invariants (p. 67 of his paper). Here $(d) = 0$ certainly appears to involve $(f) = 0$; but this is only the case either if $\varpi_1^2 = \varpi_2^2$, or if $k = 0$. Neither of these conditions enables us to satisfy the fundamental property stated above; the true condition is not easily expressed in terms of the coefficients which are there employed; for, if the true condition be satisfied, $k = \infty$, while $k\varpi_1$, $k\varpi_2$ are finite.

The Theorem of Residuation, Noether's Theorem, and the Riemann-Roch Theorem. *By* F. S. MACAULAY. Received March 28th, 1899. Read April 13th, 1899.

The following paper is more in the nature of an essay than of a rigorous investigation. Its object is to advance and explain general notions rather than to give incontrovertible proofs of all the statements made. Sections I. and II. contain a discussion of the most general aspect of the Theorem of Residuation, and lead, in Section III., to an analytical and generalized interpretation of results previously deduced geometrically. (*Proc. Lond. Math. Soc.*, Vol. XXIX., pp. 673–695.)

The fundamental idea of the paper is a very simple one, viz., that the whole intersection of two given curves C_l, C_m at a common point A may be resolved into an equivalent aggregate, α, of simple points, no matter how complex the forms of the two curves at A may be. These α points are brought into evidence analytically by the fact that they supply α independent linear equations for the coefficients of a general algebraic curve of sufficiently high order. It is highly probable that the α equations could always be written in such a form, and arranged in such order, that each new equation, interpreted in connexion with those which precede it and apart from those which succeed it, expresses the condition that the curve passes through a new point, or, more strictly, possesses a property equivalent to that of passing through a new point. Such an arrangement is not, however, attempted *per se* in the paper.

The fundamental notion of a complex point being equivalent to an aggregate of simple points is in no sense a novel one; but its very simplicity has been considered as liable to lead to erroneous deductions. That there is, however, much inherent possibility of usefulness in the idea cannot reasonably be disputed. In particular, the resolution of a complex point into its equivalent simple points affords a means of viewing the theorem of residuation in its most general and extended aspect.

Several lengthy paragraphs of proof or explanation in Section I. have been relegated to footnotes, in order to obscure as little as possible the sequence of ideas.

I. *The General Theorem of Residuation.*

We assume as our starting point the fundamental theorem that if the whole intersection of two given algebraic plane curves C_l, C_m consists of *lm separate* points, then the equation of any other curve C_n which passes through these *lm* points is capable of being written in the form*

$$C_n \equiv C_l S_{n-l} + C_m S_{n-m} = 0.$$

Conversely, any curve whose equation is of this form passes through the *lm* points, as is evident.

If, however, C_l, C_m have a common multiple point at A (with or without contact), then, although they have at A, strictly speaking, but one common point, their whole intersection at A is equivalent to a certain definite number of simple points, which have become absorbed in the single point A. If we imagine an infinitesimal change imposed on the two curves C_l, C_m, by an infinitesimal variation of their coefficients† (including, if desirable, the adding of terms with infinitesimal coefficients beyond those of highest order in C_l, C_m), then the whole intersection of the two curves becomes in general changed to simple and separate points, the infinitesimally displaced curves having nowhere any absolute contact.

If now any curve obtained by a like infinitesimal change of C_n passes through all the separate points, we have at once

$$C_n \equiv C_l S_{n-l} + C_m S_{n-m},$$

since this identity will hold for the infinitesimally displaced curves, by our original theorem.

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* Noether, *Mathematische Annalen*, Vol. II., p. 314. In order to exclude apparent exceptions to the theorem it should be assumed that all the points of intersection of C_l, C_m are in the finite region. (See footnote, p. 18.)

† The method of infinitesimal variation in the coefficients is employed by Halphen (*l.c.*, footnote, p. 21). The method has been challenged, on insufficient ground, in my estimation, as lacking clearness and rigour.

Hence the two following statements, properly interpreted, are absolutely equivalent, and will be hereafter treated as such:—

(i.) *C_n passes through the whole intersection of C_l, C_m.*

(ii.) *C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$.*

We may also express the equivalence as follows:—

It cannot, in any valid sense, be said that C_n passes through the whole intersection of C_l, C_m unless C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$.

The meaning of statement (i.) is that there exists *one* pair of curves which are infinitesimal displacements of C_l, C_m and intersect wholly in separate points, to which corresponds *a* curve which is an infinitesimal displacement of C_n and passes through all the separate points. This being true of one such displacement of C_l, C_m is true of any, since it at once gives

$$C_n \equiv C_l S_{n-l} + C_m S_{n-m}.$$

The general theorem of residuation is contained, in an undeveloped or embryonic form, in the absolute equivalence of statements (i.) and (ii.) above. Hence an investigation of all the properties involved in the theorem of residuation may be made to rest on an investigation of the properties of the curves which are of the form $C_l S_{n-l} + C_m S_{n-m}$.*

* More definitely, it depends on an investigation of the conditions which must be satisfied by three unknown curves S, S', S'' in order that $CS + C'S' + C''S''$ may be identically zero, C', C'' being any two given curves, and C any curve of a given linear system, the conditions for S having to be independent of the arbitrary parameters involved in C (p. 20).

