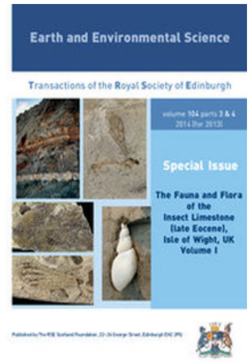

I.—*On Reciprocal Figures, Frames, and Diagrams of Forces*

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TRANSACTIONS.

I.—*On Reciprocal Figures, Frames, and Diagrams of Forces.* By J. CLERK MAXWELL, F.R.SS. L. & E. (Plates I. II. III.)

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Two figures are reciprocal when the properties of the first relative to the second are the same as those of the second relative to the first. Several kinds of reciprocity are known to mathematicians, and the theories of Inverse Figures and of Polar Reciprocals have been developed at great length, and have led to remarkable results. I propose to investigate a different kind of geometrical reciprocity, which is also capable of considerable development, and can be applied to the solution of mechanical problems.

A Frame may be defined geometrically as a system of straight lines connecting a number of points. In actual structures these lines are material pieces, beams, rods, or wires, and may be straight or curved; but the force by which each piece resists any alteration of the distance between the points which it joins acts in the straight line joining those points. Hence, in studying the equilibrium of a frame, we may consider its different points as mutually acting on each other with forces whose directions are those of the lines joining each pair of points. When the forces acting between the two points tend to draw them together, or to prevent them from separating, the action along the joining line is called a Tension. When the forces tend to separate the points, or to keep them apart, the action along the joining line is called a Pressure.

If we divide the piece joining the points by any imaginary section, the resultant of the whole internal force acting between the parts thus divided will be mechanically equivalent to the tension or pressure of the piece. Hence, in order to exhibit the mechanical action of the frame in the most elementary manner, we may draw it as a skeleton, in which the different points are joined by straight lines, and we may indicate by numbers attached to these lines the tensions or pressures in the corresponding pieces of the frame.

The diagram thus formed indicates the state of the frame in a way which is geometrical as regards the position and direction of the forces, but arithmetical as regards their magnitude.

But, by assuming that a line of a certain length shall represent a force of a certain magnitude, we may represent every force completely by a line. This is done in Elementary Statics, where we are told to draw a line from the point of application of the force in the direction in which the force acts, and to cut off as many units of length from the line as there are units of force in the force, and finally to mark the end of the line with an arrow-head, to show that it is a force and not a piece of the frame, and that it acts in that direction and not the opposite.

By proceeding in this way, we should get a system of arrow-headed forces superposed on the skeleton of the frame, two equal and opposite arrows for every piece of the frame.

To test the equilibrium of these forces at any point of concourse, we should proceed by the construction of the parallelogram of forces, beginning with two of the forces acting at the point, completing the parallelogram, and drawing the diagonal, and combining this with the third force in the same way, till, when all the forces had been combined, the resultant disappeared. We should thus have to draw three new lines, one of which is an arrow, in taking in each force after the first, leaving at last not only a great number of useless lines, but a number of new arrows, not belonging to the system of forces, and only confusing to any one wishing to verify the process.

To simplify this process, we are told to construct the Polygon of Forces, by drawing in succession lines parallel and proportional to the different forces, each line beginning at the extremity of the last. If the forces acting at the point are in equilibrium, the polygon formed in this way will be a closed one.

Here we have for the first time a true Diagram of Forces, in which every force is not only represented in magnitude and direction by a straight line, but the equilibrium of the forces is manifest by inspection, for we have only to examine whether the polygon is closed or not. To secure this advantage, however, we have given up the attempt to indicate the position of the force, for the sides of the polygon do not pass through one point as the forces do. We must, therefore, give up the plan of representing the frame and its forces in one diagram, and draw one diagram of the frame and a separate diagram of the forces. By this method we shall not only avoid confusion, but we shall greatly simplify mechanical calculations, by reducing them to operations with the parallel ruler, in which no useless lines are drawn, but every line represents an actual force.

A Diagram of Forces is a figure, every line of which represents in magnitude and direction the force acting along a piece of the frame.

To express the relation between the diagram of the frame and the diagram of forces, the lines of the frame should each be indicated by a symbol, and the corresponding lines of the diagram of forces should be indicated by the same symbol, accented if necessary.

We have supposed the corresponding lines to be parallel, and it is necessary that they should be parallel when the frame is not in one plane; but if all the pieces of the frame are parallel to one plane, we may turn one of the diagrams round a right angle, and then every line will be perpendicular to the corresponding line.

