfrac{d\eta}{d\xi}$ to annihilate f, whatever the linear transformation may have been, it is only necessary for ξ', $\eta'+\xi'\dfrac{d\eta'}{d\xi'}$, $\eta'\dfrac{d\eta'}{d\xi'}$ to annihilate F.

We may therefore state as follows. In order that a function of the derivatives may become a homogeneous function of degree zero of the new derivatives after every linear transformation of the variables, or in order that it may become an isobaric function of weight zero, it is necessary that it have the four properties of being free from the first derivative y_1, homogeneous of degree zero, isobaric of weight zero, and annihilated by $\eta\dfrac{d\eta}{d\xi}$ or V; in other words, it must be an absolute differential invariant of the general linear group. But, in order that it may become, after every linear transformation of the variables, a function free from the first derivative, or a function annihilated by V, it is only necessary for it to have the three properties of being free from y_1, being of zero sum of degree and weight throughout, and being annihilated by V; in other words, it need be only an absolute pure reciprocant (or linear function of such), *i.e.*, an absolute differential invariant (or linear function of such) of a linear group whose generality is limited by the one relation $a_1b_2-a_2b_1$ among the coefficients—a sub-group of the general linear group.*

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* By "absolute differential invariant" of a continuous group, I, in this paper, mean, with Lie, a function of derivatives, and it might be also of the variables—though these last do not occur in the cases of groups here considered—which is absolutely unaltered in form by the substitutions of the group and its extensions. A function which persists in form but for a factor is called a "relative or non-absolute differential invariant." By "absolute pure reciprocant," I mean, with

8. It is an interesting subject of investigation whether there are MacMahon operators which transform, by the general linear scheme, into themselves, as there are for mere interchange of the variables and some other very special schemes. Were there such, and were the coefficients in them free from the constants in the scheme, then any such operator would generate absolute differential invariants from absolute differential invariants. But the answer appears, for

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Sylvester, a function of second and higher derivatives which persists in form but for the assumption of a factor which is a power of -1 when dependent and independent variables are interchanged. A non-absolute pure reciprocant persists in form after such interchange except for a factor which is not a mere power of -1.

It is a prevalent, but mistaken, impression that absolute pure reciprocants are identical with absolute differential invariants of the general linear group. What Sylvester proved (*Amer. Jour.*, Vol. VIII., pp. 248, &c.) with regard to the differential invariancy of a pure reciprocant R for the general linear substitution is that, if homogeneous (of degree i), and consequently isobaric (of weight w), it becomes $(a_1 b_2 - a_2 b_1)^i (a_1 + b_1 y_1)^{-w-i} R$, with my notation as above: his w is different. If it be an *absolute* reciprocant, what he has proved at an earlier stage is that $w+i=0$, but not that $i=0$, $w=0$ separately. There is absolute invariancy for the substitution only if $i=0$ as well as $w+i=0$. There are, however, absolute pure reciprocants of all degrees i, and even non-homogeneous ones. Sylvester calls those of degree zero *plenarily absolute* to indicate that they are absolutely invariant for any linear group. Absolute pure reciprocants which are not of degree zero are, if homogeneous, differential invariants of any linear group, but are absolute differential invariants only for linear groups in which generality of constants is limited by the one relation $a_1 b_2 - a_2 b_1 = 1$.

Again, what I myself proved (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 388) with regard to the differential invariancy of pure cyclicants or ternary reciprocants for the general linear substitution of $lx+my+nz+p$, &c., for x, &c., is that there is persistence in the case of a homogeneous one but for a factor of the form

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}^i \left\{ lm' - l'm - (mn' - m'n)\frac{\partial z}{\partial x} - (nl' - n'l)\frac{\partial z}{\partial y} \right\}^{-i-\frac{1}{2}w}.$$

The second index here vanishes when the pure cyclicant is absolute; but the first i, as a rule, does not. Thus absolute pure cyclicants include, but are not only coextensive with, all absolute differential invariants of the general linear group in three variables. Linear functions of them are identical with absolute differential invariants of a sub-group of the general linear group, namely, of the *special* subgroup in which the generality of coefficients is limited by the one relation

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = 1.$$

Those who, like myself, in 1886 and following years, followed Sylvester in researches on reciprocants and allied classes of differential invariants, and showed, it must be confessed, imperfect acquaintance with Sophus Lie's grand work on Continuous Groups and Differential Invariants in general, have reason to be grateful to Herr O. A. Stöckert for a recent memoir, *Ueber die Beziehungen der Reciprokantentheorie zur allgemeinen Theorie der Differentialinvarianten* (Chemnitz, 1895), in which he elucidates very instructively Lie's theory in its bearing on Sylvester's and our own. In that memoir the error occurs (pp. 30, 32) of attributing to Sylvester the opinion that an absolute pure reciprocant is necessarily of zero degree and weight. The proof given fails on p. 32 at line 11.

practical purposes, negative. It is true that inspection of the symbolic equivalences of the last article gives us one such persistent operator of the first order

$$\xi\frac{d\eta}{d\xi}-\eta\equiv\xi'\frac{d\eta'}{d\xi'}-\eta',$$

i.e., $$y_2\frac{\partial}{\partial y_2}+2y_3\frac{\partial}{\partial y_3}+3y_4\frac{\partial}{\partial y_4}+\ldots\equiv y_2'\frac{\partial}{\partial y_2'}+2y_3'\frac{\partial}{\partial y_3'}+3y_4'\frac{\partial}{\partial y_4'}+\ldots,$$

so that the persistence (19) of my memoir in the *Phil. Trans.* (*loc. cit.* p. 25) is one which holds in general; but this has merely the effect of multiplying a homogeneous isobaric function by the excess of its weight over its degree. It expresses the persistence in value of the characteristic $w-i$, but gives, in cases of interest, nothing new by its operation (a conclusion drawn by means of it which, though of interest, is not new, is the case for $q=2$ of one in § 17 below). The more general operator $\xi^{m-1}\eta^{m-1}\left\{\xi\frac{d\eta}{d\xi}-\eta\right\}$ [*loc. cit.*, (17)], which is persistent in the special theory, is not so in the present general one except for $m=n=1$.

Nor do the operators $\xi\frac{d\eta}{d\xi}+\eta$, $\xi^{m-1}\eta^{m-1}\left\{\xi\frac{d\eta}{d\xi}+\eta\right\}$, which are negatively persistent in the special theory [*loc. cit.*, (20), (18)], yield directly anything of much interest. in general. If we take the general operator of the first order,

$$\lambda\xi+\mu\eta+\nu\xi\frac{d\eta}{d\xi}+\varpi\eta\frac{d\eta}{d\xi},$$

and try to find all values of λ, μ, ν, ϖ for which the form persists but for a factor, upon substitution from the last article, we obtain, besides the persistent $\xi\frac{d\eta}{d\xi}-\eta$ as above, only the equivalences

$$-2a_2\xi+(a_1-b_2)\left(\eta+\xi\frac{d\eta}{d\xi}\right)+2b_1\eta\frac{d\eta}{d\xi}$$
$$\equiv -2a_2\xi'+(a_1-b_2)\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)+2b_1\eta'\frac{d\eta'}{d\xi'}$$

and $$(a_1-b_2)\left(a_2\xi-b_1\eta\frac{d\eta}{d\xi}\right)+2a_2b_1\left(\eta+\xi\frac{d\eta}{d\xi}\right)$$
$$\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\left(a_2\xi+b_1\eta\frac{d\eta}{d\xi}\right)$$
$$\equiv\frac{\{a_1+b_2\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\}^2}{4(a_1b_2-a_2b_1)}\left\{(a_1-b_2)\left(a_2\xi'-b_1\eta'\frac{d\eta'}{d\xi'}\right)\right.$$
$$\left.+2a_2b_1\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\left(a_2\xi'+b_1\eta'\frac{d\eta'}{d\xi'}\right)\right\}.$$

which, involving as they do a_1, b_1, a_2, b_2 as coefficients, give us no information as to persistence for all linear transformations, so that they are only matters of curiosity in general. It is the latter, however, which, for the case of $a_1 = b_2 = 0$, $a_2 = b_1 = 1$, *i.e.*, for the case of mere interchange of variables x, y, gives the negative persistence of $\eta + \xi \dfrac{d\eta}{d\xi}$, *i.e.*,

$$2y \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots$$

[*loc. cit.*, (20)]. It also gives for that case the negative persistence of $\xi + \eta \dfrac{d\eta}{d\xi}$, *i.e.*, of

$$(1-y_1^2)\frac{\partial}{\partial y_1} + y_1\left(2y_1 \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots\right) + V,$$

and the former gives the positive persistence of $\eta \dfrac{d\eta}{d\xi} - \xi$, *i.e.*, of

$$-(1+y_1^2)\frac{\partial}{\partial y_1} + y_1\left(2y_1 \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots\right) + V,$$

two facts not expressly introduced in the memoir to which reference is made.

9. The digression may be pardonable if I consider for a moment some properties of operators of the types $\xi^n\eta^m$, $\xi^n\eta^m \dfrac{d\eta}{d\xi}$ without reference to transformations. To find their *alternants* will really be to do otherwise what MacMahon has already done (*Proc. Lond. Math. Soc.*, Vol. XVIII., pp. 66–69), but the use of the present compact symbolic forms so simplifies the process, and exhibits the results in such suggestive shape, that I believe there to be justification.