The necessary and sufficient number of independent linear equations that the coefficients of C_n must satisfy in order that C_n may be of the form $C_l S_{n-l} + C_m S_{n-m}$ (C_l, C_m *having no common factor*) is exactly lm if $n \geqslant l+m$, and

$$lm - \tfrac{1}{2}(l+m-n-1)(l+m-n-2)$$

if n is less than $l+m$ but not less than l or m. *This is true no matter how general or specialized the forms of C_l, C_m may be, subject to the condition mentioned*, and is proved in the *Proc. Lond. Math. Soc.*, Vol. XXVI., p. 503. One result of this theorem is that none of the equations among the coefficients of C_n which express the condition that C_n passes through the whole intersection of C_l, C_m are lost by virtue of the fact of any number of the points of intersection of C_l, C_m becoming absorbed in a single point.

In connexion with the above theorem there is, however, a paradox, which, like other geometrical paradoxes, leads to important consequences. If the curve C_n is degenerate, *i.e.*, if $C_n \equiv C_{n'} C_{n''}$, then the total number of independent equations may be very much reduced. The independent linear equations for the coefficients of C_n will not be linear in the coefficients of $C_{n'}$, $C_{n''}$, and will not be independent if C_l, C_m have any common multiple points. The reason is that, if C_l, C_m have a common multiple point at A, the form of their whole intersection at A may be assumed to some extent arbitrarily. If, for example, $C_{n'}$ be subject to the single condition of passing through A, C_l and C_m may be assumed to have i of their simple points of intersection at A on $C_{n'}$, i being the smaller of the two orders of the multiple points of C_l, C_m at A. On the other hand, if another curve pass through these i points, it will be subject to, not 1, but i, conditions, $C_{n'}$ being known. It is to be understood that $C_{n'}$ and $C_{n''}$ are mutually connected.

This method of procedure, simple as it may appear at first sight, has not, so far as I know, been anywhere elaborated. This may be regarded as a justification for attempting the imperfect elaboration which follows.

In the rest of this paper we shall assume that C_l, C_m have no common point at an infinitely great distance from the origin.* If this is not actually the case, we can choose a line, $ax+by=1$, which neither passes through, nor infinitely near to, any common point of C_l, C_m, and then linearly transform C_l, C_m by substituting $x' : y' : ax'+by'-1$ for $x : y : 1$. This ensures that the desired property shall hold for the transformed curves, and we may deal with these in the place of C_l, C_m.

Our next consideration is that of a curve C_n, or rather the general curve $C = \lambda' C' + \lambda'' C'' + \lambda''' C''' + \ldots$ of a given linear system C', C'', ..., which is not of the form $C_l S_{n-l} + C_m S_{n-m}$. We require an answer to the following question:—What is the number of the lm points of

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* Two of the many reasons for making this assumption are the following:—

(i.) In varying the coefficients, whereby coincident points are changed to separate points, it is sometimes necessary to add infinitesimal terms to C_l, C_m extending beyond the terms of highest order in C_l, C_m. When this is the case the infinitesimally displaced curves will have a common group of asymptotic points in addition to a common group of lm points corresponding to the whole intersection of C_l, C_m. When none of the lm points are asymptotic, the two groups of points are absolutely distinct, one group being entirely in the finite region, and the other being entirely in the infinite region. If, on the other hand, some of the lm points are asymptotic, the two sets of asymptotic points would have to be dissociated from one another, which might prove a troublesome matter.

(ii.) On p. 677 of Vol. XXIX. of the *Proc. Lond. Math. Soc.* the value found for ρ_p, as the number of arbitrary coefficients in u_p, is only valid if all the N points are in the finite region, a limitation which previously escaped notice. This interpretation of the value of ρ_p constitutes, in our method, the connecting link between the geometrical and analytical formulation of results (pp. 27, 28). Hence, in using this value for ρ_p, without modification, it is essential that none of the N points should escape to an infinite distance.

The disadvantage of the condition that C_l, C_m are to have no common asymptotic points is that the actual labour of any operations connected with them may be thereby materially increased.

The case in which two curves have no common asymptotic points bears a close analogy to the "simple" case of the intersection of two curves at a common multiple point which have no contact of branches. There is, however, this difference, that we know nothing about two given curves in the region of *absolute* infinity (as distinguished from the infinite region), whereas we do know the course of a given curve as it emerges from the infinitely small region surrounding a multiple point.

