

THE RELATION BETWEEN "POTENTIAL TEMPERATURE" AND "ENTROPY."<sup>1</sup>

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IN 1888 the late Professor von Helmholtz incidentally introduced the term "waergehalt" in connection with his investigations,<sup>2</sup> "On Atmospheric Motions." According to him the "waergehalt" or the actual heat contained in a given mass of air is to be measured by the absolute temperature which the mass would assume if it were brought adiabatically to the normal or standard pressure. It remained for the late Professor von Bezold, however, to perceive the full significance of this term and to reveal its important bearing in the discussion of meteorological phenomena.

As the quantity really involved in this new term is not a quantity of heat, von Bezold suggested that the term be replaced by the evidently more appropriate one of "potential temperature."<sup>3</sup> This met with von Helmholtz's approval.

With the aid of this happy idea of "potential temperature" von Bezold was enabled to draw in a simple and beautiful manner a number of important conclusions governing thermodynamic phenomena taking place in the atmosphere. Thus, for example, he found that:

"Strict adiabatic changes of state in the atmosphere leave the potential temperature unchanged, whereas pseudo-adiabatic ones invariably increase the same, the increase being in proportion to the amount of aqueous evaporation."

Von Bezold called attention to the fact that this law bears a strik-

<sup>1</sup> Presented before the Philosophical Society of Washington, March 16, 1907.

<sup>2</sup> Sitzungsberichte Berliner Akademie, 1888, V. XLVI., p. 652, "Ueber atmosphärische Bewegungen," see translation in Abbe's *Mechanics of the Earth's Atmosphere*, Washington, 1891, p. 83. The symbol  $\theta$  is used to denote the "Waergehalt."

<sup>3</sup> Sitzb. Berliner Akad., 1888, V. XLVI., p. 1189, "Zur Thermodynamik der Atmosphäre"; also in von Bezold's "Gesammelte Abhandlungen," Vieweg und Sohn, Braunschweig, 1906, p. 128. A translation will be found in Abbe's *Mechanics*, etc., p. 243.

ing resemblance to the well-known theorem of Clausius, now commonly known as the second law of thermodynamics, viz.: "that the entropy strives towards a maximum;" but, he says, "it is not identical with it."

The purpose of this paper is to examine into the precise relationship between the two functions "potential temperature" and "entropy" and to see whether any use could be made advantageously of the former in the treatment of certain thermodynamic problems as well as to ascertain wherein the potential temperature law fails to give full expression of the second law of thermodynamics. To my knowledge no application has as yet been made of the new term in treatises on thermodynamics. The substance of this paper was communicated to the American Association for the Advancement of Science at the Springfield meeting in 1895, but publication pending opportunity for further elaboration was deferred.

*The "potential temperature" of a body is defined as the absolute temperature assumed when the body is brought adiabatically to standard pressure.*

Defining the thermodynamic state per *unit of mass* of a body by the three variables,  $T$ , the absolute temperature,  $v$ , the volume per unit of mass,  $p$ , the pressure supposed uniform, the following characteristic equation subsists between them:  $T = f(v, p)$ .

If the body be brought now adiabatically to standard pressure  $p_0$ , then the temperature assumed at the end of the process is the so-called *potential temperature* as above defined and is designated by the symbol  $\theta$ . Hence,

$$\theta = f(v, p_0). \quad (1)$$

For a perfect gas, since  $kT = pv$ ,  $k$  being a constant for any particular gas,

$$\theta = \frac{p_0}{k} \cdot v = k_0 \cdot v, \quad (2)$$

*or the potential temperature for any particular gas is directly proportional to the volume* and, hence, as von Bezold showed, the potential temperature readily admits of a graphical representation on the usual  $pv$  diagram, being simply proportional to the  $v$  abscissæ of points of intersection of the line of standard pressure,  $p = p_0$ , with the adiabats.

Hence, were it possible to express the entropy function for perfect gases directly in terms of potential temperature, we should likewise have for certain cases an easy graphical representation of the entropy function.