If any number of lines meet at the same point in the frame, the corresponding lines in the diagram of forces form a closed polygon.

It is possible, in certain cases, to draw the diagram of forces so that if any number of lines meet in a point in the diagram of forces, the corresponding lines in the frame form a closed polygon.

In such cases, the two diagrams are said to be reciprocal in the sense in which we use it in this paper. If either diagram be taken as representing the frame, the lines of the other diagram will represent a system of forces which, if applied along the corresponding pieces of the frame, will keep it in equilibrium.

The properties of the "triangle" and "polygon" of forces have been long known, and a "diagram" of forces has been used in the case of the "funicular polygon," but I am not aware of any more general statement of the method of drawing diagrams of forces before Professor RANKINE applied it to frames, roofs, &c., in his "Applied Mechanics," p. 137, &c. The "polyhedron of forces," or the proposition that forces acting on a point perpendicular and proportional to the areas of the faces of a polyhedron are in equilibrium, has, I believe, been enunciated independently at various times, but the application of this principle to the construction of a diagram of forces in three dimensions was first made by Professor RANKINE in the "Philosophical Magazine," Feb. 1864. In the "Philosophical Magazine" for April 1864, I stated some of the properties of reciprocal figures, and the conditions of their existence, and showed that any plane rectilinear figure which is a perspective representation of a closed polyhedron with plane faces has a reciprocal figure. In Sept. 1867, I communicated to the British Association a method of drawing the reciprocal figure, founded on the theory of reciprocal polars.

I have since found that the construction of diagrams of forces in which each force is represented by one line, had been independently discovered by Mr W. P. TAYLOR, and had been used by him as a practical method of determining the forces acting in frames for several years before I had taught it in King's College, or even studied it myself. I understand that he is preparing a statement of the application of the method to various kinds of structures in detail, so that it can be made use of by any one who is able to draw one line parallel to another.

Professor FLEEMING JENKIN, in a paper recently published by the Society, has fully explained the application of the method to the most important cases occurring in practice.

In the present paper I propose, first, to consider plane diagrams of frames and of forces in an elementary way, as a practical method of solving questions about the stresses in actual frameworks, without the use of long calculations.

I shall then discuss the subject in a theoretical point of view, and give a method of defining reciprocal diagrams analytically, which is applicable to figures either of two or of three dimensions.

Lastly, I shall extend the method to the investigation of the state of stress in a continuous body, and shall point out the nature of the function of stress first discovered by the Astronomer Royal for stresses in two dimensions, extending the use of such functions to stresses in three dimensions.

On Reciprocal Plane Rectilinear Figures.

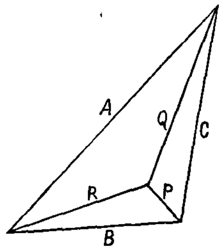
Definition.—Two plane rectilinear figures are reciprocal when they consist of an equal number of straight lines, so that corresponding lines in the two figures are at right angles, and corresponding lines which meet in a point in the one figure form a closed polygon in the other.

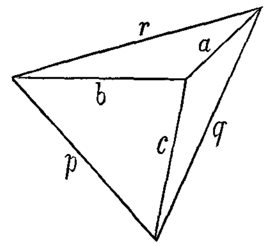
Note.—It is often convenient to turn one of the figures round in its own plane 90°. Corresponding lines are then parallel to each other, and this is sometimes more convenient in comparing the diagrams by the eye.

Since every polygon in the one figure has three or more sides, every point in the other figure must have three or more lines meeting in it. Since every line in the one figure has two, and only two, extremities, every line in the other figure must be a side of two, and only two, polygons. If either of these figures be taken to represent the pieces of a frame, the other will represent a system of forces such that, these forces being applied as tensions or pressures along the corresponding pieces of the frame, every point of the frame will be in equilibrium.

The simplest example is that of a triangular frame without weight, ABC, jointed at the angles, and acted on by three forces, P, Q, R, applied at the angles. The directions of these three forces must meet in a point, if the frame is in equilibrium. We shall denote the lines of the figure by capital letters, and those of the reciprocal figure by the corresponding small letters; we shall denote points by the lines which meet in them, and polygons by the lines which bound them.