It is well to note that the n-ic excess of a given point-group N is that number of the N equations supplied by the point-group for a general n-ic which are identically satisfied, owing to the values of the coefficients of the terms of the n-ic beyond the n^{th} order being all zero. In other words, the n-ic excess of a point-group is that number of the points whose effect for an n-ic is lost or nugatory, owing to the fact that the form of any algebraic curve at absolute infinity is indeterminate.

intersection of C_l, C_m which the general curve C of the system may be said to pass through? If C passes through no multiple points common to C_l, C_m, the answer is of course obvious, although the case where C, C_l, C_m have all three contact of a higher order than the first has to be carefully treated. But, if C, C_l, C_m have one or more common multiple points, the answer is not obvious. The answer in this case depends on whether the given linear system C is defined by the geometrical conditions of passing through a given point-base,* or by the analytical conditions of being comprised in a given linear form. For analytical theory an answer may be given as follows:—

The maximum number of (simple) points which the general curve C of a given linear system C', C'', ... may be said to have in common with C_l, C_m is $N = lm - N'$, where N' is the number of independent linear equations which must be satisfied by the coefficients of a general polynomial S, of order not less than $l+m-2$, in order that CS may be of the form $C_l S' + C_m S''$, *i.e.*, in order that $C'S$, $C''S$, ... may one and all be of the form $C_l S' + C_m S''$.†

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* A point-base of a simple kind is defined in the footnote, *Proc. Lond. Math. Soc.*, Vol. XXIX., p. 676; but in this paper the term is used in its most general meaning. A set of a independent linear equations among the coefficients of a general polynomial S, giving to the curve S a property equivalent to that of passing through a simple points all situated at A, determines a *base-point* at A of *degree* a and *order* equal to the order of the multiple point which the curve S must have at A. *A base-point may thus be defined as a point-group collected at a point, and may have properties just as varied as those of a point-group in respect to order, degree, excess, and defect.* A *point-base* is made up of base-points, its degree being the sum of the degrees of its base-points, and its order being the order of the lowest curve which passes through it. If the base-points are all simple points finitely separated, the point-base is called a point-group.

Examples of point-bases (N, N') occur in this section, and examples of base-points in the next section. Base-points which determine no directions, much less curvatures, beyond those which are inherent in the specification of a higher singularity, may be called *simple* base-points. They are of two kinds:—(i.) the ordinary i-point ($i \geq 1$), of degree $\frac{1}{2}i(i+1)$ and order i, which gives an ordinary i-fold point to any curve, and (ii.) the k-point of degree $\Sigma\frac{1}{2}i(i+1)$ and order k (k being the greatest of the i's), specifying the component i-points of a higher singularity, with the directions and curvatures, &c., which determine the situations of the component i-points relative to the point itself.

† The reasoning by which this conclusion seems to be justified is as follows:— We imagine such infinitesimal changes to be made in C_l, C_m, C', C'', ... that the curves to which C_l, C_m are changed have lm separate points of intersection in the finite region, while the whole set of curves to which C_l, C_m, C', C'', ... are changed have the *greatest possible* number N of these lm separate points in common. It seems probable that the infinitesimal terms to be added to C_l, C_m need not extend beyond the terms of orders l and m; but the truth of this is not evident, and we therefore suppose that C_l, C_m are changed to $C_{l'}$, $C_{m'}$, where $l' \geq m' \geq m$. The curves $C_{l'}$, $C_{m'}$ therefore intersect in $l'm' - lm$ points in the infinite region, besides the lm points in the finite region. Taking now C', C'', ... to denote the curves to which the original C', C'', ... are changed, we may suppose, by continuing the

The points common to C_l, C_m through which C does not pass (N' in number) are the points through which S does pass, by virtue of the N' independent linear equations satisfied by the coefficients of S. The theorem of residuation treated analytically thus leads to the extremely difficult problem of determining the most general linear system S, of order not less than $l+m-2$, which satisfies the identity

$$CS \equiv C_l S' + C_m S''$$

for all values of the arbitrary parameters involved in C. Theoretically, however, S is absolutely determinate, and without ambiguity, since the determination depends only on the solution of linear equations. Also, having determined S, we can determine the most general system K, of order not less than $l+m-2$, which satisfies the identity

$$KS \equiv C_l S' + C_m S''$$

for all values of the arbitrary parameters involved in S.* The linear system K constitutes the "complete" system, through N, which contains the given system C. The equations for the coefficients of K will express the fact that K passes through the $N = lm - N'$ points common to C_l, C_m and the system C. If we substitute the coefficients

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infinitesimal terms of C', C'', ... far enough beyond their original terms of highest order, that C', C'', ... not only all pass through the N points, but also through the *whole* of the $l'm' - lm$ points; for the conditions of their passing through these affect only the coefficients of their terms of higher order, that is, terms with infinitesimal coefficients which may be chosen in any way desirable, these terms extending to an order as high as we please. The conditions that $C'S$, $C''S$, ... should each be of the form $C_{l'} S' + C_{m'} S''$ now only require that S should pass through the remaining $N' = lm - N$ of the lm points. The coefficients of S have then only to satisfy N' conditional equations. These equations are not only independent, but must continue to remain independent when all the infinitesimal parts of the coefficients of $C_{l'}$, $C_{m'}$, C', C'', ... become zero, provided only that S is of sufficiently high order. This number N' is therefore the irreducible minimum of independent equations which must exist for S, *i.e.*, it is the same as the number N' in the text. This proves our theorem.