Let us denote by $(A\xi + B\xi^2 + \dots)(A'\xi + B'\xi^2 + \dots)$ the operator which results from the performance of the operation represented by the first written factor on the back of that represented by the second. The part of the resultant operator which does not involve symbols of second partial differentiation will be obtained in symbolic form by making the left-hand operator act only on A', B', ..., dealing with the ξ in the right-hand operator as if it were a constant. Thus, in particular,

$$\begin{aligned}(A\xi + B\xi^2 + C\xi^3 + \dots)\,\eta &= (A\xi + B\xi^2 + C\xi^3 + \dots)(y_1\xi + y_2\xi^2 + y_3\xi^3 + \dots) \\ &= A\xi + B\xi^2 + C\xi^3 + \dots + R,\end{aligned}$$

where R involves $\frac{\partial^2}{\partial y_1^2}, \frac{\partial^2}{\partial y_1 \partial y_2}, \ldots$. So

$$(A\xi+B\xi^2+C\xi^3+\ldots)\,\xi^n\eta^m = m\xi^n\eta^{m-1}(A\xi+B\xi^2+C\xi^3+\ldots)+R';$$

and $$(A\xi+B\xi^2+C\xi^3+\ldots)\,\xi^n\eta^m\frac{d\eta}{d\xi} = m\xi^n\eta^{m-1}\frac{d\eta}{d\xi}(A\xi+B\xi^2+C\xi^3+\ldots)$$
$$+\xi^n\eta^m\frac{d}{d\xi}(A\xi+B\xi^2+C\xi^3+\ldots)+R''.$$

Now, ϑ, ϕ being any two linear operators, we call as usual $\vartheta\phi-\phi\vartheta$ the alternant of ϑ and ϕ, and write it $(\vartheta.\phi)$. The terms R in $\vartheta\phi$ and $\phi\vartheta$ are the same. Hence we at once deduce

$$\begin{aligned}(\xi^\nu\eta^\mu.\xi^n\eta^m) &= \xi^\nu\eta^\mu.\xi^n\eta^m-\xi^n\eta^m.\xi^\nu\eta^\mu\\ &= m\xi^n\eta^{m-1}\xi^\nu\eta^\mu-\mu\xi^\nu\eta^{\mu-1}\xi^n\eta^m\\ &= (m-\mu)\,\xi^{n+\nu}\eta^{m+\mu-1},\end{aligned}$$

$$\begin{aligned}\left(\xi^\nu\eta^\mu.\xi^n\eta^m\frac{d\eta}{d\xi}\right) &= m\xi^n\eta^{m-1}\frac{d\eta}{d\xi}\xi^\nu\eta^\mu+\xi^n\eta^m\frac{d}{d\xi}(\xi^\nu\eta^\mu)-\mu\xi^\nu\eta^{\mu-1}\xi^n\eta^m\frac{d\eta}{d\xi}\\ &= m\xi^{n+\nu}\eta^{m+\mu-1}\frac{d\eta}{d\xi}+\nu\xi^{n+\nu-1}\eta^{m+\mu},\end{aligned}$$

$$\begin{aligned}\left(\xi^\nu\eta^\mu\frac{d\eta}{d\xi}.\xi^n\eta^m\frac{d\eta}{d\xi}\right) &= m\xi^n\eta^{m-1}\frac{d\eta}{d\xi}\xi^\nu\eta^\mu\frac{d\eta}{d\xi}+\xi^n\eta^m\frac{d}{d\xi}\left(\xi^\nu\eta^\mu\frac{d\eta}{d\xi}\right)\\ &\quad-\mu\xi^\nu\eta^{\mu-1}\frac{d\eta}{d\xi}\xi^n\eta^m\frac{d\eta}{d\xi}-\xi^\nu\eta^\mu\frac{d}{d\xi}\left(\xi^n\eta^m\frac{d\eta}{d\xi}\right)\\ &= (\nu-n)\,\xi^{n+\nu-1}\eta^{m+\mu}\frac{d\eta}{d\xi}.\end{aligned}$$

Thus the alternant of two MacMahon operators of given orders $(m+n, \mu+\nu)$ is a MacMahon operator whose order $(m+n+\mu+\nu-1)$ is one less than the sum of the orders of the two.

Moreover, the alternant of two operators of the same type $\left(\xi^n\eta^m \text{ or } \xi^n\eta^m\frac{d\eta}{d\xi}\right)$ is an operator of the same type. The alternant of two of different types $\left(\xi^\nu\eta^\mu \text{ and } \xi^n\eta^m\frac{d\eta}{d\xi}\right)$ is as a rule the sum of two operators, one of each type.

It is easy to see, by consideration of the indices and coefficients in the three types of alternant equalities that η is the only operator of positive integral order of the type $\xi^n\eta^m$ which cannot be expressed as an alternant of operators of the same type, and that $\xi\frac{d\eta}{d\xi}$ is the only one of the type $\xi^n\eta^m\frac{d\eta}{d\xi}$ which cannot be expressed as an alternant of operators of its type. It is also easy to see that the alternant $\left(\xi.\eta\frac{d\eta}{d\xi}\right)$ is the only one of the type $\left(\xi^{\nu}\eta^{\mu}.\xi^n\eta^m\frac{d\eta}{d\xi}\right)$ which cannot be linearly expressed in terms of alternants of the two types $(\xi^{\nu}\eta^{\mu}.\xi^n\eta^m)$, $\left(\xi^{\nu}\eta^{\mu}\frac{d\eta}{d\xi}.\xi^n\eta^m\frac{d\eta}{d\xi}\right)$.

The alternants of operators of the first order are themselves of the first order, *i.e.*, are linear functions of themselves, a fact which expresses that, as mentioned earlier, they possess the group property. They are

$$(\xi.\eta) = \xi,$$

$$\left(\xi.\xi\frac{d\eta}{d\xi}\right) = \xi,$$

$$\left(\xi.\eta\frac{d\eta}{d\xi}\right) = \eta+\xi\frac{d\eta}{d\xi},$$

$$\left(\eta.\xi\frac{d\eta}{d\xi}\right) = 0,$$

$$\left(\eta.\eta\frac{d\eta}{d\xi}\right) = \eta\frac{d\eta}{d\xi},$$

$$\left(\xi\frac{d\eta}{d\xi}.\eta\frac{d\eta}{d\xi}\right) = \eta\frac{d\eta}{d\xi}.$$

It is, moreover, evident hence that, as also stated earlier, ξ, $\eta+\xi\frac{d\eta}{d\xi}$, $\eta\frac{d\eta}{d\xi}$ form a sub-group, as their alternants in pairs are linear in themselves.

What the group and sub-group are has been already indicated (§ 7).

10. To return to transformations by the extended linear scheme (1). The transformation of MacMahon operators of positive integral order is what we have so far considered in application of the method of § 3. But the method is one which, as we saw, applies to all linear operators whatever. One simple MacMahon operator of zero order may be mentioned in passing, namely,

$$\frac{d\eta}{d\xi}-y_1 \equiv 2y_2\frac{\partial}{\partial y_1}+3y_3\frac{\partial}{\partial y_2}+4y_4\frac{\partial}{\partial y_3}+\dots,$$

whose effect is that of $\frac{d}{dx}$, the symbol of total differentiation with regard to x in so far as $y_1, y_2, y_3, \ldots$ are functions of x in virtue of the relation connecting x and y. After substitution for y_1 in terms of y'_1, by § 1, its transformation becomes by the general rule

$$\frac{1}{a_1+b_1y'_1}\left\{\frac{d\eta'}{d\xi'}-y'_1\right\},$$

which accords with the facts of total differentiation.

We may transform linear operators whose symbolical expressions involve $\frac{d^2\eta}{d\xi^2}, \frac{d^3\eta}{d\xi^3}, \ldots$, in so far as we know the expressions for those derivatives in terms of $\frac{d\eta'}{d\xi'}, \frac{d^2\eta'}{d\xi'^2}, \frac{d^3\eta'}{d\xi'^3}, \ldots$, when ξ, η, ξ', η' are actual increments of x, y, x', y'. The expressions in question are of the same form as those (§ 2) for $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ in terms of $\frac{dy'}{dx'}, \frac{d^2y'}{dx'^2}, \frac{d^3y'}{dx'^3}, \ldots$. Thus, for instance, since

$$\frac{d^2\eta}{d\xi^2}\equiv(a_1b_2-a_2b_1)\frac{d^2\eta'}{d\xi'^2}\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)^{-3}\equiv(a_1b_2-a_2b_1)\frac{d^2\eta'}{d\xi'^2}\left(\frac{d\xi}{d\xi'}\right)^{-3},$$

we have, by § 3, that the transformed equivalent of

$$\xi^n\frac{d^2\eta}{d\xi^2}=1.2y_2\frac{\partial}{\partial y_n}+2.3y_3\frac{\partial}{\partial y_{n+1}}+3.4y_4\frac{\partial}{\partial y_{n+2}}+\ldots \quad (n>0)$$

is, symbolically, the expansion in powers of ξ' of

$$\xi^n\frac{d^2\eta'}{d\xi'^2}\left(\frac{d\xi}{d\xi'}\right)^{-2}=\left\{a_1\xi'+b_1(y'_1\xi'+y'_2\xi'^2+\ldots)\right\}^n$$

$$\times\left\{1.2y'_2+2.3y'_3\xi'+3.4y'_4\xi'^2+\ldots\right\}\left\{a_1+b_1(y'_1+2y'_2\xi'+3y'_3\xi'^2+\ldots)\right\}^{-2},$$

and, actually, the result of replacing, in the expansion,

$$\xi'^r \text{ by } \frac{\partial}{\partial y'_r} \quad (r=1, 2, 3, \ldots, \infty);$$

and more generally that the transformed equivalent of

$$\xi^n\eta^m\left(\frac{d\eta}{d\xi}\right)^p\left(\frac{d^2\eta}{d\xi^2}\right)^q$$

is, symbolically,

$$(a_1b_2-a_2b_1)^{q-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m$$

$$\times\left(a_2+b_2\frac{d\eta'}{d\xi'}\right)^p\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)^{1-p-3q}\left(\frac{d^2\eta'}{d\xi'^2}\right)^q.$$