Here, then, are three lines, A, B, C, forming a triangle, and three other lines, P, Q, R, drawn from the angles and meeting in a point. Of these forces let that along P be given. Draw the first line p of the reciprocal diagram parallel to P, and of a length representing, on any convenient scale, the force along P. The forces along P, Q, R are in equilibrium, therefore, if from one extremity of p we draw q parallel to Q, and from the other extremity r parallel to R, so as to form a triangle pqr, then q and r will represent on the same scale the forces along Q and R.





To determine whether these forces are tensions or pressures, make a point travel along p in the direction in which the force in P acts on the point of concourse of PQR, and let the point travel in the same direction round the polygon pqr. Then, the direction in which the point travels along any side of the polygon will be the direction in which the force acts along the corresponding piece of the frame on the point of concourse. If it acts from the point of concourse, the force is a tension; if towards it, it is a pressure.

The other extremity of P meets B and C, and the forces along these three pieces are in equilibrium. Hence, if we draw a triangle, having p for one side and lines parallel to B and C for the others, the sides of this triangle will represent the three forces.

Such a triangle may be described on either side of p, the two together would form a parallelogram of forces; but the theory of reciprocal figures indicates that only one of these triangles forms part of the diagram of forces.

The rule for such cases is as follows :—Of the two extremities of p, one corresponds to the closed figure PRB, and the other to the closed figure PQC, these being the polygons of which P is a side in the first figure.

We must, therefore, draw b parallel to B from the intersection of p and r, and not from the other extremity, and we must draw c parallel to C from the intersection of p and q.

We have now a second triangle, pbc, corresponding to the forces acting at the point of concourse of P, B, C. To determine whether these forces are tensions or pressures, we must make a point travel round pbc, so that its course along p is in the opposite direction to its course round pqr, because the piece P acts on the points PBC and PQR with equal and opposite forces.

If we now consider the equilibrium of the point of concourse of QC and A, we shall find that we have determined two of these forces by the lines q and c, and that the third force must be represented by the line a which completes the triangle qca.

We have now constructed a complete diagram of forces, in which each force is represented by a single line, and in which the equilibrium of the forces meeting at any point is expressed visibly by the corresponding lines in the other figure forming a closed polygon.

There are in this figure six lines, having four points of concourse, and forming four triangles. To determine the direction of the force along a given line at any point of concourse, we must make a point travel round the corresponding polygon in the other figure in a direction which is positive with respect to that polygon. For this purpose it is desirable to name the polygons in a determinate order of their sides, so arranged that, when we arrive at the same side in naming the two polygons which it divides, we travel along it in opposite directions. For instance, if *pqr* be one of the polygons, the others are *pbc*, *qca*, *rab*.

Note.—It may be observed, that after drawing the lines *p*, *q*, *r*, *b*, *c* with the parallel ruler, the line *a* was drawn by joining the points of concourse of *q*, *r* and *b*, *c*; but, since it represents the force in A, *a* is parallel to A. Hence the following geometrical theorem :—

If the lines PQR, drawn from the angles of the triangle ABC, meet in a point, then if *pqr* be a triangle with its corresponding sides parallel to P, Q, R, and if *a*, *b*, *c* be drawn from its corresponding angles parallel to A, B, C, the lines *a*, *b*, *c* will meet in a point.

A geometrical proof of this is easily obtained by finding the centres of the four circles circumscribing the triangles ABC, AQR, BRP, CPQ, and joining the four centres thus found by six lines.

These lines meet in the four centres, and are perpendicular to the six lines, A, B, C; P, Q, R; but by turning them round 90° they become parallel to the corresponding lines in the original figure.

The diagram formed in this way is definite in size and position, but any figure similar to it is a reciprocal diagram to the original figure. I have explained the construction of this, the simplest diagram of forces, more at length, as I wish to show how, after the first line is drawn and its extremities fixed on, every other line is drawn in a perfectly definite position by means of the parallel ruler.

In any complete diagram of forces, those forces which act at a given point in the frame form a closed polygon. Hence, there will be as many closed polygons in the diagram as there are points in the frame. Also, since each piece of the frame acts with equal and opposite forces on the two points which form its extremities, the force in the diagram will be a side of two different polygons. These polygons might be drawn in any positions relatively to each other; but, in the diagrams here considered, they are placed so that each force is represented by one line, which forms the boundary between the two polygons to which it belongs.

If we regard the polygons as surfaces, rather than as mere outlines, every polygon will be bounded at every point of its outline by other polygons, so that the whole assemblage of polygons will form a continuous surface, which must either be an infinite surface or a closed surface.