Also the condition that KS should be of the form $C_{l'} S' + C_{m'} S''$ only requires that K should pass through the N points and the $l'm' - lm$ points. But, assuming K (like C', C'', ...) to have infinitesimal terms proceeding far enough, the conditional equations corresponding to the $l'm' - lm$ points will affect only the infinitesimal coefficients of K, and the only equations among the finite coefficients of K are those supplied by the N points. It is to be noticed that K and S are not mutually connected.

* The number of independent equations for the coefficients of K will be N. This is proved in the last footnote. If K were taken of less order than $l+m-2$, the number of independent equations might be less than N. See, however, the last paragraph of the paper, p. 30.

The fact that only $N = lm - N'$ equations have to be satisfied by the coefficients of K, notwithstanding that S has all but N' of its coefficients arbitrary, is a special property. This property ought to be capable of direct analytical proof; and the same remark applies to properties mentioned later, pp. 24–26.

of any curve of the system C for the coefficients of K in the N equations, we shall obtain a set of N identities among the coefficients of C_l, C_m, C', C'', ... which will not of course be linear in any of these coefficients.

If now we take C_m as base-curve, the two point-bases N, N' (footnote, p. 19) are residual, having C_l for their connecting curve; while N is the point-base of highest degree on C_m through which C_l, C', C'', ... all pass. The general system S through N' cuts C_m for the rest in the whole series of point-bases coresidual to N. So also the general system K through N (which includes the linear system C) cuts C_m for the rest in the whole series of point-bases residual to N. Finally any two curves of the systems K, S through N, N' cut C_m again in two residual point-bases for which S' is the connecting curve. This is the general theorem of residuation on the base-curve C_m.

II. *Noether's Fundamental Theorem.*

Denoting by C_l, C_m, C_n given non-homogeneous polynomials in two variables x, y, of orders l, m, n, and by S, S', &c., unknown polynomials to be chosen as desired, we may enunciate Noether's "fundamental theorem in the theory of Algebraic Functions" as follows:*—

The necessary and sufficient conditions that C_n may be capable of being written in the form

$$C_l S_{n-l} + C_m S_{n-m}$$

are that for each and every point of intersection $x = a$, $y = b$ of the two curves $C_l = 0$, $C_m = 0$ there should exist a curve

$$C_n - C_l S' - C_m S'' = 0$$

which has a t-fold point at $x = a$, $y = b$, the number t having any

* The following papers in the *Mathematische Annalen* directly discuss Noether's theorem:—

Vol. VI., 1873, pp. 351–359 (M. Noether); XXVII., 1886, pp. 527–536 (A. Voss); XXX., 1887, pp. 401–409 (L. Stickelberger); pp. 410–417 (Noether); XXXIV., 1889, pp. 447–449 (E. Bertini); pp. 450–453 (Noether); XXXIX., 1891, pp. 129–141 (A. Brill); XL., 1892, pp. 140–144 (Noether); XLII., 1893, pp. 601–604 (H. F. Baker).

One of the most interesting proofs of Noether's theorem is that by M. Halphen in the *Bulletin de la Société Mathématique de France*, Vol. V., 1877, pp. 160–163 (reproduced in Clebsch-Benoist, *Leçons sur la Géométrie*, Vol. II., 1880, pp. 49–51). Halphen, however, assumes a result which appears to require proof. This proof has been supplied by A. Berry in the *Proc. Lond. Math. Soc.*, Vol. XXX., pp. 271–276. (See also second footnote, p. 16.)

integral value not less than a certain minimum, which minimum depends on the character of the whole intersection of the two curves $C_l = 0$, $C_m = 0$ *at the point* $x = a$, $y = b$. S', S'' *may be different for different points.*

A short explanation will serve to make the theorem clear. In the first place the conditions of the theorem are obviously *necessary*, no matter how large t may be; for this is at once seen by taking $S' \equiv S_{n-l}$, $S'' \equiv S_{n-m}$. The only question then is as to the *sufficiency* of the conditions.

Consider the simplest example to which the theorem applies; viz., when C_l, C_m are single-branched and do not touch at the common point a, b, so that their whole intersection at a, b consists of one simple point. By taking $t = 1$, it is seen that the conditions of the theorem require that C_n should pass through the point a, b. And the conditions of the theorem require no more than this; for it can be easily proved, by taking the point a, b as origin and the tangents to C_l, C_m as axes of coordinates, that, provided only C_n passes through the origin, S', S'' can be so chosen that the curve

$$C_n - C_l S' - C_m S'' = 0$$

has a multiple point of any desired order (from 1 upwards) at the origin. In this simplest case of all the minimum value of t is therefore unity.

So, in the most general case, however complex the character of the whole intersection of C_l, C_m at the point a, b may be, Noether proves it to be sufficient, in order to know that C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$, that S', S'' can be found such that the curve

$$C_n - C_l S' - C_m S'' = 0$$

has a multiple point of sufficiently high order at a, b, with similar conditions for each point of intersection of the curves C_l, C_m.