An interesting class of examples of such transformations is afforded by our knowledge of pure reciprocants. If $R\left(\frac{d^2\eta}{d\xi^2}, \frac{d^3\eta}{d\xi^3}, \frac{d^4\eta}{d\xi^4}, \ldots\right)$ be a pure reciprocant in ξ, η of degree i and weight (sum of indices of differentiation in each term) w, we know (§ 7, footnote) that its expression in terms of ξ', η' (regarded as actual and not symbolic) is

$$(a_1 b_2 - a_2 b_1)^i \left(a_1 + b_1 \frac{d\eta'}{d\xi'}\right)^{-w-i} R\left(\frac{d^2\eta'}{d\xi'^2}, \frac{d^3\eta'}{d\xi'^3}, \frac{d^4\eta'}{d\xi'^4}, \ldots\right).$$

Hence the transformed equivalent of the operator whose symbolic expression is the expansion of

$$\xi^n R\left(\frac{d^2\eta}{d\xi^2}, \frac{d^3\eta}{d\xi^3}, \frac{d^4\eta}{d\xi^4}, \ldots\right) \quad (n>0),$$

where η denotes $y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots$, is, symbolically, the expansion of

$$(a_1 b_2 - a_2 b_1)^{i-1} (a_1\xi' + b_1\eta')^n \left(a_1 + b_1 \frac{d\eta'}{d\xi'}\right)^{1-w-i} R\left(\frac{d^2\eta'}{d\xi'^2}, \frac{d^3\eta'}{d\xi'^3}, \frac{d^4\eta'}{d\xi'^4}, \ldots\right),$$

where η' denotes $y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \ldots$.

II.

11. We now proceed to the more general analogous theory when there are q variables supposed connected by one relation. Let $q-1$ of them, chosen as the independent variables, be called $x_1, x_2, \ldots, x_{q-1}$, and the single dependent one y. After the preceding exposition of the first case of $q = 2$, a much slighter general presentation will suffice than would have been necessary had a different order been chosen, and the inevitable imperfections of a notation for partial differential coefficients, and the necessary avoidance of extensively writing out explicit forms of operators dealt with, will, it is hoped, cause but little obscurity.

The notation $y_{rst\ldots}$ will be used as denoting

$$\frac{1}{r!\,s!\,t!\ldots r_{q-1}!}\,\frac{\partial^{r+s+t+\ldots+r_{q-1}}}{\partial x_1^r \partial x_2^s \partial x_3^t \ldots \partial x_{q-1}^{r_{q-1}}}\,y.$$

There are supposed to be always $q-1$ suffixes, each of which may be zero or any positive integer, while their sum is 1 at least.

Let $\xi_1, \xi_2, \xi_3, \ldots, \xi_{q-1}$ be independent simultaneous finite increments of the $q-1$ independent variables, and η the corresponding increment of y. It is supposed that Taylor's theorem applies to y as a function of $x_1, x_2, \ldots, x_{q-1}$ for values considered, so that

$$\eta = \sum_{r+s+t+\ldots+r_{q-1}=1}^{r=\infty,\, s=\infty,\, t=\infty,\, \ldots,\, r_{q-1}=\infty} y_{rst\ldots}\, \xi_1^r \xi_2^s \xi_3^t \ldots \xi_{q-1}^{r_{q-1}}.$$

As well as using $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ in an actual sense, we shall also use them purely as symbols with a meaning now to be explained. The symbol η will always mean the above expansion in terms of the symbols ξ. These latter symbols will have no individual meaning except when standing alone in the first power in an ultimate expansion. A power or product of symbols ξ in an ultimate expansion will have a meaning as a whole, namely,

$$\xi_1^r \xi_2^s \xi_3^t \ldots \xi_{q-1}^{r_{q-1}} \text{ will mean } \frac{\partial}{\partial y_{rst\ldots r_{q-1}}},$$

a symbol of partial differentiation, the supposition being that it is acting on a function of the various partial derivatives of y, and, it may be, of the variables $y, x_1, x_2, \ldots, x_{q-1}$ in addition. Thus, for instance,

$$\xi_1, \xi_2, \xi_3, \xi_2^2, \xi_1^2\xi_3, \ldots$$

will mean respectively

$$\frac{\partial}{\partial y_{100\ldots}}, \frac{\partial}{\partial y_{010\ldots}}, \frac{\partial}{\partial y_{001\ldots}}, \frac{\partial}{\partial y_{020\ldots}}, \frac{\partial}{\partial y_{201\ldots}}, \ldots.$$

By an operator such as

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \frac{\partial \eta}{\partial \xi_1},$$

for instance, we shall mean the operator obtained by substituting in the product for η its expansion in terms of the ξs as above, performing the partial differentiation with regard to ξ_1, expanding the resulting product in powers and products of powers of $\xi_1, \xi_2, \ldots, \xi_{q-1}$, writing every such power or product of powers after its multiplier in the expansion, and finally substituting for each product $\xi_1^r \xi_2^s \xi_3^t \ldots$ as it occurs ultimately the corresponding symbol of partial differentiation as above.

12. We shall have to consider largely operators of the q types

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m,\ \xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \frac{\partial \eta}{\partial \xi_1},\ \ldots,\ \xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \frac{\partial \eta}{\partial \xi_{q-1}},$$

where $n_1, n_2, \ldots, n_{q-1}, m$ are positive, integral, or zero, and $\Sigma n + m \not< 1$. Such operators are for the case $q = 2$ the MacMahon operators of the preceding section, and, for the next case $q=3$ the analogous ternary operators which I have used on a previous occasion (*Phil. Trans.*, *loc. cit.*, pp. 36, &c.). We speak of an operator which is a linear function with constant coefficients of operators of these types for which $\Sigma n+m$ is constant as being of order $\Sigma n+m$.

The alternants of pairs of operators of these types are with ease written down as in § 9. There are three classes of fundamental alternant equalities: viz.,

$$(\xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}.\,\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^{m})$$
$$=(m-\mu)\,\xi_1^{n_1+\nu_1}\xi_2^{n_2+\nu_2}\ldots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu-1},$$

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}.\,\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$
$$=m\xi_1^{n_1+\nu_1}\xi_2^{n_2+\nu_2}\ldots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu-1}\frac{\partial\eta}{\partial\xi_s}$$
$$+\nu_s\xi_1^{n_1+\nu_1}\ldots\xi_s^{n_s+\nu_s-1}\ldots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu},$$

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}.\,\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$
$$=m\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^{m-1}\frac{\partial\eta}{\partial\xi_s}\xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}$$
$$+\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial}{\partial\xi_s}\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}\right)$$
$$-\mu\xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu-1}\frac{\partial\eta}{\partial\xi_r}\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}$$
$$-\xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial}{\partial\xi_r}\left(\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$
$$=\nu_s\xi_1^{n_1+\nu_1}\ldots\xi_s^{n_s+\nu_s-1}\ldots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_r}$$
$$-n_r\xi_1^{n_1+\nu_1}\ldots\xi_r^{n_r+\nu_r-1}\ldots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_s}$$

of which the last includes in particular $(r=s)$

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}.\,\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_r}\right)$$
$$=(\nu_r-n_r)\,\xi_1^{n_1+\nu_1}\ldots\xi_r^{n_r+\nu_r-1}\ldots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_r}.$$

On inspection of these identities we at once see that the alternant of two operators of given orders $\Sigma\nu+\mu$, $\Sigma n+m$ is an operator of the same kind whose order $\Sigma\nu+\Sigma n+\mu+m-1$ is one less than the sum of those given orders.