In the  $p$  $v$  diagram opposite let  $aa'$  and  $bb'$  represent portions of two adiabats, and  $o'a'b'$  be the line of standard pressure  $p = p_0$ .

Suppose the initial thermodynamic state of the body experimented upon be represented by the point  $a$  and some process  $ab$  be carried out. According to definition, the potential temperature,  $\theta_a$ , in the state  $a$  will be the temperature at the point along the adiabat  $aa'$  where it is intersected by the line of standard pressure. But according to equation (2) the temperature at, this point,  $a'$ , is proportional to the volume, *i. e.*, to  $o'a'$ . Similarly the potential temperature in the state  $b$  will be proportional to the abscissa  $o'b'$ . Hence if measured on the same scale,  $o'a'$  and  $o'b'$  will represent directly for the same substance the respective potential temperatures. It is thus easy to represent graphically at any stage of the process  $ab$  the corresponding potential temperature.

If it is desired to determine the numerical value of the potential temperature, this can be done with the aid of the equation of the adiabat thus :

$$\theta_a = \theta_{a'} = T_a \left( \frac{v_a}{v_{a'}} \right)^{\epsilon-1} = T_a \left( \frac{p_0}{p_a} \right)^{\frac{\epsilon-1}{\epsilon}},$$

or

$$\theta_a = \left( \frac{p_0 v_a}{k} \right)^{\frac{\epsilon-1}{\epsilon}} T_a^{\frac{1}{\epsilon}} = k' v_a^{\frac{\epsilon-1}{\epsilon}} T_a^{\frac{1}{\epsilon}}. \quad (3)$$

$$[\epsilon = 1.41].$$

For a perfect gas, the entropy,  $s$ , *per, unit of mass* may be expressed by the following equation :<sup>1</sup>

$$s = \int -\frac{dh}{T} = c_v \log T + k \log v + \text{const.}, \quad (4)$$

<sup>1</sup> See, *e. g.*, Planck's Thermodynamics.

$c_p$  and  $c_v$  are, respectively, the specific heats at constant pressure and at constant volume;  $k$  is a constant for any particular gas. Utilizing equation (3) and remembering that

$$\varepsilon = \frac{c_p}{c_v} \quad \text{and} \quad k = (c_p - c_v)$$

we get

$$s = c_p \log \theta + (\varepsilon - 1) \log \frac{k}{p_0} + \text{const.},$$

or

$$s = c_p \log \theta + \text{const.} \quad (5)$$

This gives us the relation sought between potential temperature and entropy. Since  $c_p$  is invariably a positive quantity, it follows at once that for any process the potential temperature varies in precisely the same direction as the entropy. If the entropy is increased, as it invariably is for irreversible processes in accordance with the second law of thermodynamics, then is the potential temperature likewise increased. When the entropy remains constant, as for reversible processes, *e. g.*, a strict adiabatic process, then the potential temperature likewise remains constant. In other words as far as perfect gases are concerned it is possible to express the entropy function in its simplest form by means of a quantity — the potential temperature — not only readily interpretable but also easy of direct graphical representation.

Owing to the intimate relationship between entropy and potential temperature the term “entropic temperature” might appear as being possibly a more suggestive one for von Helmholtz’s “waermegehalt” than that of “potential temperature,” but it may hardly seem advisable now since von Bezold’s extensive use of the latter term to recommend a change.

*Cyclical Process.* — By turning back to the diagram, it will be seen that the change in potential temperature in going from  $a$  to  $b$  is precisely the same as from  $a'$  to  $b'$ , *i. e.*, the same as for a simple expansion process under constant pressure. Hence, in carrying out the cyclical process  $abb'a'$ , it will readily be seen that the sum total of the potential temperature changes is zero, just as in the case of the sum total of the entropy changes.