The conditions of the theorem require that for every point of intersection of C_l, C_m there should exist two curves S', S'' such that C_n *is the same as* $C_l S' + C_m S''$ *to any degree of approximation;* and, this being so, the condition is satisfied for *every* point in the plane. Looked at in this light the significance of the theorem is readily comprehended.

The only modifications that have been made in the theorem since Noether first gave it (*l.c.*, Vol. VI.) relate to the determination of the minimum value of t. The knowledge of the minimum is of some interest, but has not yet been proved to be of any essential importance, except in the two cases (ii.) and (iv.) below.

(i.) The number t need not exceed the number, α, of simple points to which the whole intersection of C_l, C_m at a, b is equivalent (Brill, *l.c.*). This is the minimum value of t if, and only if, one at least of the two curves C_l, C_m is single-branched at the point a, b; but in this case simpler conditions can be substituted for those of the theorem, viz., that C_n should have contact of order $\alpha-1$ at a, b with the single-branched curve.

(ii.) If C_l, C_m have respectively i-fold and j-fold points at a, b, and have no contact (so that ij is the number of simple points to which their whole intersection at a, b is equivalent), the minimum value of t is $i+j-1$ (Noether, *l.c.*, Vol. VI.).

This is called the "simple" case.*

(iii.) If C_l, C_m have i-fold and j-fold points at a, b, and their whole intersection at a, b is equivalent to $ij+\beta$ simple points, the minimum value of t does not exceed $i+j+\beta-1$. (Bertini, *l.c.*)

(iv.) If C_l, C_m have multiple points of higher singularity at a, b which can be resolved into ordinary multiple points (including ordinary cusps) common to C_l, C_m, of which any one pair of corresponding components is i-fold for C_l and j-fold for C_m, with the corresponding whole intersection equivalent to ij simple points, then it is sufficient that $C_n - C_l S' - C_m S''$ should have a multiple point of higher singularity at a, b whose corresponding component is of order $i+j-1$. (Noether, *l.c.*, Vol. XXXIV.)

As regards its application to the theorem of residuation Noether's theorem seems open to criticism. Noether possibly did not regard his theorem from this point of view when he first gave it, but subsequently both he and others have so regarded it. It should at least be made clear that the conditions in the theorem supply a *theoretical* rather than a *practical* test; but the direct contrary seems to be implied in much that is written on the subject. Although a great step towards the general theorem of residuation, it does not advance the whole way. The theorem only gives us a test for answering the question whether C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$ or not, whereas, as we have seen in Section I., we want to know the conditions which S must satisfy in order that CS (C being partially or wholly

* The "simple" case requires only that the two curves have no contact at the common multiple point. The two multiple points may be of any kind of singularity provided this condition holds.

given) may be of the form $C_l S' + C_m S''$. The latter question includes the former, but the former does not include the latter.

It will be seen that Noether's theorem supplies us with a sufficiency of *local* tests for deciding, in the most general case, whether C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$ or not; or rather, as we should prefer to say, it provides a local test for deciding whether C_n does or does not pass through the whole intersection of C_l, C_m at a common multiple point. The local tests are quite independent of one another; and, if satisfied at every point of intersection of C_l, C_m, the final result follows that C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$. We proceed to explain how much is theoretically required for the satisfying of Noether's tests.

The number of simple points to which the whole intersection of C_l, C_m at a common multiple point is equivalent may be found by taking the multiple point as origin and equating coefficients in the identity

$$C_l S' + C_m S'' \equiv \Sigma,$$

where Σ, S', S'' are general ordinary power series, *i.e.* unterminated series arranged in ascending positive integral powers of x, y. The number of independent linear equations that result for the coefficients of Σ alone is the number of simple points required.*

For the equating of coefficients, write the identity

$$C_l S' + C_m S'' \equiv \Sigma$$

in the form

$$C_l (u_0' + u_1' + u_2' + \ldots) + C_m (u_0'' + u_1'' + u_2'' + \ldots) \equiv u_0 + u_1 + u_2 + \ldots,$$

where u_p, as usual, stands for a homogeneous polynomial in x, y of order p. Arrange the equations in sets, the p^{th} set coming from the equating of coefficients of terms of order $p-1$. The $(p+1)^{\text{th}}$ set of equations will involve the coefficients of u_p, u'_{p-i}, u''_{p-j} (i, j being the orders of the multiple points of C_l, C_m at the origin) together with the coefficients of Σ, S', S'' which have appeared in the p previous

* Assuming that the number of equations, a_1, for the coefficients of Σ does not exceed the number, a, of the simple points of intersection of C_l, C_m which are absorbed at the origin, it can be proved that $a_1 = a$ as follows:—If we substitute for Σ a general polynomial S_n of sufficiently high order $n \geq l+m-2$ (S', S'' being still power series), the equations for the coefficients of S_n will be a_1 in number; and, if we add to these all the other equations for S_n corresponding to the other points of intersection of C_l, C_m, we shall obtain a total of Σa_1 equations, *i.e.* a number of equations not exceeding Σa or lm. But these equations, in their totality, require S_n to be of the form $C_l S_{n-l} + C_m S_{n-m}$, by Noether's theorem, and are therefore equivalent to lm independent equations (footnote, p. 17). Hence Σa_1 is not less than lm, and assuming it is not greater than lm, from above, we have $\Sigma a_1 = lm = \Sigma a$; therefore $a_1 = a$, $\beta_1 = \beta$, &c.