In particular, the alternant of two operators of the first order is itself an operator of the first order. The linearly independent operators of the first order are q^2 in number: viz., $q-1$ operators ξ_r $(r=1, 2, \ldots, q-1)$, one operator η, $q-1$ operators of the type $\xi_1\frac{\partial\eta}{\partial\xi_1}$, $(q-1)(q-2)$ operators of the type $\xi_1\frac{\partial\eta}{\partial\xi_2}$, and $q-1$ operators $\eta\frac{\partial\eta}{\partial\xi_r}$ $(r=1, 2, \ldots, q-1)$. The alternant identities for them in pairs are of the following types, the number of each type being written after every one, an alternant $(\vartheta.\phi)$ not being reckoned as distinct from its negative $(\phi.\vartheta)$;

$$(\xi_1.\xi_2)=0,\ \tfrac{1}{2}(q-1)(q-2); \qquad (\xi_1.\eta)=\xi_1,\ q-1;$$

$$\left(\xi_1.\xi_1\frac{\partial\eta}{\partial\xi_1}\right)=\xi_1,\ q-1; \qquad \left(\xi_1.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=0, (q-1)(q-2);$$

$$\left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\xi_2,\ (q-1)(q-2); \qquad \left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3); \qquad \left(\xi_1.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\xi_1\frac{\partial\eta}{\partial\xi_1}+\eta,\ q-1;$$

$$\left(\xi_1.\eta\frac{\partial\eta}{\partial\xi_2}\right)=\xi_1\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2); \qquad \left(\eta.\xi_1\frac{\partial\eta}{\partial\xi_1}\right)=0,\ q-1;$$

$$\left(\eta.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2); \qquad \left(\eta.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_1},\ q-1;$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=-\xi_1\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\quad\xi_2\frac{\partial\eta}{\partial\xi_1},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_1},\ q-1; \qquad \left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\xi_2\frac{\partial\eta}{\partial\xi_2}-\xi_1\frac{\partial\eta}{\partial\xi_1},\ \tfrac{1}{2}(q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_1\frac{\partial\eta}{\partial\xi_3}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_1}\right)=\xi_3\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_4}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3)(q-4);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3);$$

$$\left(\eta\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2).$$

The whole number of these equalities is $\frac{1}{2}q^2(q^2-1)$.

As the alternants in pairs of the q^2 linearly independent operators of the first order are linear with constant coefficients in the operators themselves, except such of them as vanish, they are the operators appertaining to the infinitesimal transformations of a continuous substitution group of q^2 parameters. It will be seen later (§ 16) that the group in question is that which consists of the extensions of the general linear group of substitutions for the q variables $x_1, x_2, \ldots, x_{q-1}, y$.

Moreover, we notice that the $q-1$ operators ξ, the $(q-1)(q-2)$ operators $\xi_r\frac{\partial\eta}{\partial\xi_s}$ $(r\neq s)$, and the $q-1$ operators $\eta\frac{\partial\eta}{\partial\xi}$ all occur as alternants of operators; but that the operator η and the $q-1$ operators $\xi_r\frac{\partial\eta}{\partial\xi_r}$ only occur in alternants in the connexions $\eta+\xi_r\frac{\partial\eta}{\partial\xi_r}$ and

$$\xi_r\frac{\partial\eta}{\partial\xi_r}-\xi_s\frac{\partial\eta}{\partial\xi_s}=\left(\eta+\xi_r\frac{\partial\eta}{\partial\xi_r}\right)-\left(\eta+\xi_s\frac{\partial\eta}{\partial\xi_s}\right).$$

The q^2-1 operators

$$\xi,\ \xi_r\frac{\partial\eta}{\partial\xi_s}\ (r\neq s),\ \eta\frac{\partial\eta}{\partial\xi},\ \eta+\xi_r\frac{\partial\eta}{\partial\xi_r}\quad (r=1,2,3,\ldots,q-1)$$

are then seen to be a set whose alternants are linear with constant coefficients in themselves. These are consequently the operators appertaining to the infinitesimal transformations of a sub-group of the general group of extensions. What we see in § 16 will also show us that this sub-group is the group of extensions of the group of linear substitutions in which the generality of the coefficients of $x_1, x_2, \ldots, x_{q-1}, y$ is limited by the one relation that the value of their determinant is unity. It will also be proved that this system of q^2-1 operators is the system of annihilators of absolute pure q-ary reciprocants, *i.e.*, that, as in the cases of $q=2$, $q=3$, such reciprocants are, with a reservation which will appear, identical with absolute differential invariants of the linear group in which generality of coefficients is limited by the one relation just stated.

13. We proceed to consider the transformation of operators linear in symbols of partial differentiation with regard to the partial derivatives $y_{rst\ldots}$ of y with regard to $x_1, x_2, \ldots, x_{q-1}$, consequent upon the q general linear equations of transformation of the q variables

$$x_i = a_{i1}x_1'+a_{i2}x_2'+\ldots+a_{i.q-1}x_{q-1}'+a_{iq}y'+c_i\quad (i=1,2,\ldots,q-1),$$

$$y = a_{q1}x_1'+a_{q2}x_2'+\ldots+a_{q.q-1}x_{q-1}'+a_{qq}y'+c_q.$$

Let $\xi_1, \xi_2, \ldots, \xi_{q-1}$ be actual finite increments of $x_1, x_2, \ldots, x_{q-1}$, and η the actual consequent increment of y, given as an expansion in terms of the ξs by Taylor's theorem, as in § 11. Let $\xi_1', \xi_2', \ldots, \xi_{q-1}', \eta'$ be the corresponding increments of the new variables. Then

$$\xi_i = a_{i1}\xi_1'+a_{i2}\xi_2'+\ldots+a_{i.q-1}\xi_{q-1}'+a_{iq}\eta'\quad (i=1,2,\ldots,q-1),$$

$$\eta = a_{q1}\xi'+a_{q2}\xi_2'+\ldots+a_{q.q-1}\xi_{q-1}'+a_{qq}\eta';$$

which, if Δ denote the determinant of q^2 constituents $|\,a_{ij}\,|$, and if A_{mn} be its first minor corresponding to a_{mn}, may also be written

$$\Delta\xi_i' = A_{1i}\xi_1+A_{2i}\xi_2+\ldots+A_{q-1.i}\xi_{q-1}+A_{qi}\eta\quad (i=1,2,\ldots,q-1),$$

$$\Delta\eta' = A_{1q}\xi_1+A_{2q}\xi_2+\ldots+A_{q-1.q}\xi_{q-1}+A_{qq}\eta.$$

We have also $\qquad \eta' = \Sigma y'_{rst\ldots}\xi_1'^r\xi_2'^s\xi_3'^t\ldots\xi_{q-1}'^{r_{q-1}},$

a summation exactly corresponding to the already written

$$\eta = \Sigma y_{rst\ldots}\xi_1^r\xi_2^s\xi_3^t\ldots\xi_{q-1}^{r_{q-1}}.$$

The formulæ of linear transformation of the variables lead to formulæ for the expression of all derivatives $y_{rst\ldots}$ in terms of derivatives $y'_{rst\ldots}$, and conversely. These formulæ express what are called the extensions of the linear transformation. It is unnecessary to wait to prove here the well-known fact that, because the scheme of transformation of the variables is linear, its extensions do not involve the variables explicitly, but only the derivatives. Thus in the accented equivalent of $\frac{\partial}{\partial y_{rst\ldots}}$ in terms of symbols of partial differentiation with regard to accented letters there will be no occurrence of any $\frac{\partial}{\partial x'}$ or of $\frac{\partial}{\partial y'}$.

If, then, we are transforming an operator

$$\Sigma\left(P_r\frac{\partial}{\partial x_r}\right)+Q\frac{\partial}{\partial y}+\Sigma\left(A_{rst\ldots}\frac{\partial}{\partial y_{rst\ldots}}\right),$$

supposed to act on a function f of $x_1, x_2, \ldots, x_{q-1}, y$, and the derivatives $y_{rst\ldots}$, into its equivalent in form suitable for action on the equivalent function F of the accented variables and derivatives, we may substitute for the first part

$$\Sigma\left(P_r\frac{\partial}{\partial x_r}\right)+Q\frac{\partial}{\partial y}$$

from the formulæ

$$\Delta\frac{\partial}{\partial x_i}=A_{i1}\frac{\partial}{\partial x'_1}+A_{i2}\frac{\partial}{\partial x'_2}+\ldots+A_{i.q-1}\frac{\partial}{\partial x'_{q-1}}+A_{iq}\frac{\partial}{\partial y'}$$

$$(i=1, 2, \ldots, q-1),$$

$$\Delta\frac{\partial}{\partial y}=A_{q1}\frac{\partial}{\partial x'_1}+A_{q2}\frac{\partial}{\partial x'_2}+\ldots+A_{q.q-1}\frac{\partial}{\partial x'_{q-1}}+A_{qq}\frac{\partial}{\partial y'};$$

and transform the remaining part exactly as if it acted on a function of the derivatives only.

14. Now, in the q equations of transformation of the last article for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ in terms of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}, \eta'$, suppose η replaced by its expansion in terms of the ξs with coefficients like $y_{rst\ldots}$, and η' replaced by its similar expansion in terms of the ξ's. We have then q equations connecting the $q-1$ quantities ξ, the $q-1$ quantities ξ', the derivatives $y_{rst\ldots}$, and the derivatives $y'_{rst\ldots}$. As the one relation supposed to exist among y and $x_1, x_2, \ldots, x_{q-1}$ is of perfectly arbitrary

form, and may be varied as we please, we may consider the derivatives $y_{rst\ldots}$ as an infinite number of quite independent variables. Also $\xi_1, \xi_2, \ldots, \xi_{q-1}$ are quite independent of these derivatives and of one another. The other quantities involved in the equations we are considering are dependent variables; viz., $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ are functions of $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and η, which is a function of the others and the derivatives given by its expansion, and the accented derivatives $y'_{rst\ldots}$ are functions of the unaccented derivatives determined by the extensions of the linear transformation. The equations which we are regarding, with η and η' replaced in them by their expansions, are in appearance q equations for the determination of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ in terms of $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and the derivatives. But $q-1$ equations suffice for the purpose. Hence any one of the q must be a consequence of the rest, and the expressions for accented derivatives in terms of unaccented.

