On Differential Equations, Total and Partial; and on a New Soluble Class of the First and an Exceptional Case of the Second. By Sir James Cockle, F.R.S.

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1. The contents of the seven sections of this paper are substantially as follows:—§1 and §2 are preliminary; §2, consisting chiefly of matter intended to be illustrative, embraces Binary forms (such as $Pdx+Qdy$), with some reference to Ternaries (such as $Pdx+Qdy+Rdz$); §3 is on a First, and well-marked, Species of Exceptional Single Solution of Ternaries; §4 is a Digression on Exact Ternary Differentials, and on the process of Monge; §5 is on Exceptional Single Solutions of Ternaries, considered generally; §6 is on the same subject, considered in relation to partial differential equations; §7 is on Quaternaries. There are some points into which I should like to have entered more fully, but the paper has already run to some length.

§ 1. *Notation.—Lemmas.*

2. Unless the contrary appear, the following notation will be preserved throughout this paper. The equations, wherein p and q are in general functions of x, y, and z,

$$\Delta = \frac{d}{dx} + p\frac{d}{dz}, \quad \text{and} \quad \nabla = \frac{d}{dy} + q\frac{d}{dz}$$

define the operators Δ and ∇, while $\Box$ standing alone is given by

$$\Box = \nabla p - \Delta q,$$

which, developed, gives

$$\Box = \frac{dp}{dy} - \frac{dq}{dx} + q\frac{dp}{dz} - p\frac{dq}{dz}.$$

3. From the compounded operator $\nabla\Delta - \Delta\nabla$ (and the same thing would be true of an operator similarly compounded of any two operators each of the form $\Sigma p_n \frac{d}{dx_n}$) all will disappear which would disappear if $\frac{d}{dx}$, $\frac{d}{dy}$, and $\frac{d}{dz}$ were symbols of quantity; *ex. gr.*, all such symbols as $\frac{d^2}{d\xi d\eta}$, and that whether ξ and η be equal or unequal. And

$$\nabla\Delta - \Delta\nabla = \Box\frac{d}{dz}.$$

4. When $\square$ stands not alone, but in this combination, viz., $\square_3(P, Q, R)$, then

$$\square_3(P, Q, R) = P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right),$$

so that the solitary $\square$ means, in fact,

$$-\square_3(p, q, -1).$$

5. Since

$$P\theta\left(Q\frac{d\theta}{dz} - R\frac{d\theta}{dy}\right) + Q\theta\left(R\frac{d\theta}{dx} - P\frac{d\theta}{dz}\right) + R\theta\left(P\frac{d\theta}{dy} - Q\frac{d\theta}{dx}\right)$$

vanishes identically, and since always

$$A\frac{d\theta}{da} = \frac{d}{da}(A\theta) - \theta\frac{dA}{da},$$

consequently $\quad \square_3(P\theta, Q\theta, R\theta) - \theta^2\square_3(P, Q, R)$

vanishes identically; *i.e.*, we have identically

$$\square_3(P\theta, Q\theta, R\theta) = \theta^2\square_3(P, Q, R).$$

6. The symbols A, E, and O will be taken to refer to x, y, and z respectively; A denoting a function of x only, E a function of y only, and O a function of z only. In conformity with this notation, $Æ$ will denote a function of x and y only, $Œ$ a function of y and z only, and we may take Θ to represent a function of x and z only.

§ 2. *On certain Properties of Binaries.—Reference to Ternaries.*

7. If the expressions

$$dz - (\Theta + p)\,dx \quad \text{and} \quad dz - p\,dx$$

have a common integrating factor μ, then μ satisfies

$$-\frac{d}{dz}(\mu\Theta + \mu p) = \frac{d\mu}{dx} = -\frac{d}{dz}(\mu p),$$

whence $\quad 0 = -\dfrac{d}{dz}(\mu\Theta), \quad \text{or} \quad \mu = \dfrac{1}{A\Theta};$

and $\quad \dfrac{d}{dx}\dfrac{1}{A\Theta} = -\dfrac{d}{dz}\dfrac{p}{A\Theta}$

may be put under the form

$$\Delta\left(\frac{1}{A\Theta}\right) + \frac{1}{A\Theta}\frac{dp}{dz} = 0,$$

or, developing, reducing, and transposing,

$$\frac{dp}{dz} - \Delta \log \Theta = \frac{1}{A}\frac{dA}{dx},$$

and, if the common integrating factor exists, the sinister of the last equation will of course be a function of x only. We speak of a common integrating factor even when one of the expressions is already an exact differential.