sets of equations. If the $(p+1)^{th}$ set is the first which does not supply any new equations for the coefficients of Σ alone, that is, if the first $p+1$ sets are such that the coefficients of S', S'' cannot all be eliminated so as to result in an equation which involves the coefficients of u_p, *then, and not till then, the equating of coefficients may stop.* From the first p sets the coefficients of S', S'' may then be eliminated so as to give all the equations which hold for the coefficients of Σ alone. It goes without saying that this method cannot in general be carried out practically.

The number p is the minimum value of t mentioned in the enunciation of Noether's theorem. It seems impracticable, and of no great consequence, to find a simple and general analytical formula for it.

The above is an extension to the general case of a method (or illustration) employed by Noether for the "simple" case in the *Math. Ann.*, Vol. VI. Noether assumes without proof that the equations, taken in sets, actually determine the coefficients of S', S'' in terms of those of Σ. This is true in the "simple," but not in the general, case. For example, if

$$C_l \equiv y^2 + ax^3 + \ldots, \qquad C_m \equiv y^2 + bx^3 + \ldots,$$

the first two sets of the equations only involve u_0 and the coefficients of u_1, which are zero; the third set brings in u_0' and u_0'', but does not determine both; the fourth set determines u_0' and u_0'', but not the newly introduced coefficients of u_1' and u_1''; while the fifth set is the first which does not supply any new equations for the coefficients of Σ alone. The minimum value of t is 4 in this example.

By determining all the independent equations for the coefficients of Σ, and making them all to hold when $C'S, C''S, C'''S, \ldots$ are substituted for Σ, we obtain all the independent equations which must hold for the coefficients of S; and we thus determine the most general system S such that any curve of the system CS passes through the whole intersection of C_l, C_m at the origin. The number of independent equations for the coefficients of S will always be less than the number for Σ, provided all the curves of the system C pass through the origin. The difference gives the number of simple points common to C_l, C_m at the origin through which the general curve of the system C may be supposed to pass. Also the difference will be the number of independent equations that must hold for the coefficients of a general polynomial K in order that KS may be of the form Σ, or $C_l S' + C_m S''$, to the necessary degree of approximation at the origin. In other words, the number of independent equations for the coefficients of K together with the number of independent equations for the coefficients of S will be equal to the number of independent equations for the coefficients of Σ, or the number of simple points to which the whole intersection of C_l, C_m at the origin is equivalent. From these systems S and K, which satisfy local conditions at one place only, we can proceed to those systems S and K which satisfy the like conditions at as many places as we please, and, in particular, at all places, as on p. 20.

The theorem of residuation is, however, essentially of a geometrical character; and the problems it suggests are not likely to be completely solved without the free use of geometrical methods. We can solve, by the aid of geometry, the problem of the determination of the general linear systems S and K for (at least) that case in which the linear system C reduces to a single fixed curve. I hope to prove this, and other statements made in this section, in a later paper.

III. *The Generalized Riemann-Roch Theorem.*

By the generalized Riemann-Roch theorem we mean the theorem given on pp. 526, 527 of Vol. XXVI. of the *Proc. Lond. Math. Soc.*, which is restated in a more convenient form at the beginning of the footnote on p. 688 of Vol. XXIX. The proof in Vol. XXVI. holds for the general case if the theorem of residuation is assumed to hold generally. The theorem is applied below to the general case.

We suppose the general curve C of the given linear system C', C'', ... to have no fixed constituent. We may then choose two curves C_l, C_m of the system, whose orders are as low as possible, so as to have no common constituent; and we assume their whole intersection to lie in the finite region.

We have explained in Section I. that the least degree N' of the whole point-base common to C_l, C_m which is not common to all the members of the linear system is the number of independent linear equations that must be satisfied by the coefficients of a general polynomial S, of order $l+m-2$, in order that the identity

$$CS \equiv C_l S' + C_m S''$$

may hold for all values of the arbitrary parameters involved in C. It may be observed in passing that the identity of the polynomials CS and $C_l S' + C_m S''$ is a very different thing from the identity of the same expressions when, as in Section II., S, S', S'' are unterminated power series. The latter case only requires the equating of coefficients from the beginning, arriving at a stage where it may stop; but the former case requires the equating of coefficients to the end bringing no new unknown coefficients into the new equations after a certain time.

We imagine now that all the coefficients of S', S'' have been eliminated and that we have all the resulting equations for the coefficients of S.