Let $y_{rst\ldots}$ alone among the quantities now regarded as independent variables receive an infinitesimal increment $\delta y_{rst\ldots}$, and let $\delta\xi'_1, \delta\xi'_2, \ldots, \delta\xi'_{q-1}$ be the consequent increments of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$.

It is most convenient to regard the q linear equations connecting the ξs and η with the ξ's and η' as before us in their second forms—those which express accented letters in terms of unaccented. These give us at once

$$\Delta . \delta\xi'_i = A_{qi}\xi_1^r\xi_2^s\xi_3^t \ldots \delta y_{rst\ldots} \quad (i = 1, 2, \ldots, q-1)$$

and

$$\Delta\left\{\frac{\partial\eta'}{\partial\xi'_1}\delta\xi'_1 + \frac{\partial\eta'}{\partial\xi'_2}\delta\xi'_2 + \ldots + \frac{\partial\eta'}{\partial\xi'_{q-1}}\delta\xi'_{q-1} + \sum_{\rho, \sigma, \tau, \ldots}\left[\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}\xi_1'^{\rho}\xi_2'^{\sigma}\xi_3'^{\tau}\ldots\right]\delta y_{rst\ldots}\right\}$$

$$= A_{qq}\xi_1^r\xi_2^s\xi_3^t \ldots \delta y_{rst\ldots},$$

where the summation with regard to $\rho, \sigma, \tau, \ldots$ covers all zero and positive integral values, such that $\rho+\sigma+\tau+\ldots \not< 1$. The fact that the first $q-1$ of these equations must lead to the last gives us at once the identity

$$\sum_{\rho, \sigma, \tau, \ldots}\left\{\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}\xi_1'^{\rho}\xi_2'^{\sigma}\xi_3'^{\tau}\ldots\right\}$$

$$= \Delta^{-1}\left\{A_{qq} - A_{q1}\frac{\partial\eta'}{\partial\xi'_1} - A_{q2}\frac{\partial\eta'}{\partial\xi'_2} - \ldots - A_{q.q-1}\frac{\partial\eta'}{\partial\xi'_{q-1}}\right\}\xi_1^r\xi_2^s\xi_3^t\ldots.$$

Here Δ is the determinant $|a_{ij}|$ of the q^2 coefficients in the linear expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$, and each A_{rs} is a first minor $\frac{\partial\Delta}{\partial a_{rs}}$ of that determinant.

Had we in finding this identity proceeded by consideration of the linear expressions for unaccented in terms of accented letters instead of *vice versa*, the right-hand side would in the first place have presented itself in the form

$$\Delta^{-1}\frac{d(\xi_1, \xi_2, \ldots, \xi_{q-1})}{d(\xi'_1, \xi'_2, \ldots, \xi'_{q-1})}\xi_1^r \xi_2^s \xi_3^t \ldots,$$

where, in the Jacobian functional determinant,

$$\frac{d\xi_r}{d\xi'_s} \text{ means } \frac{\partial\xi_r}{\partial\xi'_s}+\frac{\partial\xi_r}{\partial\eta'}\frac{\partial\eta'}{\partial\xi'_s}, \quad i.e., \; a_{rs}+a_{rq}\frac{\partial\eta'}{\partial\xi'_s}.$$

This latter form, whose identity with the former is easily verified, is occasionally the most convenient for the expression of results.

Now suppose the right-hand member of the identity arrived at to be expanded in terms of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ by means of the linear expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and the expansion for η'. The various coefficients in the development obtained must be severally equal to the corresponding coefficients in the left-hand expansion.

We thus have expressions in terms of accented derivatives of y' for all partial derivatives $\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}$; and these must be those which would be obtained from the extensions of the formulæ of linear transformation, were they actually exhibited.

But let us use otherwise the fact of the absolute identity of the two expansions before us. We may substitute for every product $\xi_1'^{\rho}\xi_2'^{\sigma}\xi_3'^{\tau}\ldots$ any quantity or symbol we please in both of them, and the identity will still remain one. Let us then substitute for them according to the symbolism explained in § 11; *i.e.*, for $\xi_1'^{\rho}\xi_2'^{\sigma}\xi_3'^{\tau}\ldots$ on both sides the corresponding $\frac{\partial}{\partial y'_{\rho\sigma\tau\ldots}}$, the supposition being that it is acting on a function of the derivatives $y'_{\rho\sigma\tau\ldots}$ or some of them, and it may be also of $x'_1, x'_2, \ldots, x'_{q-1}, y'$. The left-hand expansion then becomes

$$\sum_{\rho,\sigma,\tau,\ldots}\left\{\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}\frac{\partial}{\partial y'_{\rho\sigma\tau\ldots}}\right\},$$

i.e., simply

$$\frac{\partial}{\partial y_{rst\ldots}},$$

regarded as operating on the equivalent function of derivatives $y_{rst...}$, and it may be also of $x_1, x_2, \ldots, x_{q-1}, y$.

Accordingly, the transformation of operators of partial differentiation $\frac{\partial}{\partial y_{rst...}}$ is before us, and is expressed by the rule: Replace such an operator symbolically by the corresponding product $\xi_1^r \xi_2^s \xi_3^t \ldots$; multiply it by

$$\Delta^{-1}\left\{A_{qq}-A_{q1}\frac{\partial\eta'}{\partial\xi_1'}-A_{q2}\frac{\partial\eta'}{\partial\xi_2'}-\ldots-A_{q.q-1}\frac{\partial\eta'}{\partial\xi_{q-1}'}\right\};$$

express the result in terms of $\xi_1', \xi_2', \ldots, \xi_{q-1}'$ by the linear formulæ and the expansion for η'; expand in powers and products of powers of $\xi_1', \xi_2', \ldots, \xi_{q-1}'$; and, finally, for each power or product $\xi_1'^p \xi_2'^\sigma \xi_3'^\tau \ldots$ which occurs in the expansion, write the corresponding operator of partial differentiation $\frac{\partial}{\partial y'_{p\sigma\tau...}}$.

Moreover, the transformation of any linear partial differential operator (see § 13),

$$\Sigma\left(A_{rst...}\frac{\partial}{\partial y_{rst...}}\right),$$

is at once effected in like manner. Write it symbolically

$$\Sigma\left(A_{rst...}\xi_1^r \xi_2^s \xi_3^t \ldots\right);$$

multiply it throughout by the factor just written; and complete the process of expansion and interpretation of symbolic products as before.

15. We will now consider in particular the transformation of the operators of § 12. We may deal with such operators together by considering the operator

$$\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^m\left\{\lambda_q+\lambda_1\frac{\partial\eta}{\partial\xi_1}+\lambda_2\frac{\partial\eta}{\partial\xi_2}+\ldots+\lambda_{q-1}\frac{\partial\eta}{\partial\xi_{q-1}}\right\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}, \lambda_q$ are q arbitrary constants. The symbolism is that explained in § 11.

By immediate application of the rule just arrived at, the transformation of this operator is symbolically

$$\Delta^{-1}\left\{A_{qq}-A_{q1}\frac{\partial\eta'}{\partial\xi_1'}-A_{q2}\frac{\partial\eta'}{\partial\xi_2'}-\ldots-A_{q.q-1}\frac{\partial\eta'}{\partial\xi_{q-1}'}\right\}$$
$$\times\left\{\lambda_q+\lambda_1\frac{\partial\eta}{\partial\xi_1}+\lambda_2\frac{\partial\eta}{\partial\xi_2}+\ldots+\lambda_{q-1}\frac{\partial\eta}{\partial\xi_{q-1}}\right\}\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^m;$$

but this may be written in better form.

Identification of the Pfaffian equations

$$d\eta - \frac{\partial\eta}{\partial\xi_1}\,d\xi_1 - \frac{\partial\eta}{\partial\xi_2}\,d\xi_2 - \ldots - \frac{\partial\eta}{\partial\xi_{q-1}}\,d\xi_{q-1} = 0,$$

$$d\eta' - \frac{\partial\eta'}{\partial\xi_1'}\,d\xi_1' - \frac{\partial\eta'}{\partial\xi_2'}\,d\xi_2' - \ldots - \frac{\partial\eta'}{\partial\xi_{q-1}'}\,d\xi_{q-1}' = 0,$$

where the ξs and η are actual quantities linearly expressed in terms of the ξ's and η', gives

$$\frac{a_{qq} - a_{1q}\dfrac{\partial\eta}{\partial\xi_1} - a_{2q}\dfrac{\partial\eta}{\partial\xi_2} - \ldots - a_{q-1.q}\dfrac{\partial\eta}{\partial\xi_{q-1}}}{1}$$

$$= \frac{a_{qi} - a_{1i}\dfrac{\partial\eta}{\partial\xi_1} - a_{2i}\dfrac{\partial\eta}{\partial\xi_2} - \ldots - a_{q-1.i}\dfrac{\partial\eta}{\partial_{\,q-1}}}{-\dfrac{\partial\eta'}{\partial\xi_i'}} \quad (i = 1, 2, \ldots, q-1)$$

$$= \frac{-\Delta\dfrac{\partial\eta}{\partial\xi_i}}{A_{iq} - A_{i1}\dfrac{\partial\eta'}{\partial\xi_1'} - A_{i2}\dfrac{\partial\eta'}{\partial\xi_2'} - \ldots - A_{i.q-1}\dfrac{\partial\eta'}{\partial\xi_{q-1}'}} \quad (i = 1, 2, \ldots, q-1)$$

$$= \frac{\Delta}{A_{qq} - A_{q1}\dfrac{\partial\eta'}{\partial\xi_1'} - A_{q2}\dfrac{\partial\eta'}{\partial\xi_2'} - \ldots - A_{q.q-1}\dfrac{\partial\eta'}{\partial\xi_{q-1}'}}.$$