8. If the two expressions

$$dz - (\Theta + p)\,dx \quad \text{and} \quad dz - pdx$$

have a common solution, *i.e.*, if there be a substitution which will reduce each to zero, then that solution is, or is contained in, $\Theta = 0$. Take, for example,

$$dz + 2\sqrt{-z}\,.\,dx \quad \text{and} \quad dz + 2x\,dx.$$

9. When two expressions have both a common integrating factor and a common solution, the solution makes the factor infinite. But no necessary inference can thence be drawn as to singularity. Thus, the common solution of

$$dz - (x + \sqrt{x^2 - 2z})\,dx \quad \text{and} \quad dz - xdx$$

is singular with respect to the first expression, and particular with respect to the second; while the common solution of

$$dz - \{1 + (z - x)\}\,dx \quad \text{and} \quad dz - dx$$

is a particular primitive of both.

10. Applying a portion of the foregoing to ternaries, the condition that the two expressions

$$dx + zdy + ydz \quad \text{and} \quad dx + zdy - \frac{x}{z}\,dz$$

should have a common solution is $\frac{x}{z} + y = 0$. And this is sufficient, for the first expression is $d\,(x + zy)$, and the second is $zd\left(\frac{x}{z} + y\right)$, the complete integrals are $x + zy = c$ and $\frac{x}{z} + y = c$, and the common solution is given by $c = 0$.

§ 3. *On a First, and well-marked, Species of Exceptional Single Solution of Ternaries.*

11. Let N, M and L be the integrating factors of $Xdx + Ydy + Zdz$, when x, y, and z are respectively taken as constant. Then, to say that $\square_3$ vanishes identically, is in effect to say that all (both) the three (substantially two) equations $N = M = L$ can be satisfied.

But it is sometimes possible, even when $\square_2$ does not so vanish, to satisfy one of the equations, say $N=M$, and in such case M (or N) can be at once determined.

The result of such determination is to give to $M(Xdx+Ydy+Zdz)$ either of the forms, indifferently,

$$\frac{dV}{dz}dz+\frac{dV}{dx}dx+MYdy, \text{ or } \frac{dW}{dz}dz+\frac{dW}{dy}dy+MXdx,$$

which may be replaced by

$$dV-V_2dy, \text{ or } dW-W_2dx.$$

12. Subtracting one of these equivalent forms from the other, we have the identity $\quad d(V-W)=V_2dy-W_2dx,$

and, the sinister being an exact differential, the dexter is also, and V_2 and W_2 are functions of x and y only, satisfying the identity

$$\frac{dV_2}{dx}+\frac{dW_2}{dy}=0.$$

Hence, if we represent V_2 by $\frac{d\text{Æ}_2}{dy}$, and therefore W_2 by $-\frac{d\text{Æ}_2}{dx}$, the two forms will become $dV-\frac{d\text{Æ}_2}{dy}dy$, and $dW+\frac{d\text{Æ}_2}{dx}dx$.

13. To determine M (or N), we have

$$\frac{d(ZM)}{dx}=\frac{d(XM)}{dz} \text{ and } \frac{d(ZM)}{dy}=\frac{d(YM)}{dz},$$

or, writing for a few moments

$$M=\frac{\mu}{Z},\ \frac{X}{Z}=-p, \text{ and } \frac{Y}{Z}=-q,$$

$$\frac{d\mu}{dx}=-\frac{d(p\mu)}{dz}, \text{ and } \frac{d\mu}{dy}=-\frac{d(q\mu)}{dz}.$$

Hence $$\frac{d\mu}{dx}=-p\frac{d\mu}{dz}-\mu\frac{dp}{dz},$$

or $$\frac{dp}{dz}=\Delta\log\frac{1}{\mu} \quad \text{.............................(1)},$$

so $$\frac{dq}{dz}=\nabla\log\frac{1}{\mu} \quad \text{.............................(2)}.$$

Operating on (1) with ∇ and on (2) with Δ, we have, subtracting the results, $\quad \frac{d}{dz}(\nabla p-\Delta q)=(\nabla\Delta-\Delta\nabla)\log\frac{1}{\mu},$

or
$$\frac{d\Box}{dz} = \Box \frac{d}{dz} \log \frac{1}{\mu},$$
whence
$$\frac{1}{\mu} = Æ\Box,$$
where $Æ$ is a function of x and y only. This relation is necessary, but not sufficient, for the solution of (1) and (2). Substituting therein, we find
$$\frac{dp}{dz} - \Delta \log \Box = \frac{1}{Æ} \frac{dÆ}{dx} \text{ ....(i.)},$$
$$\frac{dq}{dz} - \nabla \log \Box = \frac{1}{Æ} \frac{dÆ}{dy} \text{ ....(ii.)};$$
or, replacing p and q by $-\frac{X}{Z}$ and $-\frac{Y}{Z}$ respectively, and remembering that $\Box = -\Box_2 \left(-\frac{X}{Z}, -\frac{Y}{Z}, -1\right)$
$$= -\Box_2 \left(-\frac{X}{Z}, -\frac{Y}{Z}, -\frac{Z}{Z}\right) = \frac{-\Box_2 (X, Y, Z)}{Z^2},$$
we have
$$-\frac{d}{dz}\left(\frac{X}{Z}\right) - \left(\frac{d}{dx} - \frac{X}{Z}\frac{d}{dz}\right) \log \left\{-\frac{\Box_2 (X, Y, Z)}{Z^2}\right\} = \frac{1}{Æ}\frac{dÆ}{dx} \text{...(i.)},$$
$$-\frac{d}{dz}\left(\frac{Y}{Z}\right) - \left(\frac{d}{dy} - \frac{Y}{Z}\frac{d}{dz}\right) \log \left\{-\frac{\Box_2 (X, Y, Z)}{Z^2}\right\} = \frac{1}{Æ}\frac{dÆ}{dy} \text{...(ii.)};$$
and, when the conditions (i.) and (ii.) are identically satisfied, they are sufficient, and $Xdx + Ydy + Zdz$ can be put, indifferently, under either of the forms
$$\frac{-Æ\Box_2}{Z}\left(dV - \frac{dÆ_2}{dy} dy\right) \text{ or } \frac{-Æ\Box_2}{Z}\left(dW + \frac{dÆ_2}{dx} dx\right),$$
where $\Box_2$ is written for $\Box_2 (X, Y, Z)$.