We suppose the N' equations for the coefficients of S to be solved in the following way:—Suppose $S \equiv u_0+u_1+u_2+\ldots+u_{l+m-2}$. The coefficient u_0 may be determined in terms of the remainder from one of the equations, and its value substituted in the rest, so that we get a new set of $N'-1$ equations from which u_0 has been eliminated. From this new set of equations the coefficients of u_1 may be determined, and their values substituted in the rest of the equations. From the new set of equations the coefficients of u_2 may be determined, and their values substituted in the rest of the equations; and so on. Suppose that all the coefficients of $u_0+u_1+\ldots+u_{p-1}$ are determinable in this manner, but that all the coefficients of u_p cannot be so determined. This last must happen if $N' < \frac{1}{2}(p+1)(p+2)$; and it may also happen if $N' \geqslant \frac{1}{2}(p+1)(p+2)$.

The fact that all the coefficients of $u_0+u_1+\ldots+u_{p-1}$ are determinable in terms of the remaining coefficients of S accounts for $\frac{1}{2}p(p+1)$ of the N' equations, so that there are $N'-\frac{1}{2}p(p+1)$ equations among the coefficients of $u_p+\ldots+u_{\ +m-2}$. Also, if all the coefficients of u_p are not determinable in terms of the coefficients of $u_{p+1}+\ldots+u_{l+m-2}$, then there are a certain number ρ_p of the coefficients of u_p which are arbitrary;* while there will be $N'+\rho_p-\frac{1}{2}(p+1)(p+2)$ equations among the coefficients of $u_{p+1}+\ldots+u_{l+m-2}$. Solving this new set of equations for the coefficients of u_{p+1}, it will be found that there are a certain number ρ_{p+1} of these which are arbitrary, where $\rho_{p+1} > \rho_p$. *The limits of possibility of the value of* ρ_{p+1} *are* ρ_p+1 *and* $p+2$. Proceeding in this method of solving, the whole N' equations will in time become exhausted (say) when a certain number of the

* The way in which this may happen, even when $N' \geqslant \frac{1}{2}(p+1)(p+2)$, is that some of the coefficients of u_p cannot be determined separately, but only in sets of two or more. A set being determined may be said to determine one in the set, whichever one we like, leaving the rest in the set arbitrary. If the elimination of all the coefficients of u_p should result in the complete disappearance of some of the coefficients of $u_{p+1}+\ldots+u_{l+m-2}$ from the $N'+\rho_p-\frac{1}{2}(p+1)(p+2)$ equations, such coefficients would be arbitrary.

coefficients of u_q have been determined in terms of the arbitrary coefficients of u_q, and the coefficients of $u_{q+1}+\ldots+u_{l+m-2}$, which last are therefore *all* arbitrary. We ought then to have

$$N'+\rho_p+\rho_{p+1}+\ldots+\rho_q=\tfrac{1}{2}(q+1)(q+2),$$

and
$$q+1>\rho_q>\rho_{q-1}>\ldots>\rho_p>0.$$

The N' equations become exhausted as soon as $\rho_{q+1}=q+2$. This would not, however, be the case if C_l, C_m had any asymptotic points in common, through which the general curve C of the linear system did not pass; for then there would be equations in which only the coefficients of the terms of highest order in S would be involved, even if the order of S exceeded $l+m-2$ by any amount, and very possibly also other equations in which only the coefficients of the terms of the two highest orders in S would be involved, and so on.

The lowest curve through the point-base N' is of order p, since in u_p there are one or more arbitrary coefficients; and the order of the point-base is therefore p,* and its degree N'. It must not, however, be supposed that N' is the simplest point-base derivable from N. This last would be found by drawing the two curves of lowest order through N, having no common constituent, to intersect again. The order of the point-base N, which is the order of the lowest curve through N, is given below; but we have no theorem at present which determines with certainty the order of the other lowest curve. The orders, degrees, and forms of the several base-points of N' may be found by the methods of Section II., by transferring the origin to each base-point in succession.

We take now the numbers

$$\ldots\ 0,\ 0,\ \rho_p,\ \rho_{p+1},\ \ldots\ \rho_{q-1},\ \rho_q,\ q+2,\ q+3,\ \ldots,$$

which are the differences of the successive defects of the point-base N'; and write down their successive differences, or second differences of the defects, viz.

$$\ldots\ 0,\ 0,\ \delta_p,\ \delta_{p+1},\ \ldots\ \delta_{q-1},\ \delta_q,\ \delta_{q+1},\ 1,\ 1,\ 1,\ \ldots\ .$$

This series of numbers constitutes the *characterization* of the point-base N', which we express by dropping the zeros at the beginning

* The order p of the point-base N' cannot exceed, but may be less than, the smaller of the two numbers l, m, since these are the orders of two curves passing through the point-base.

and units at the end, changing the suffixes to $1, 2, \ldots a$, and writing

$$N' = (\delta_1, \delta_2, \ldots \delta_a),$$

where $$p+a = q+2 = \Sigma\delta, \quad \delta_a > 1.$$

The generalized Riemann-Roch theorem then gives us the characterization of N, which is as follows (assuming $m \geqslant l$) :*—