Thus

$$\left\{A_{qq} - A_{q1}\frac{\partial\eta'}{\partial\xi_1'} - A_{q2}\frac{\partial\eta'}{\partial\xi_2'} - \ldots - A_{q.q-1}\frac{\partial\eta'}{\partial\xi_{q-1}'}\right\}\frac{\partial\eta}{\partial\xi_i}$$

$$= -\left\{A_{iq} - A_{i1}\frac{\partial\eta'}{\partial\xi_1'} - A_{i2}\frac{\partial\eta'}{\partial\xi_2'} - \ldots - A_{i.q-1}\frac{\partial\eta'}{\partial\xi_{q-1}'}\right\}.$$

Applying this fact for each value of i, we see that the transformed operator above may be written

$$\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^m\left\{\lambda_q' + \lambda_1'\frac{\partial\eta'}{\partial\xi_1'} + \lambda_2'\frac{\partial\eta'}{\partial\xi_2'} + \ldots + \lambda_{q-1}'\frac{\partial\eta'}{\partial\xi_{q-1}'}\right\},$$

where

$$\Delta\lambda_i' = A_{1i}\lambda_1 + A_{2i}\lambda_2 + \ldots + A_{q-1.i}\lambda_{q-1} + A_{qi}(-\lambda_q) \;\; (i = 1, 2, \ldots, q-1),$$

$$\Delta(-\lambda_q') = A_{1q}\lambda_1 + A_{2q}\lambda_2 + \ldots + A_{q-1.q}\lambda_{q-1} + A_{qq}(-\lambda_q),$$

or, more simply,

$$\lambda_i = a_{i1}\lambda_1' + a_{i2}\lambda_2' + \dots + a_{i.q-1}\lambda_{q-1}' + a_{iq}(-\lambda_q') \quad (i=1,2,\dots,q-1),$$

$$-\lambda_q = a_{q1}\lambda_1' + a_{q2}\lambda_2' + \dots + a_{q.q-1}\lambda_{q-1}' + a_{qq}(-\lambda_q'),$$

so that $\lambda_1, \lambda_2, \dots, \lambda_{q-1}, -\lambda_q$ express in terms of $\lambda_1', \lambda_2', \dots, \lambda_{q-1}', \lambda_q'$, as do quantities cogredient with $\xi_1, \xi_2, \dots, \xi_{q-1}, \eta$. We shall exemplify the use of this fact later.

For the present our object is to draw a conclusion from the form of the transformed operator without paying special attention to this connexion between the λs and λ's. We have, in particular, that the transformations of

$$\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^m \quad \text{and} \quad \xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^m\frac{\partial\eta}{\partial\xi_r}$$

are of the above form, *i.e.*, since we can express $\xi_1, \xi_2, \dots, \xi_{q-1}, \eta$ linearly in terms of $\xi_1', \xi_2', \dots, \xi_{q-1}', \eta'$, are linear functions of operators of the same two forms as themselves with different values of $n_1, n_2, \dots, n_{q-1}, m$, and r. The sum $\Sigma n+m$ is, however, the same for each part of the transformed operators as in the operators subjected to transformation, in virtue of the linearity of the expressions for $\xi_1, \xi_2, \dots, \xi_{q-1}, \eta$. Consequently, an operator of any order $\Sigma n+m$ transforms into a linear function with constant coefficients of operators of the same order $\Sigma n+m$.

In particular, the complete system of operators of the first order (§ 12) transforms into the complete system of operators of the first order, a fact in accord with the next article.

16. I will now utilize the method of transformation before us to prove that, as stated earlier, the q^2 independent operators of the first order are the operators appertaining to the q^2 independent infinitesimal substitutions of the group with q^2 parameters, which is constituted by the extensions of the general linear group with $q(q+1)$ parameters; so that the writing down of explicit forms of these operators of infinitesimal substitution to any extent is reduced to mere multiplication of multinomials, and, indeed, to the mere squaring of one multinomial η.

Consider, first, a substitution which leaves $x_1, x_2, \dots, x_{q-1}$ unaltered, but replaces y by

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_{q-1}x_{q-1} + (1+\alpha_q)\,y + \gamma,$$

where $\alpha_1, \alpha_2, \dots, \alpha_q, \gamma$ are infinitesimal constants. Here γ does not affect the derivatives $y_{rst\ldots}$. The increment which it gives to a function f of variables and derivatives is only $\gamma \dfrac{\partial}{\partial y} f$.

Again, $a_1, a_2, \ldots, a_{q-1}$ affect each only one, a first, derivative. Thus, for instance, a_1 alters only $y_{100\ldots}$, changing it into $y_{100\ldots}+a_1$. Thus the $a_1, a_2, \ldots, a_{q-1}$ increments of f are respectively

$$a_1\left(x_1\frac{\partial}{\partial y}+\frac{\partial}{\partial y_{100\ldots}}\right)f,\quad a_2\left(x_2\frac{\partial}{\partial y}+\frac{\partial}{\partial y_{010\ldots}}\right)f,\quad a_3\left(x_3\frac{\partial}{\partial y}+\frac{\partial}{\partial y_{001\ldots}}\right),\quad \ldots,$$

i.e., $$a_1\left(x_1\frac{\partial}{\partial y}+\xi_1\right)f,\quad a_2\left(x_2\frac{\partial}{\partial y}+\xi_2\right)f,\quad \ldots,\quad a_{q-1}\left(x_{q-1}\frac{\partial}{\partial y}+\xi_{q-1}\right)f.$$

Once more, the existence of a_q changes $y_{rst\ldots}$ into $(1+a_q)\,y_{rst\ldots}$. Its effect is then to give f the increment

$$a_q\left(y\frac{\partial}{\partial y}+\Sigma y_{rst\ldots}\frac{\partial}{\partial y_{rst\ldots}}\right)f,$$

i.e., $$a_q\left(y\frac{\partial}{\partial y}+\eta\right)f.$$

Again, consider a substitution which leaves y and $x_2, x_3, \ldots, x_{q-1}$ unaltered, but replaces x_1 by

$$(1+\beta_1)\,x_1+\beta_2x_2+\beta_3x_3+\ldots+\beta_{q-1}x_{q-1}+\beta_qy+\gamma',$$

where the βs and γ' are infinitesimal constants. The consequent increment of f is in just the same way what, in a notation which regards $y, x_2, x_3, \ldots, x_{q-1}$ as independent variables and x_1 as dependent, would be written

$$\left\{\gamma'\frac{\partial}{\partial x_1}+\beta_2\left(x_2\frac{\partial}{\partial x_1}+\xi_2\right)+\beta_3\left(x_3\frac{\partial}{\partial x_1}+\xi_3\right)+\ldots\right.$$
$$\left.\ldots+\beta_{q-1}\left(x_{q-1}\frac{\partial}{\partial x_1}+\xi_{q-1}\right)+\beta_q\left(y\frac{\partial}{\partial x_1}+\eta\right)+\beta_1\left(x_1\frac{\partial}{\partial x_1}+\xi_1\right)\right\}f;$$

but we have here to express the symbolic operators by their equivalents when, as in fact, y is dependent and $x_1, x_2, \ldots, x_{q-1}$ independent. In other words, we have to transform the operator as in the last two articles, taking

$$x_2 = x_2',\quad x_3 = x_3',\quad \ldots,\quad x_{q-1} = x_{q-1}',\quad y = y',\quad x_1 = x_1',$$

where, on the left, x_1 is dependent, and, on the right, y'; and then remove accents.