14. Had we wished to satisfy $M = L$, we should have, in place of the last set of formulæ marked (i.) and (ii.), the following, viz.,
$$-\frac{d}{dx}\left(\frac{Y}{X}\right) - \left(\frac{d}{dy} - \frac{Y}{X}\frac{d}{dx}\right) \log \left(-\frac{\Box_2}{X^2}\right) = \frac{1}{Œ}\frac{dŒ}{dy} \text{.........(iii.)},$$
$$-\frac{d}{dx}\left(\frac{Z}{X}\right) - \left(\frac{d}{dz} - \frac{Z}{X}\frac{d}{dx}\right) \log \left(-\frac{\Box_2}{X^2}\right) = \frac{1}{Œ}\frac{dŒ}{dz} \text{.........(iv.)},$$
for $Xdx + Ydy + Zdz$ and $\Box_2 (X, Y, Z)$ are unaffected by the simultaneous cyclical interchanges (X, Y, Z), (x, y, z), and N, M, L correspond respectively with the cases of x, y, z constant. If, then, (i.) and (ii.) represent the phase (X, Y, Z) (x, y, z), the phase (Y, Z, X) (y, z, x)

will be represented by (iii.) and (iv.), and the phase (Z, X, Y) (z, x, y) by

$$-\frac{d}{dy}\left(\frac{Z}{Y}\right)-\left(\frac{d}{dz}-\frac{Z}{Y}\frac{d}{dy}\right)\log\left(-\frac{\square_2}{Y^3}\right)=\frac{1}{\Theta}\frac{d\Theta}{dz} \quad \text{.........(v.)},$$

$$-\frac{d}{dy}\left(\frac{X}{Y}\right)-\left(\frac{d}{dx}-\frac{X}{Y}\frac{d}{dy}\right)\log\left(-\frac{\square_2}{Y^3}\right)=\frac{1}{\Theta}\frac{d\Theta}{dx} \quad \text{.........(vi.)},$$

which system answers to $L=N$. The three equations $N=M$, $M=L$, and $L=N$ may indeed be appropriately symbolized by $(x)=(y)$, $(y)=(z)$, and $(z)=(x)$.

By an independent process I have arrived at the results

$$\frac{1}{\mu}=\frac{\Theta\square}{q}=\frac{(-\Theta)(-\square)}{q},$$

and
$$\frac{1}{q}\left(\frac{dp}{dy}-\frac{dq}{dx}\right)-\left(\frac{p}{q}\frac{d}{dy}-\frac{d}{dx}\right)\log\frac{-\square}{q}=-\frac{1}{\Theta}\frac{d\Theta}{dx},$$

$$\frac{1}{q}\left(\frac{dq}{dz}-\nabla\log\frac{\square}{q}\right)=\frac{1}{\Theta}\frac{\Theta}{dz},$$

wherein, however, $p=-\frac{X}{Z}$, and $q=-\frac{Y}{Z}$. On making these substitutions, and reducing, the latter results coincide with (v.) and (vi), and thus we have a verification; in connection with which I remark that $Æ$, $Œ$, and Θ may each be affected with an arbitrary constant multiplier; for $d\log(Cv)=d\log v$.

The transformation here indicated is that of $Xdx+Ydy+Zdz$ into

$$-\frac{\Theta\square_3}{Y}\left(dU+\frac{d\Theta_3}{dz}dz\right) \text{ or } -\frac{\Theta\square_3}{Y}\left(dW-\frac{d\Theta_3}{dx}dx\right).$$

16. So for $M=L$ the transformation indicated is that of $Xdx+Ydy+Zdz$ into

$$-\frac{Œ\square_3}{X}\left(dU-\frac{dŒ_3}{dz}dz\right) \text{ or } -\frac{Œ\square_3}{X}\left(dV+\frac{dŒ_3}{dy}dy\right).$$