The diagram cannot be infinite, because it is made up of a finite number of finite lines representing finite forces. It must, therefore, be a closed surface returning on itself, in such a way that every point in the plane of the diagram either does not belong to the diagram at all, or belongs to an even number of sheets of the diagram.

Any system of polygons, which are in contact with each other externally, may be regarded as a sheet of the diagram. When two polygons are on the same side of the line, which is common to them, that line forms part of the common boundary of two sheets of the diagram. If we reckon those areas positive, the boundary of which is traced in the direction of positive rotation round the area, then all the polygons in each sheet will be of the same sign as the sheet, but those sheets which have a common boundary will be of opposite sign. At every point in the diagram there will be the same number of positive as of negative sheets, and the whole area of the positive sheets will be equal to that of the negative sheets.

The diagram, therefore, may be considered as a plane projection of a closed polyhedron, the faces of the polyhedron being surfaces bounded by rectilinear polygons, which may or may not, as far as we yet know, lie each in one plane.

Let us next consider the plane projection of a given closed polyhedron. If any of the faces of this polyhedron are not plane, we may, by drawing additional lines, substitute for that face a system of triangles, each of which is necessarily in a plane. We may, therefore, consider the polyhedron as bounded by plane faces. Every angular point of this polyhedron will be defined by its projection on the plane and its height above it.

Let us now take a fixed point, which we shall call the origin, and draw from it a perpendicular to the plane. We shall call this line the axis. If we then draw from the origin a line perpendicular to one of the faces of the polyhedron, it will cut the plane at a point which may be said to correspond to the projection of that face. From this point draw a line perpendicular to the plane, and take on this line a point whose distance from the plane is equal to that of the intersection of the axis with the face of the polyhedron produced, but on the other side of the plane. This point in space will correspond to the face of the polyhedron. By repeating this process for every face of the polyhedron, we shall find for every face a corresponding point with its projection on the plane.

To every edge of the polyhedron will correspond the line which joins the points corresponding to the two faces which meet in that edge. Each of these lines is perpendicular to the projection of the other; for the perpendiculars from the origin to the two faces, lie in a plane perpendicular to the edge in which they meet, and the projection of the line corresponding to the edge is the intersection of this plane with the plane of projection. Hence, the edge is perpendicular to the projection of the corresponding line. The projection of the edge is therefore perpendicular to the projection of the corresponding line, and therefore to the corresponding line itself. In this way we may draw a diagram on the plane of projection, every line of which is perpendicular to the corresponding line in the original figure, and so that lines which meet in a point in the one figure form a closed polygon in the other.

If, in a system of rectangular co-ordinates, we make $z=0$ the plane of projection, and $x=0$ $y=0$ $z=-c$ the fixed point, then if the equation of a plane be

$$z = \mathrm{A}x + \mathrm{B}y + \mathrm{C}\,,$$

the co-ordinates of the corresponding point will be

$$\xi = c\mathrm{A} \qquad \eta = c\mathrm{B} \qquad \zeta = -\,\mathrm{C}\,,$$

and we may write the equation

$$c(z+\zeta) = x\xi + y\eta\ .$$

If we suppose ξ, η, ζ given as the co-ordinates of a point, then this equation, considering x, y, z as variable, is the equation of a plane corresponding to the point.

If we suppose x, y, z the co-ordinates of a point, and ξ, η, ζ as variable, the equation will be that of a plane corresponding to that point.

Hence, if a plane passes through the point xyz, the point corresponding to this plane lies in the plane corresponding to the point xyz.

These points and planes are reciprocally polar in the ordinary sense with respect to the paraboloid of revolution

$$2cz = x^2 + y^2\ .$$

We have thus arrived at a construction for reciprocal diagrams by considering each as a plane projection of a plane-sided polyhedron, these polyhedra being reciprocal to one another, in the geometrical sense, with respect to a certain paraboloid of revolution.

Each of the diagrams must fulfil the conditions of being a plane projection of a plane-sided polyhedron, for if any of the sides of the polyhedron of which it is the projection are not plane, there will be as many points corresponding to that side as there are different planes passing through three points of the side, and the other diagram will be indefinite.

Relation between the Number of Edges, Summits, and Faces of Polyhedra.