Now, here,
$$\Delta = \begin{vmatrix} 1, & 0, & 0, & \dots, & 0, & 0 \\ 0, & 1, & 0, & \dots, & 0, & 0 \\ 0, & 0, & 1, & \dots, & 0, & 0 \\ \dots & \dots & \dots & & & \dots \\ 0, & 0, & 0, & \dots, & 0, & 1 \\ 0, & 0, & 0, & \dots, & 1, & 0 \end{vmatrix} = -1,$$

and the other factor which has to be introduced as well as Δ^{-1} may, as we saw in § 14, be calculated as the Jacobian of the old independent Greek letters (in this case $\xi_2, \xi_3, \dots, \xi_{q-1}, \eta$) with regard to the new independent ones (in this case $\xi_2', \xi_3', \dots, \xi_{q-1}', \xi_1'$). It is then

$$\begin{vmatrix} 1, & 0, & 0, & \dots, & 0, & 0 \\ 0, & 1, & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots, & 1, & 0 \\ \frac{\partial \eta'}{\partial \xi_2'}, & \frac{\partial \eta'}{\partial \xi_3'}, & \frac{\partial \eta'}{\partial \xi_4'}, & \dots, & \frac{\partial \eta'}{\partial \xi_{q-1}'}, & \frac{\partial \eta'}{\partial \xi_1'} \end{vmatrix} = \frac{\partial \eta'}{\partial \xi_1'}.$$

Thus we have to substitute for $\xi_2, \xi_3, \dots, \xi_{q-1}, \eta, \xi_1$, their equivalents with y dependent $-\xi_2 \frac{\partial \eta'}{\partial \xi_1'}$, $-\xi_3 \frac{\partial \eta'}{\partial \xi_1'}$, ..., $-\xi_{q-1} \frac{\partial \eta'}{\partial \xi_1'}$, $-\eta \frac{\partial \eta'}{\partial \xi_1'}$, $-\xi_1 \frac{\partial \eta'}{\partial \xi_1'}$, where $\xi_2, \xi_3, \dots, \xi_{q-1}, \eta, \xi_1$ are the same as $\xi_2', \xi_3', \dots, \xi_{q-1}', \eta', \xi_1'$. Doing so, and removing the accents which have been used only for temporary convenience, the aggregate of the second set of infinitesimal increments of f becomes

$$\Big\{ \gamma' \frac{\partial}{\partial x_1} + \beta_2 \Big(x_2 \frac{\partial}{\partial x_1} - \xi_2 \frac{\partial \eta}{\partial \xi_1} \Big) + \beta_3 \Big(x_3 \frac{\partial}{\partial x_1} - \xi_3 \frac{\partial \eta}{\partial \xi_1} \Big) + \dots$$
$$\dots + \beta_{q-1} \Big(x_{q-1} \frac{\partial}{\partial x_1} - \xi_{q-1} \frac{\partial \eta}{\partial \xi_1} \Big) + \beta_q \Big(y \frac{\partial}{\partial x_1} - \eta \frac{\partial \eta}{\partial \xi_1} \Big) + \beta_1 \Big(x_1 \frac{\partial}{\partial x_1} - \xi_1 \frac{\partial \eta}{\partial \xi_1} \Big) \Big\} f.$$

In these two partial aggregates of infinitesimal substitutions, symbolic operators of all types ξ_r, η, $\xi_r \frac{\partial \eta}{\partial \xi_r}$, $\xi_r \frac{\partial \eta}{\partial \xi_s} (r \neq s)$, $\eta \frac{\partial \eta}{\partial \xi_s}$ have been introduced, every one which has occurred, occurring once only; and no operators, having reference to derivatives $y_{rst\dots}$, of any other kind are present. It is clear that we complete the entire aggregate of infinitesimal substitutions by giving next x_2 its most general infinitesimal increment, keeping $x_1, x_3, \dots, x_{q-1}, y$ unaltered; then x_3

in like manner, and so on. No new type of operator is thus introduced; but every one of each type, as above, is just once presented. The proof for the case of x_1, which has been given, applies to all the others with alterations of suffixes.

We thus have it clearly before us that the $q(q+1)$ operators appertaining to the independent infinitesimal substitutions of the general linear group and its extensions are

(i.) one of the type $\dfrac{\partial}{\partial y}$,

(ii.) $q-1$ of the type $\dfrac{\partial}{\partial x_r}$,

(iii.) $q-1$ of the type $x_r \dfrac{\partial}{\partial y}+\xi_r$,

(iv.) one of the type $y\dfrac{\partial}{\partial y}+\eta$,

(v.) $q-1$ of the type $x_r\dfrac{\partial}{\partial x_r}-\xi_r\dfrac{\partial \eta}{\partial \xi_r}$,

(vi.) $(q-1)(q-2)$ of the type $x_r\dfrac{\partial}{\partial x_s}-\xi_r\dfrac{\partial \eta}{\partial \xi_s}$ $(r \neq s)$,

(vii.) $q-1$ of the type $y\dfrac{\partial}{\partial x_s}-\eta\dfrac{\partial \eta}{\partial \xi_s}$.

The q^2 operators appertaining to the group which consists of the extensions only are given by (iii.) to (vii.) inclusive with the $\dfrac{\partial}{\partial y}$, $\dfrac{\partial}{\partial x_r}$, $\dfrac{\partial}{\partial x_s}$ parts omitted.

As to the sub-group of the general linear group in which the generality of the constants is limited by the one condition

$$\Delta \equiv | a_{ij} | = 1,$$

the fact of its having one infinitesimal substitution fewer is expressed by the vanishing of the sum of the infinitesimal multipliers of the q operators (iv.) and (v.). The conclusions which follow have been stated in the latter part of § 12.

17. Let us now think of differential invariants, with a view to arriving at a proof that the absolute differential invariants of this

last-mentioned sub-group—an invariant sub-group (*Lie*) or self-conjugate sub-group (*Burnside*) of the general linear group—are identical with absolute pure q-ary reciprocants; or, more precisely, that absolute pure q-ary reciprocants are coextensive with these absolute differential invariants of the sub-group, which are homogeneous or entirely of odd or entirely of even degree.

Notice that it is not a necessity, but a convention, to regard the differential invariants of the sub-group, or the pure reciprocants, as homogeneous,* just as it is not a necessity, but a convention, to regard ordinary invariants,† except those of a single quantic, as homogeneous. The convention is reasonable; for the separate homogeneous parts of a non-homogeneous differential invariant or pure reciprocant are themselves differential invariants or pure reciprocants, just as in the case of ordinary invariants. It will illustrate the usefulness of our present method to prove this for absolute differential invariants of the self-conjugate sub-group.

The following is an equivalence of symbolic operators for the general linear scheme :—

$$\xi_1\frac{\partial\eta}{\partial\xi_1}+\xi_2\frac{\partial\eta}{\partial\xi_2}+\dots+\xi_{q-1}\frac{\partial\eta}{\partial\xi_{q-1}}-\eta\equiv\xi_1'\frac{\partial\eta'}{\partial\xi_1'}+\xi_2'\frac{\partial\eta'}{\partial\xi_2'}+\dots+\xi_{q-1}'\frac{\partial\eta'}{\partial\xi_{q-1}'}-\eta'.$$

To prove it we apply § 15. The operator which transforms into

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* In the second of my papers on ternary reciprocants the unjustifiable remark was made (*Proc. Lond. Math. Soc.*, Vol. xviii., p.157) that a ternary reciprocant is *necessarily* homogeneous, and was based on the unjustifiable statement that this is so for ordinary invariants of a system of binary quantics.

† An ordinary q-ary invariant or covariant (non-absolute) is an absolute invariant of a group holohedrically isomorphous with a linear homogeneous group of q variables in which the generality of coefficients is limited by the one relation $\Delta = 1$, the special sub-group of the general linear homogeneous group. For instance, when the variables in $(a_0, a_1, \dots, a_p)(x, y)^p$ are substituted for, according to this special group, the group of substitutions for the coefficients involves the same independent parameters, and the infinitesimal transformations, namely, in the one case $y\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y}$, $y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}$, and, in the other,

$$a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+\dots,\quad pa_1\frac{\partial}{\partial a_0}+(p-1)a_2\frac{\partial}{\partial a_1}+\dots,\quad pa_0\frac{\partial}{\partial a_0}+(p-2)a_1\frac{\partial}{\partial a_1}+\dots,$$

have the same composition. An absolute invariant or covariant is an absolute invariant of a group holohedrically isomorphous with the general linear homogeneous group.

$\xi_r' \dfrac{\partial \eta'}{\partial \xi_r'}$ is, by immediate application of that article,

$$\xi_r' \left\{ a_{1r} \frac{\partial \eta}{\partial \xi_1} + a_{2r} \frac{\partial \eta}{\partial \xi_2} + \dots + a_{q-1.r} \frac{\partial \eta}{\partial \xi_{q-1}} - a_{qr} \right\}$$

$$(r = 1, 2, \dots, q-1),$$

and that which transforms into $-\eta'$ is

$$\eta' \left\{ a_{1q} \frac{\partial \eta}{\partial \xi_1} + a_{2q} \frac{\partial \eta}{\partial \xi_2} + \dots + a_{q-1.q} \frac{\partial \eta}{\partial \xi_{q-1}} - a_{qq} \right\};$$

addition of which gives, by means of the linear expressions for $\xi_1, \xi_2, \dots, \xi_{q-1}, \eta$ in terms of $\xi_1', \xi_2', \dots, \xi_{q-1}', \eta'$, the identity which has been written.

From this equivalence, a persistence in form of an operator, it follows that

$$\xi_1 \frac{\partial \eta}{\partial \xi_1} + \xi_2 \frac{\partial \eta}{\partial \xi_2} + \dots + \xi_{q-1} \frac{\partial \eta}{\partial \xi_{q-1}} - \eta$$

is an operator which, acting on an absolute differential invariant of a linear group, produces another absolute differential invariant. It annihilates one of the general linear group, but not, as a rule, one of the sub-group.