17. Recurring to the first form of (i.) and (ii.), and multiplying (i.) into dx and (ii.) into dy, add the products, and add and subtract $\frac{d\square}{\square dz}dz$ to and from the result. We shall obtain

$$\frac{dp}{dz}dx+\frac{dq}{dz}dy-\frac{d\square}{\square}+\frac{1}{\square}\frac{d\square}{dz}(dz-pdx-qdy)=\frac{1}{Æ}dÆ \quad \text{......(A)},$$

which gives, on integration, $\log Æ$. If we wish to find $d\log(Æ\square)$

at once, and write for a moment $\frac{1}{\square}\frac{d\square}{dz}=\frac{1}{\rho}$, then we have

$$\frac{1}{\rho}\left\{dz+\left(\rho\frac{dp}{dz}-p\right)dx+\left(\rho\frac{dq}{dz}-q\right)dy\right\}=d\log(Æ\square)\quad\text{...(B)},$$

and, the dexter of (B) being an exact differential, the sinister is so, and integration will give log ($Æ\square$). And (B) is very readily applicable to $dz-pdx-qdy$ when p and q (or p or q) are linear in z.

18. Cases in which, when one of the systems (i.) and (ii.), or (iii.) and (iv.), or (v.) and (vi.), is identically satisfied, $\square=0$ solves $dz-pdx-qdy$, exist, and some are given in this paper. The mark of such a solution is, that it accompanies one of the systems, and at the same time discloses another solution (or rather two other solutions) of Monge's form. This species of solution seems to be suggested when we suppose one, and only one, of the relations $N=M=L$ to be capable of being fulfilled, and so to lie in the pathway of research.

19. It will, I think, be found convenient to give a name to the functions $\square$ and $\square_2$. Let us call them Discriminoids. I find that the solution of the following example, taken from a paper "On Ternary Differential Equations," by Mr. Robert Rawson, ("Manchester Procedings," Vol. XVI., p. 114), and wherein w, v are functions of x, y only (w, however, not being of the form $fxFy$); viz., the solution of

$$\left(\frac{dw}{dy}z-\frac{dv}{dy}\right)dy+\left(\frac{v}{w}\frac{dw}{dx}-\frac{dv}{dx}\right)dx+wdz,$$

belongs to this First Species of Discriminoidal Solution. I shall give another such example further on.

§4. *Digression on Exact Ternary Differentials and on the Process of Monge.*

20. The necessity that the three conditions

$$\frac{dP}{dy}=\frac{dQ}{dx},\ \frac{dQ}{dz}=\frac{dR}{dy},\ \text{and}\ \frac{dR}{dx}=\frac{dP}{dz}$$

should be fulfilled in order that $Pdx+Qdy+Rdr$ should be an exact differential, is at once recognized. The sufficiency of these conditions may be easily proved. They, in effect, demand that $Pdx+Qdy+Rdz$ should be capable of being placed under any of the three equivalent forms $dU+\left(R-\frac{dU}{dz}\right)dz$; $dV+\left(Q-\frac{dV}{dy}\right)dy$; $dW+\left(P-\frac{dW}{dx}\right)dx$ indifferently. Subtracting the second from the first, the third from the second and the first from the third successively, we obtain, after transposition, the following identities :—

$$d(U-V) = \left(Q-\frac{dV}{dy}\right)dy-\left(R-\frac{dU}{dz}\right)dz,$$

$$d(V-W) = \left(P-\frac{dW}{dx}\right)dx-\left(Q-\frac{dV}{dy}\right)dy,$$

$$d(W-U) = \left(R-\frac{dU}{dz}\right)dz-\left(P-\frac{dW}{dx}\right)dx.$$

But, the sinisters being exact differentials, the dexters are so. Consequently $Q-\frac{dV}{dy}$ and $R-\frac{dU}{dz}$ are free from x,

$P-\frac{dW}{dx}$ and $Q-\frac{dV}{dy}$ are free from z,

$R-\frac{dU}{dz}$ and $P-\frac{dW}{dx}$ are free from y;

in other words, $R-\frac{dU}{dz}$ is a function of z only, $Q-\frac{dV}{dy}$ is a function of y only, and $P-\frac{dW}{dx}$ a function of x only. This establishes the sufficiency.

21. I question whether Monge's process can be advantageously applied to the finding of single solutions involving all the variables. For, first, in applying it we may lose a solution; secondly, there is a better procedure; and, thirdly, it may be doubted whether his solution is, in a proper sense, general. If by any means we can reduce $Pdx+Qdy+Rdz$ to the form $dV-\rho dW$; then $V=\phi(W)$, combined with $\rho=\phi'(W)$, would be a dual solution having apparently as strong a claim to generality as that of Monge. Using Mr. Harley's cyclical symbol Σ', form $\Sigma' zdx$ or $zdx+xdy+ydz$; then, using the integrating factor $\frac{1}{zx}$ for the first two terms, we reduce $\Sigma' zdx$ to

$$d\left(\log x+\frac{y}{z}\right)+\frac{y}{z}\left(\frac{1}{z}+\frac{1}{x}\right)dz;$$

whence the solution of Monge, viz.,

$$\log x+\frac{y}{z}=\phi(z), \quad -\frac{y}{z}\left(\frac{1}{z}+\frac{1}{x}\right)=\phi'(z).$$