It is manifest that after a closed surface has been divided into separate faces by lines drawn upon it, every new line drawn from a point in the system, either introduces one new point into the system, or divides a face into two parts, according as it is drawn to an isolated point, or to a point already connected with the system. Hence the sum of points and faces is increased by one for every new line. If the closed surface is acyclic, or simply connected,* like that of a solid body without any passage through it, then, if from any point we draw a closed curve on the surface, we divide the surface into two faces. We have here one line, one point, and two faces. Hence, if e be the number of lines, s the number of points, and f the number of faces, then in general

$$e - s - f = m$$

when m remains constant, however many lines be drawn. But in the case of a simple closed surface

$$m = -2 .$$

If the closed surface is doubly connected, like that of a solid body with a hole through it, then if we draw one closed curve round the hole, and another closed curve through the hole, and round one side of the body, we shall have $e = 2$, $s = 1$, $f = 1$, so that $n = 0$. If the surface is n-ly connected, like that of a solid with $n - 1$ holes through it, then we may draw n closed curves round the $n - 1$ holes and the outside of the body, and $n - 1$ other closed curves each through a hole and round the outside of the body.

We shall then have $4(n - 1)$ segments of curves terminating in $2(n - 1)$ points and dividing the surface into two faces, so that $e = 4(n - 1)$, $s = 2\,(n - 1)$, and $f = 2$, and

$$e - s - f = 2n - 4 ,$$

and this is the general relation between the edges, summits, and faces of a polyhedron whose surface is n-ly connected.

The plane reciprocal diagrams, considered as plane projections of such

* See RIEMANN, Crelle's Journal, 1857, *Lehrsätze aus der analysis situs*, for space of two dimensions; also CAYLEY on the Partitions of a Close, Phil. Mag. 1861; HELMHOLTZ, Crelle's Journal, 1858, *Wirbelbewegung*, for the application of the idea of multiple continuity to space of three dimensions; J. B. LISTING, Göttingen Trans., 1861, *Der Census Räumlicher Complexe*, a complete treatise on the subject of Cyclosis and Periphraxy.

On the importance of this subject see GAUSS, Werke, v. 605, "Von der *Geometria Situs* die LEIBNITZ ahnte und in die nur einem Paar Geometern (EULER und VANDERMONDE) einen schwachen Blick zu thun vergönnt war, wissen und haben wir nach anderthalbhundert Jahren noch nicht viel mehr wie nichts."

Note added March 14, 1870.—Since this was written, I have seen LISTING's *Census*. In his notation, the surface of an n-ly connected body (a body with $n - 1$ holes through it) is $(2n - 2)$ cyclic. If $2n - 2 = K_2$ expresses the degree of cyclosis, then LISTING's general equation is—

$$s - (e - K_1) + (f - K_2 + \varpi_2) - (v - K_3 + \varpi_3 - w) = 0 ,$$

where s is the number of points, e the number of lines, K_1 the number of endless curves, f the number of faces, K_2 the number of degrees of cyclosis of the faces, ϖ_2 the number of periphractic or closed faces, v the number of regions of space, K_3 their number of degrees of cyclosis, ϖ_3 their number of degrees of periphraxy or the number of regions which they completely surround, and w is to be put $= 1$ or $= 0$, according as the system does or does not extend to infinity.

polyhedra, have the same relation between the numbers of their lines, points, and polygons. It is manifest that since

$$e_1 = e_2\,, \qquad s_1 = f_1\,, \qquad \text{and } f_1 = s_2\,,$$

where the suffixes refer to the first and second diagrams respectively

$$n_1 = n_2\,,$$

or the two diagrams are connected to the same degree.

On the Degrees of Freedom and Constraint of Frames.

To determine the positions of s points in space, with reference to a given origin and given axes, $3s$ data are required; but since the position of the origin and axes involve 6 data, the number of data required to determine the relative position of s points is $3s - 6$.

If, therefore, the lengths of $3s - 6$ lines joining selected pairs of a system of s points be given, and if these lengths are all independent of each other, then the distances between any other pair of points will be determinate, and the system will be rigidly connected.

If, however, the lines are so chosen that those which join pairs of points of a system of s' of the points are more than $3s' - 6$ in number, the lengths of these lines will not be independent of each other, and the lines of this partial system will only give $3s' - 6$ independent data to determine the complete system.