Now absolute differential invariants of the sub-group we are considering have $\eta + \xi_1 \dfrac{\partial \eta}{\partial \xi_1}$, $\eta + \xi_2 \dfrac{\partial \eta}{\partial \xi_2}$, ... for annihilators (§ 12). Hence the effect on any such of $\xi_r \dfrac{\partial \eta}{\partial \xi_r}$ (for any r) is the same as that of $-\eta$. It follows that $-q\eta$, and consequently η, is an operator which generates absolute differential invariants of the sub-group from others. Now the effect of η is to multiply any homogeneous function by its degree. If, then,

$$H_i + H_j + H_k + \dots$$

is an absolute differential invariant with parts of degrees $i, j, k, \dots$, it follows in succession that

$$iH_i + jH_j + kH_k + \dots,$$

$$i^2 H_i + j^2 H_j + k^2 H_k + \dots,$$

&c., &c.,

are also absolute differential invariants; and, consequently, that H_i, H_j, H_k, ... separately are. We lose, then, no generality, as to complete systems, if we confine attention to differential invariants which are homogeneous.

Let us now consider what a homogeneous absolute differential invariant of our sub-group becomes when a linear substitution is applied for which $\Delta = -1$ instead of $+1$. Such a substitution may be effected by two stages: (1) a change of sign in y, and (2) a substitution of determinant $+1$. The first stage does not alter the differential invariant if its degree be even, but changes its sign if its degree be odd. The second stage makes no alteration in it, being merely the performance of a substitution for which it is invariant. Thus a function of derivatives which does not contain parts of both odd and even degrees, and which is an absolute differential invariant for a linear group for which $\Delta = +1$, is, at most, changed in sign by a substitution for which $\Delta = -1$.

18. In my paper "On Ternary and n-ary Reciprocants" (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 191), the only one in which cases of n (*i.e.*, q) > 3 have been introduced, I gave only a provisional definition of a q-ary reciprocant, expressing no confidence that the best form of presentation was given. It now seems better to adopt the following definition for the class of q-ary reciprocants which is of importance in the present connexion. A function of the derivatives of y with regard to $x_1, x_2, \ldots, x_r, \ldots, x_{q-1}$ is called an absolute q-ary reciprocant when it is equal, but for a constant factor, to the same function of the derivatives of x_r with regard to $x_1, x_2, \ldots, y, \ldots, x_{q-1}$, for each of the $q-1$ values of r. Here y occupies the old position of x_r in the series of independent variables. The reciprocant is called *pure* when first derivatives do not occur in it. All this is in strict accord with Sylvester's definition of ordinary reciprocants. The constant factor is either ± 1, as the conclusions which follow will establish.

We notice, first, that any absolute differential invariant, which does not contain parts of both odd and even degrees, of the linear group with generality of coefficients limited by the one relation $\Delta = 1$, is an absolute q-ary reciprocant. For the substitution which effects a change of independent variable as above is (as in § 16) one for which $\Delta = -1$, and the effect of this on the differential invariant is, at most, to change its sign, by the preceding article. One which contains parts both of odd and of even degrees is not a q-ary reciprocant, but the sum of two such of opposite characters.

We have, further, to prove conversely that any absolute pure q-ary reciprocant is an absolute differential invariant for the special linear group. We shall see this by showing that it has the full system of annihilators of absolute differential invariants of the group.

The property of absolute pure q-ary reciprocants which suffices to give the entire conclusion is that any one, and all its equivalents arising from interchanges of the dependent with one of the independent variables, are free from first derivatives.

That the reciprocant R_y itself (y dependent) is free from first derivatives is expressed by saying that it has the $q-1$ annihilators

$$\xi_1, \xi_2, \xi_3, \ldots, \xi_{q-1}.$$

That its equivalent with x_1 dependent and y occupying the place of x_1 is free from first derivatives is, in like manner, expressed by saying that it has $q-1$ annihilators which, were x_1 dependent variable, would be

$$\eta, \xi_2, \xi_3, \ldots, \xi_{q-1}.$$

Now, as in § 16, these annihilators, expressed in the notation which has meaning when y is dependent, are, respectively,

$$-\eta\frac{\partial\eta}{\partial\xi_1}, \quad -\xi_2\frac{\partial\eta}{\partial\xi_1}, \quad -\xi_3\frac{\partial\eta}{\partial\xi_1}, \quad \ldots, \quad -\xi_{q-1}\frac{\partial\eta}{\partial\xi_1}.$$

In like manner, expressing that the equivalents of R_y with $x_2, x_3, \ldots, x_{q-1}$ in succession as the one dependent variable are free from first derivatives, we get $q-2$ other sets of $q-1$ annihilators of R_y, but no new types are introduced. So far, then, the information is that R_y has

(i.) $q-1$ annihilators $\xi_1, \xi_2, \ldots, \xi_{q-1}$,

(ii.) $q-1$ annihilators $\eta\dfrac{\partial\eta}{\partial\xi_r}$ $(r = 1, 2, \ldots, q-1)$,

(iii.) $(q-1)(q-2)$ annihilators $\xi_r\dfrac{\partial\eta}{\partial\xi_s}$ $(r\neq s;\ r, s = 1, 2, \ldots, q-1)$,

i.e., together q^2-q annihilators.

But each of the facts (ii.) really gives two facts of annihilation independent of one another and of (i.) and (iii.), in consequence of R_y being free from first derivatives. Consider, for instance, the annihilation by $\eta\dfrac{\partial\eta}{\partial\xi_1}$. The terms in the expansion of $\eta\dfrac{\partial\eta}{\partial\xi_1}$ which involve the first derivative $y_{100\ldots}$ in their coefficients must by themselves annihilate R_y, as otherwise operation with them would produce terms in $\eta \frac{\partial \eta}{\partial \xi_1} R_y$ against which no other terms could cancel. Now, as η only involves $y_{100\ldots}$ in the term $y_{100\ldots}\xi_1$, these terms in $\eta \frac{\partial \eta}{\partial \xi_1}$ are $y_{100\ldots}\left(\xi_1 \frac{\partial \eta}{\partial \xi_1} + \eta\right)$. Similarly as to $\eta \frac{\partial \eta}{\partial \xi_r}$ in general. Thus, besides the $q^2 - q$ annihilators above classified, R_y must also have

(iv.) the $q-1$ annihilators $\xi_r \frac{\partial \eta}{\partial \xi_r} + \eta$, $(r = 1, 2, \ldots, q-1)$.

Consequently, our supposed absolute pure q-ary reciprocant R_y has for annihilators the full system of $q^2 - 1$ operators which belong to the infinitesimal substitutions of the group of extensions of the linear group with generality of coefficients limited by the one relation $\Delta = 1$. R_y is then an absolute differential invariant of that special linear group.

Accordingly, the statement that absolute pure q-ary reciprocants are identical with those absolute differential invariants which do not contain parts of both odd and even degrees of the special linear group is correct.

Thursday, April 7th, 1898.

Dr. E. W. HOBSON, F.R.S., Vice-President, in the Chair.

There being only five members present, and so no quorum, no meeting could be held.

The Chairman (provisionally) communicated the following papers:—

An Essay towards the Generating Functions of Ternariants: Professor Forsyth.

On Systems of One-Vectors in Space of n Dimensions: Mr. W. H. Young.

Zeroes of the Bessel Functions: Mr. H. M. Macdonald.

The following presents were received for the Library :—

"Proceedings of the Royal Society," Vol. LXII., Nos. 386, 387.

"Year-Book of the Royal Society," 1896–7, No. 1.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," v., VI; 1898.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 3; Leipzig, 1898.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. IV., No. 6; New York, March, 1898.

"Science Abstracts," Vol. I., Pts. 1, 2; London, Taylor & Francis, 1898.

Rayet, G.—"Observations Pluviométriques et Thermométriques faites dans le Département de la Gironde" de Juin 1894 à Mai 1895; Juin 1895 à Mai 1896, 8vo; Bordeaux, 1895–6.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. IV., Fasc. 2; Napoli, 1898.

"Mémoires de la Société des Sciences Physiques et Natureles de Bordeaux," Tome I., Cahier 1, 2, 1895–6; Tome II., Cahier 1, 2, 1896. "Procès-Verbaux de Séances de la Société des Sciences Physiques et Naturelles de Bordeaux," Année 1894–5; 1895–6.

"Comptes Rendus de la Section de Physique et de Chimie de Varsovie," 1895–6; 1897.

"Comptes Rendus de la Section Biologique, 1895–6," et "Procès-Verbaux des Réunions Générales," Année 7; Varsovie.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VII., Fasc. 5; Roma, 1898.

"Educational Times," April, 1898.

"Annales de la Faculté des Sciences de Toulouse," Tome XII., Fasc. 1, 2; 1898.

"Rendiconti del Circolo Matematico di Palermo," Tome XII., Fasc. 1, 2; February, 1898.

"Annals of Mathematics," Vol. XII., No. 1; Virginia, February, 1898.

"Indian Engineering," Vol. XXIII., Nos. 8, 9, 10, Feb. 19–March 5, 1898.

Nova Acta :—

"Beiträge zur Geschichte der Trigonometrie," von A. v. Braunmühl, Bd. LXXI., 1; Halle, 1897.

"Nassîr Eddîn Tûsi und Regiomontan," Bd. LXXI., 2, von A. v. Braunmühl; Halle, 1897.

"Zur Geschichte der Geometrie mit constanter Zirkelöffnung," Bd. LXXI., 3, von W. M. Kotta.

("Abh. der Kaiserl. Leop.-Carol. Deutschen Akad. der Naturforschen.")

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