But by cyclical changes we obtain, through Monge's method, two more solutions each with a claim, as strong as that of the foregoing, to be called general, viz.,

$$\log y+\frac{z}{x}=\phi(x),\quad -\frac{z}{x}\left(\frac{1}{x}+\frac{1}{y}\right)=\phi'(x),$$

and $$\log z+\frac{x}{y}=\phi(y),\quad -\frac{x}{y}\left(\frac{1}{y}+\frac{1}{z}\right)=\phi'(y).$$

Nor is this all. The aggregate of these three solutions does not appear to constitute a general solution. Assume

$$ax^2+by^2+cz^2+2dxy+2exz+2fyz=V,\quad -\rho=2\alpha x+2\beta y+2\gamma z,$$

and $$W=Ax+By+Cz,$$

wherein all the coefficients are constant, and put

$$dV-\rho dW=2\Sigma' z\,dx=2\,(zdx+xdy+ydz);$$

then we shall find that the six quantities

$$a+A\alpha,\ d+A\beta,\ b+B\beta,\ f+B\gamma,\ c+C\gamma,\ \text{and}\ e+C\alpha$$

must each be equated to zero, and the three

$$e+A\gamma,\ d+B\alpha,\ \text{and}\ f+C\beta$$

each to unity. Hence we shall be led to

$$\alpha\,(B+C)=A\,(\beta+\gamma),\quad \gamma\,(A+B)=C\,(\alpha+\beta),$$

and $$\beta\,(C+A)=B\,(\alpha+\gamma);$$

and if we make $B+C=0$, $A=0$, $\alpha+\gamma=0$, and $\beta=0$, we find $a, d, b, =0$; $e, f, =1$; $B\alpha=1$; $B\gamma=C\alpha=-1$; and $c=-C\gamma$. Now put $B=1$, $C=-1$, and we have $\alpha=1$ and $\gamma=-1$. Hence $V=-z^2+2xz+2yz$; $-\rho=2x-2z$; $W=y-z$; consequently

$$dV=2\,\{(x+y-z)\,dz+zdx+zdy\},$$

$$-\rho dW=2\,\{-(x-z)\,dz+xdy-zdy\};$$

and therefore $$dV-\rho dW=2\,(zdx+xdy+ydz).$$

Thus, for $zdx+xdy+ydz=0$, we obtain the dual solution

$$2xz+2yz-z^2=\phi\,(y-z);\quad -2\,(x-z)=\phi'(y-z);$$

and hence, by cyclical changes, the two additional dual solutions

$$2yx+2zx-x^2=\phi\,(z-x);\quad -2\,(y-x)=\phi'(z-x);$$

and $$2zy+2xy-y^2=\phi\,(x-y);\quad -2\,(z-y)=\phi'(x-y).$$

22. In these last three solutions the function ϕ is arbitrary, and if we take ϕ as an algebraical function, we may represent $zdx+xdy+ydz=0$ by curves resulting from the intersection of surfaces represented by algebraical equations. No logarithm need enter.

23. By assuming $Pdx+Qdy+Rdz = dV-\rho\,dW$, and seeking for particular solutions of the partial differential equations thence deducible, we may possibly sometimes obtain solutions not within Monge's formula. In the concluding solution of the last example, there was no need for any process whatever of integration, all the steps being purely algebraical.

5. *On Exceptional Single Solutions of Ternaries, considered generally.*

24. Changing the notation for the purposes of this section, I shall now endeavour to ascertain the conditions, necessary and sufficient, in order that $dz-Mdx-Ndy$ may, when not completely integrable, admit of a single solution involving all the variables.

25. Assume that there is such a solution, represent it by $u = 0$, and let $dz-pdx-qdy = 0$ be its (divided if necessary) differential equation. Then, if the two expressions $dz-Mdx-Ndy$ and $dz-pdx-qdy$ have a common solution, that solution gives $M=p$ and $N=q$, and is, or is contained in, either, or both, of the latter equations, one of which, however, may be an identity (*ex. gr.*, $M=p$ identically).

26. Of the two forms $\square$ and $\square_2$, which are connected by the relation $\square\,(P, Q, R) = -\square_2\,(P, Q, R)$, I shall here use $\square$ and form it as before, save that p and q are to be replaced by M and N respectively.