(i.) $N = (1^{m-l}, 2^{l-p-a}, \delta_a+1, \delta_{a-1}+1, \ldots \delta_1+1)$, if $l \geqslant p+a \ (= \Sigma\delta)$;

(ii.) $N = (1^{m-p-a}, \delta_a, \ldots \delta_{l-p+1}, \delta_{l-p}+1, \ldots \delta_1+1)$, if $p < l < p+a \leqslant m$;

(iii.) $N = (\delta_a-1, \ldots \delta_{m-p+1}-1, \delta_{m-p}, \ldots \delta_{l-p+1}, \delta_{l-p}+1, \ldots \delta_1+1)$,

if $p < l \leqslant m < p+a$;

(iv.) $N = (1^{m-l-a}, \delta_a, \delta_{a-1}, \ldots \delta_1)$, if $p = l$, $l+a \leqslant m$;

(v.) $N = (\delta_a-1, \ldots \delta_{m-l+1}-1, \delta_{m-l}, \ldots \delta_1)$, if $p = l$, $m < l+a$.

The symbols 1^{m-l}, 2^{l-p-a} in (i.) stand for 1 repeated $m-l$ times, followed by 2 repeated $l-p-a$ times. Similarly for 1^{m-p-a} in (ii.) and 1^{m-l-a} in (iv.). Cases (iv.) and (v.) are the simplified forms of (ii.) and (iii.) when $p = l$, and may be still further simplified (if $\delta_1 = 1$) by the omission of any units at the end.

The order of N' is p, which is the value of $\Sigma(\delta-1)$. The order of N is the value of $\Sigma(\delta-1)$ for N, *i.e.*, it is l in cases (i.), (ii.), (iv.), $l+m-p-a$ in case (iii.), and $m-a$ in case (v.).

The lowest order that can be chosen for S so that the equations among its coefficients may determine the point-base N' without ambiguity is, I think, $q+1 = \Sigma\delta-1$, which would leave the coefficients of the terms of highest order in S entirely arbitrary. It would be of great importance to find methods for determining the value of $q+1$, so that S might be taken of order $q+1$, instead of order $l+m-2$. The order $q+1$ will be sufficiently high if the whole system of curves of order $q+1$, whose coefficients satisfy the N' equations, could not have a point-base in common of higher degree than N'. This property holds in what appears to be the most unlikely case, viz., the linear system of curves determined by x^n, $x^{n-1}y$, $x^{n-2}y^2$, ... y^n have no point-base in common of higher degree than

* In this application we regard N', and consequently $\delta_1, \delta_2, \ldots \delta_a$, as known. The numbers p, q, a, l, m are then also known, viz., $p = \Sigma(\delta-1)$, $q = \Sigma\delta-2$, a is the number of the δ's ($\delta_a > 1$), and l, m are the orders of the two curves drawn through N' and intersecting for the rest in N.

$\frac{1}{2}n(n+1)$. Hence, for the complete specification of the point-bases N' and N, it appears that we may choose for the order of S the value $q+1$, or $\Sigma\delta-1$; and for the order of K the value of $\Sigma\delta-1$ for N, *i.e.*, $l+m-p-1$ in cases (i.), (ii.), (iii.), $m-1$ in (iv.) and (v.) if $\delta_1 > 1$, and a lower order in (iv.), and (v.) if $\delta_1 = 1$, since then we should omit all units at the end of (iv.) and (v.), and the value of $\Sigma\delta-1$ for N would diminish accordingly. The general algebraic curve through N' is $PS+P'S'+\ldots = 0$, where $S, S', \ldots$ are polynomials of order $q+1$ whose coefficients satisfy the N' equations, and $P, P', \ldots$ are polynomials with arbitrary coefficients. Similarly for the general curve through N.

Concerning the Four Known Simple Linear Groups of Order 25920, *with an Introduction to the Hyper-Abelian Linear Groups.* *By* Dr. L. E. Dickson.

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Introduction.

In a paper* giving a *résumé* of the known systems of simple groups and a table of the orders of all known simple groups not exceeding one million, I find that, apart from the order 25920, every case in which two or more simple groups of the same order exist has been completely investigated as to their simple isomorphism or non-isomorphism. The greater part of the present paper deals with the four known simple groups of order 25920, viz.,†

(1) The simple group $A(4, 3)$, defined by the decomposition of the Abelian group on four indices taken modulo 3.

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* "The Known Finite Simple Groups," *Bulletin of the American Mathematical Society*, July, 1899.

† [*Note of August* 14*th*.—I should have referred to a number of important investigations in which occur substitution-groups and groups of collineations isomorphic with the above groups of order 25920. Jordan (*Traité des Substitutions*, pp. 316–329) shows that the Galois group of the equation for the 27 lines on a general cubic surface has the order 2·25920, and proves (pp. 365–369) that it is isomorphic with the Abelian group for the trisection of hyperelliptic functions of four periods. A proof involving less calculation has been given by the writer (*Comptes Rendus*, Vol. CXXVIII., p. 873, 1899). The present article was written