We have in general :

$$s_b - s_a = c_p(\log \theta_b - \log \theta_a) = c_p(\log \theta_{b'} - \log \theta_{a'}), \quad (6)$$

or the entropy change in passing from  $a$  to  $b$  by any process whatsoever — reversible or irreversible — can be measured ideally by the temperature changes incurred in allowing the body to expand under standard pressure between the initial and final adiabats.

*For other Substances.* — If the substance acted upon be not a perfect gas we have :

$$\int_a^b ds = s_b - s_a = \int_{a'}^{b'} -\frac{dh}{T} = \int_{a'}^{b'} c_p \frac{d\theta}{\theta} = \int_a^b c_p \frac{d\theta}{\theta}. \quad (7)$$

Here  $c_p$  is not a constant as in the case of a perfect gas, but varies with temperature and may even be discontinuous, hence it is impossible, in general, to carry out the integration of the right-hand member. This we know, however, that  $c_p$  is invariably positive, *i. e.*, heat must always be *supplied* to a substance to raise its temperature under a constant pressure. Since

$$ds = c_p \frac{d\theta}{\theta}, \quad (8)$$

it follows that the sign of  $ds$  is the same as that of  $d\theta$ , so that whenever the entropy increases, the potential temperature does likewise. This, while true for cases treated, is not so, in general, as previously explained.

In the foregoing paragraphs the law of potential temperature has been deduced from that of entropy, however, an independent deduction can readily be made if desired.

For example, we may build up the law of potential temperature in precisely the same manner as in the case of the entropy law by taking typical examples of natural processes and showing that nature unaided invariably tends to increase the potential temperature.

Thus take the well-known case of the sudden expansion of a perfect gas without the performance of external work. It is very easy to show on the  $p\upsilon$  diagram, since the adiabat is a steeper curve than the isotherm, that the potential temperature in the final stage is greater than in the initial stage.

So again with the case of heat conduction. Suppose we have the same mass of the same perfect gas enclosed in each of two vessels  $a$  and  $b$  of the same size and enclosed in a non-conducting vessel. The temperature of  $a$  is greater than  $b$ . Connect now  $a$  and  $b$  thermally whereupon in accordance with nature's law heat will flow from the hotter body to the colder until the two are of the same temperature. It will be found that here again the potential temperature of the entire system at the end of the process is greater than at the beginning. This may be proven most readily thus: For a perfect gas we have from (3), when the volume remains constant:

$$\theta = k'' T^{1/\epsilon},$$

hence

$$d\theta = \frac{k''}{\epsilon} \cdot \frac{1}{T^\mu} dT,$$

where  $\mu = 1 - 1/\epsilon =$  positive quantity, since  $1/\epsilon < 1$ . Consequently, *the change in potential temperature for a given change in absolute temperature, the volume remaining constant, decreases with absolute temperature.* Hence, although the two bodies,  $a$  and  $b$ , under the conditions imposed, change in absolute temperature by the same amount, the first losing, the second gaining, because of the law just stated, the potential temperature of the colder body,  $b$ , suffers a greater increase than the decrease in potential temperature experienced by the warmer body,  $a$ , which was to be proven.

So also for imperfect gases the law of increase of potential temperature for natural processes can be established independently of the entropy principle. It is merely necessary to show that in general the adiabat is steeper than the isotherm or that the change in potential temperature varies inversely with the absolute temperature, when the volume remains constant.

Thus far it has appeared as though the potential temperature law might suffice equally as well as the entropy law. However, in all thermodynamic problems where the element of mass enters, the former law necessarily fails to give as complete a representation of the second law of thermodynamics as the entropy law. The entropy function is not alone a function of pressure and volume but also of mass whereas the potential temperature is independent of

the latter. Equation (8) shows likewise that the substitution of the obviously more convenient function — potential temperature — for entropy cannot be made in general. There are doubtless, however, a number of thermodynamic problems as was shown by von Bezold, as also in this paper, where the application of the potential temperature law may be found convenient. The main purpose of this paper, as above stated, has been to show the precise relationship between the two functions.