Thus
$$\square = \frac{dM}{dy}+N\frac{dM}{dz}-\frac{dN}{dx}-M\frac{dN}{dz}.$$

27. Now put $M=P+p$ and $N=Q+q$, and develope. We have

$$\begin{aligned}\square = \;& \frac{dP}{dy}+Q\frac{dP}{dz}-\frac{dQ}{dx}-P\frac{dQ}{dz}\\ &+q\frac{dP}{dz}+Q\frac{dp}{dz}-p\frac{dQ}{dz}-P\frac{dq}{dz}\\ &+\frac{dp}{dy}-\frac{dq}{dx}+q\frac{dp}{dz}-p\frac{dq}{dz}.\end{aligned}$$

28. But, in this section, $dz-pdx-qdy = 0$ is supposed to be the differential equation of a given primitive $u = 0$. Hence the last line of Art. 27 vanishes identically. Expunging it, and rearranging the remaining terms, we have

$$\begin{aligned}\square = \;& \frac{dP}{dy}+q\frac{dP}{dz}-\frac{dQ}{dx}-p\frac{dQ}{dz}\\ &+Q\frac{d}{dz}(P+p)-P\frac{d}{dz}(Q+q).\end{aligned}$$

29. Since p is, in this section, the equivalent of $\frac{dz}{dx}$, and q of $\frac{dz}{dy}$, the first two terms of $\square$ in Art. 28 may be written $\frac{dP}{dy}+\frac{dP}{dz}\frac{dz}{dy}$. And this last expression vanishes. For $u=0$ gives $M=p$, and $M=p$ gives $P=0$, and z is a function of the independent variables x and y. Consequently, the differential coefficient of P with respect to y, treating z in P as a function of x and y, vanishes. For like reasons the aggregate of the next two terms, viz., $-\left(\frac{dQ}{dx}+p\frac{dQ}{dz}\right)$, vanishes.

30. Thus, $u=0$ reduces $\square$ to

$$\square=Q\frac{d}{dz}(P+p)-P\frac{d}{dz}(Q+q),$$

or to

$$\square=Q\frac{dM}{dz}-P\frac{dN}{dz},$$

and $u=0$ [which indeed might be replaced by $z=f(x, y)$] is accompanied by $P=0$ and $Q=0$.

31. When P and Q both vanish, either

$$\square=0,$$

or

$$\frac{dM}{dz}=\infty,$$

or

$$\frac{dN}{dz}=\infty.$$

32. Whether any equation herein contained is or is not a solution of the proposed differential equation $(dz-Mdx-Ndy=0)$, must be determined by trial.

33. If no single solution involving *all* the variables be thus obtained, then there exists no such solution, unless, indeed, the making x or y, instead of z, the dependent variable, should lead to one.

34. The precise bearing of the foregoing on the subject of singular solution I have not investigated. But the following example will illustrate all, or most, of what precedes. I do not know that I have exhausted the illustrations. M and N having served their purpose, we may now replace them by p and q, giving, henceforth, to p and q their old and unrestricted meanings, and no longer supposing that $dz-pdx-qdy$ is completely integrable by a factor. And $\frac{dp}{dz}=\infty$ and $\frac{dq}{dz}=\infty$ replace the conditions $\frac{dM}{dz}=\infty$ and $\frac{dN}{dz}=\infty$.

35. Take the expression

$$dz-(\alpha u^m+\beta u-1)\,dx-(\gamma u^m+\delta u^n+\epsilon u-1)\,dy,$$

wherein $$u = x+y+z,$$

and $$\square = (m-n)\,\alpha\delta u^{m+n-1}+(m-1)(\alpha\epsilon-\beta\gamma)\,u^m-(n-1)\,\beta\delta u^n.$$

36. First let m and n be positive. Then

(I.) If $m = n = 1$, the expression is integrable by a factor, for $\square$ vanishes identically.

(II.) If $m = n$, then

$$\square = (m-1)(\alpha\epsilon-\beta\gamma-\beta\delta)\,u^m,$$

and $\square = 0$ gives the solution $u = 0$. The expression and its solution are of the first species (discussed in § 3) of discriminoidal solution, and

$$Æ = e^{(1-m)(\beta x+\epsilon y)}.$$

(III.) If $n = 1$, then

$$\square = (m-1)(\alpha\delta+\alpha\epsilon-\beta\gamma)\,u^m,$$

and $\square = 0$ gives the solution $u = 0$. The expression and its solution are of the same first species, and

$$Æ = e^{(1-m)[\beta x+(\delta+\epsilon)y]}.$$

(IV.) If $m = 1$, then

$$\square = (1-n)(\alpha\delta+\beta\delta)\,u^n,$$

and $\square = 0$ gives the solution $u = 0$. The species is the same, and

$$Æ = e^{(1-n)[(\alpha+\beta)x+(\gamma+\epsilon)y]}.$$

(V.) If m and n have other values or relations, and if $m+n$ be >1, then $\square = 0$ gives the solution $u = 0$, which now belongs to a second species of discriminoidal solution.

(VI.) If $m+n$ be <1, then, since $\square = \infty$ accompanies the solution $u = 0$, that solution may be said to belong to a second genus of discriminoidal solution. This solution is of course within the purview of Art. 31.

(VII.) If m and n be unequal, and $m+n = 1$, then

$$\square = (2m-1)\,\alpha\delta+(m-1)(\alpha\epsilon-\beta\gamma)\,u^m+m\beta\delta\,u^{1-m},$$

wherein m is not $\frac{1}{2}$, and m and $1-m$ are positive. Here $u = 0$ is an exceptional solution, not in any sense discriminoidal, for it does not satisfy either $\square = 0$ or $\square = \infty$. It, in fact, makes $\square = (2m-1)\,\alpha\delta$, and comes from $\dfrac{dp}{dz} = \infty$ and $\dfrac{dq}{dz} = \infty$ indifferently. For these equations are respectively equivalent to

$$-mau^{m-1}-\beta = \infty \quad \text{and} \quad -m\gamma u^{m-1}-n\delta u^{n-1}-e = \infty,$$

each of which is satisfied by $u = 0$.

37. Let $\beta = 0$ and $e = 0$, and let both m and n be negative. Here $\square = 0$ yields $u = \infty$, a discriminoidal solution of the second species of the first genus of discriminoidal solution, for the system (i.) and (ii.) can now only be satisfied by $a, \gamma, \delta, = 0$.

§ 6. *The same subject considered in relation to Partial Differential Equations.*

38. The two expressions

$$\frac{d\theta}{dz}(dz-pdx-qdy)$$

and
$$d\theta-\left(\frac{d\theta}{dx}+p\frac{d\theta}{dz}\right)dx-\left(\frac{d\theta}{dy}+q\frac{d\theta}{dz}\right)dy$$

are identical, and if we can make the latter vanish, the former will vanish also. The latter will be reduced to zero provided that we can find θ as a function of x, y, and z from the two partial differential equations

$$\frac{d\theta}{dx}+p\frac{d\theta}{dz} = 0 \quad \text{and} \quad \frac{d\theta}{dy}+q\frac{d\theta}{dz} = 0,$$

for then $\theta = 0$ (or $\theta =$ const.) will make the second expression vanish. But if we follow the process given in Boole's "Differential Equations." (1865, Suppl., p. 88), we shall merely be led, from the partial differential equations, back to a total differential equation

$$dz-pdx-qdy = 0.$$

That is to say, we shall be so led if we confine our attention to solutions of one step, by which I mean solutions which satisfy a partial differential equation solely in virtue of the values which they give to the partial differential coefficients. Boole (*ibid.* 1865, pp. 331, 332; Suppl., pp. 60, 61) does not seem to recognize any distinction between solutions of one and of two steps. Yet for a solution of two steps his functional determinant would not vanish identically.

39. By way of illustration, take

$$(z^2+xz-y^2-xy)\frac{dz}{dx}+(x^2+xy-z^2-zy)\frac{dz}{dy}-(y^2+yz-x^2-xz) = 0 \qquad \text{.........(B)},$$

and let $x+y+z = 0$. This is a solution of one step, for it gives $\frac{dz}{dx} = -1$ and $\frac{dz}{dy} = -1$, and these values reduce the sinister of (B) to an identical zero.

40. But let a be an unreal cube root of unity, and let $z+ax+a^2y=0$, which gives $\frac{dz}{dx}=-a$ and $\frac{dz}{dy}=-a^2$.

These values do not reduce the sinister to zero, but the equation itself is reduced to $(1-a)\,a\,(x+y+z)(z+ax+a^2y)=0$, which is not satisfied unless we take a second step and substitute 0 for

$$z+ax+a^2y.$$

41. If we put $xy+yz+zx=u$ and $x^2+y^2+z^2=v$, then $\phi(u,v)=0$ is the general solution of (B). Now $x+y+z$ can be expressed as a function of u and v, for it is $+\sqrt{v+2u}$. But $z+ax+a^2y$ cannot be so expressed. It is indeed a factor of $v-u$, but that is not the proposition contended for (and purporting to be demonstrated in two distinct ways) by Boole. The general proposition (if such there be) on the subject is this: viz., that, if $u=a$ and $v=b$ be two integrals of $Pp+Qq=R$, and $w=0$ be any other solution of the equation, then w can be represented as a function of u and v, or, failing that, as a factor of such a function. In the first case we have a solution of one step; in the second we have a solution of two steps.

42. Recurring to the equations of Art. 39, viz., $\Delta\theta=0$ and $\nabla\theta=0$, let us examine the consequences of supposing that they are soluble by two steps. Put $\Delta\theta=A$ and $\nabla\theta=B$, where A and B are taken to vanish with θ. Operate on the first with ∇, and on the second with Δ, and we have, on subtracting the results,

$$\Box\frac{d\theta}{dz}=\nabla A-\Delta B.$$

Now, though A and B each vanish with θ, it does not follow that $\nabla A-\Delta B$ will do so. Should it however so vanish, the case may correspond with that of the discriminoidal solution $\Box=0$. Should $A=0$ and $B=0$, make $\nabla A-\Delta B$ infinite, the case may correspond with $\Box=\infty$. Should the given relations leave $\nabla A-\Delta B$ finite, we are thrown back on the method of the section last preceding. But the existence of solutions of two steps shows that there is no necessary conflict between the two modes of viewing the question presented in this and that section respectively.

§ 7. *On Quaternaries.*

43. Although we cannot employ the whole doctrine of determinants in the present inquiries, yet we may, I think, shorten our labours by adopting to some extent their notation. Take

$$\begin{vmatrix} a, & a_2, & a_3 \\ b, & b_2, & b_3 \\ c, & c_2, & c_3 \end{vmatrix}$$

and say that it is formed on the line when it is written $aA+a_2A_2+a_3A_3$, and on the column when it is written $aA+bB+cC$. Say, moreover, that in forming a determinant we always place the symbols in the order of the columns. Then the discriminoid

$$\Box_3(P, Q, R) \quad \text{or} \quad P\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)+Q\left(\frac{dR}{dx}-\frac{dP}{dz}\right)+R\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)$$

can be represented by

$$\begin{vmatrix} P, & -\frac{d}{dx}, & P \\ Q, & -\frac{d}{dy}, & Q \\ R, & -\frac{d}{dz}, & R \end{vmatrix}$$

this quasi-determinant being formed on the column and the symbols being placed in the order of columns. Proceed a step further, and write

$$\begin{vmatrix} P, & P, & -\frac{d}{dx}, & P \\ Q, & Q, & -\frac{d}{dy}, & Q \\ R, & R, & -\frac{d}{dz}, & R \\ T, & T, & -\frac{d}{dt}, & T \end{vmatrix}$$

then, the two columns on the hither side of the column of differentials being identical, we may, in conformity with the ordinary rule, suppose the result to vanish. But, developing this last quasi-determinant, we have

$$P\begin{vmatrix} Q, & -\frac{d}{dy}, & Q \\ R, & -\frac{d}{dz}, & R \\ T, & -\frac{d}{dt}, & T \end{vmatrix} - Q\begin{vmatrix} R, & -\frac{d}{dz}, & R \\ T, & -\frac{d}{dt}, & T \\ P, & -\frac{d}{dx}, & P \end{vmatrix}$$

$$+R\begin{vmatrix} T, & -\frac{d}{dt}, & T \\ P, & -\frac{d}{dx}, & P \\ Q, & -\frac{d}{dy}, & Q \end{vmatrix} - T\begin{vmatrix} P, & -\frac{d}{dx}, & P \\ Q, & -\frac{d}{dy}, & Q \\ R, & -\frac{d}{dz}, & R \end{vmatrix} = 0.$$

44. But this expresses the fact that, if we take the quaternary

$$Pdx + Qdy + Rdz + Tdt,$$

and express the four discriminoids formed by supposing x, y, z, and t successively to be constant [viz., $\square_3(Q, R, T)$, $\square_3(R, T, P)$, $\square_3(T, P, Q)$, and $\square_3(P, Q, R)$], then those four discriminoids are connected by a linear relation, so that, if three vanish identically, so will the fourth. Now, in order to apply the contents of this paper to the ascertaining whether a given quaternary admits of a single solution containing all the variables, we proceed thus: form $\square_3(P, Q, R)$, and if it vanishes identically form $\square_3(Q, R, T)$, and if that again is null form $\square_3(R, T, P)$. If that too vanishes identically, then we may be sure that $\square_3(T, P, Q)$ also vanishes identically, and that the quaternary is completely integrable. If, however, one of the discriminoids, say $\square_3(P, Q, R)$, does not vanish identically, then seek all the exceptional solutions, discriminoidal or other, of $Pdx + Qdy + Rdz$, which contain x, y, z, and t (*i.e.*, all the four variables). If no such solution can be found, the quaternary has no single solution in all the variables. If such a solution can be found, and reduces to zero the quaternary as well as the ternary, it is the required single solution of the latter. If it does not reduce the quaternary to zero, then the quaternary has no single solution whatever. It should be remarked that any one of the contained ternaries may possibly have more suitable solutions than one, and each one of the solutions of the ternary first tried should be tested by the quaternary. The process is much the same for quinary and higher forms.

Discussion of Two Double Series arising from the Number of Terms in Determinants of Certain Forms. By J. D. H. Dickson, M.A.

[*Read March 13th*, 1879.]

The first double series arises from the number of non-vanishing terms of a determinant of n^2 elements, with one diagonal of r zero-elements.

If $u_{n,r}$ be the number of such terms in a determinant as above described, it is found, by summation, in two different ways, that

$$u_{n,r} = (n-r)\,u_{n-1,r-1} + (r-1)\,u_{n-1,r-2} \quad \ldots\ldots\ldots\ldots\ (1),$$

and

$$u_{n,r} = u_{n,r+1} + u_{n-1,r} \quad \ldots\ldots\ldots\ldots\ (2).$$

From (2), if E, F be two operators operating only on u, and such that E refers to n alone, and F to r alone, then

$$E = EF + 1 \quad \ldots\ldots\ldots\ldots\ (3);$$