In a system of s points joined by e lines, there will in general be $3s - 6 - e = p$ degrees of freedom, provided that in every partial system of s' points joined by e' lines, and having in itself p' degrees of freedom, p' is not negative. If in any such system p is negative, we may put $q = -p$, and call q the number of degrees of constraint, and there will be q equations connecting the lengths of the lines; and if the system is a material one, the stress along each piece will be a function of q independent variables. Such a system may be said to have q degrees of constraint. If p' is negative in any partial system, then the degrees of freedom of the complete system are $p - p'$, where p and p' are got from the number of points and lines in the complete and partial systems. If s points are connected by e lines, so as to form a polyhedron of f faces, enclosing a space n times connected, and if each of the faces has m sides, then

$$mf = 2e\,.$$

We have also

$$e - s - f = 2n - 4\,,$$

and

$$3s - e = p + 6\,,$$

whence

$$p = 6(1 - n) + \left(2 - \frac{6}{m}\right)e$$

If all the faces of the polyhedron are triangles, $m = 3$, and we have

$$p = 6(1 - n) .$$

If $n = 1$, or in the case of a simply connected polyhedron with triangular faces, $p=o$, that is to say, such a figure is a rigid system, which would be no longer rigid if any one of its lines were wanting. In such a figure, if made of material rods forming a closed web of triangles, the tensions and pressures in the rods would be completely determined by the external forces applied to the figure, and if there were no external force, there would be no stress in the rods.

In a closed surface of any kind, if we cover the surface* with a system of curves which do not intersect each other, and if we draw another system intersecting these, and a third system passing diagonally through the intersections of the other two, the whole surface will be covered with small curvilinear triangles, and if we now substitute for the surface a system of rectilinear triangles having the same angular points, we shall have a polyhedron with triangular faces differing infinitely little from the surface, and such that the length of any line on the surface differs infinitely little from that of the corresponding line on the polyhedron. We may, therefore, in all questions about the transformation of surfaces by bending, substitute for them such polyhedra with triangular faces.

We thus find with respect to a simply connected closed inextensible surface —1*st*, That it is of invariable form ;† 2*d*, That the stresses in the surface depend entirely on the external applied forces ;‡ 3*d*, That if there is no external force, there is no stress in the surface.

In the limiting case of the curved surface, however, a kind of deformation is possible, which is not possible in the case of the polyhedron. Let us suppose that in some way a dimple has been formed on a convexo-convex part of the surface, so that the edge of the dimple is a plane closed curve, and the dimpled part is the reflexion in this plane of the original form of the surface. Then the length of any line drawn on the surface will remain unchanged.

Now let the dimple be gradually enlarged, so that its edge continually changes its position. Every line on the surface will still remain of the same length during the whole process, so that the process is possible in the case of an inextensible surface. In this way such a surface may be gradually turned outside in, and since the dimple may be formed from a mere point, a pressure applied at a single point on the outside of an inextensible surface will not be resisted, but will form a dimple which will increase till one part of the surface comes in contact with another.

In the case of closed surfaces doubly connected, $p = -6$, that is, such sur-

* On the Bending of Surfaces, by J. CLERK MAXWELL. Cambridge Transactions, 1856.

† This has been shown by Professor JELLETT, Trans. R.I.A., vol. xxii. p. 377.

‡ On the Equilibrium of a Spherical Envelope, by J. C. MAXWELL. Quarterly Journal of Mathematics, 1867.

faces are not only rigid, but are capable of internal stress, independent of external forces, and the expression of this stress depends on six independent variables.

In a polyhedron with triangular faces, if a number of the edges be taken away so as to form a hole with e_1 sides, the number of degrees of freedom is

$$p = e_1 - 6n + 3 .$$

Hence, in order to make an n-ly connected polyhedron simply rigid without stress, we may cut out the edges till we have formed a hole having $6n - 3$ edges. The system will then be free from stress, but if any more edges be removed, the system will no longer be rigid.

Since in the limiting case of the inextensible surface, the smallest hole may be regarded as having an infinite number of sides, the smallest hole made in a closed inextensible surface connected to any degree will destroy its rigidity. Its flexibility, however, may be confined within very narrow limits.

In the case of a plane frame of s points, we have $2s$ data required to determine the points with reference to a given origin and axes; but since 3 arbitrary data are involved in the choice of origin and axis, the number of data required to determine the relative position of s points in a plane is $2s - 3$.

If we know the lengths of e lines joining certain pairs of these points, then in general the number of degrees of freedom of the frame will be

$$p = 2s - e - 3 .$$

If, however, in any partial system of s' points connected by e' lines, the quantity $p' = 2s' - e' - 3$ be negative, or in other words, if a part of the frame be self-strained, this partial system will contribute only $2s' - 3$ equations independent of each other to the complete system, and the whole frame will have $p - p'$ degrees of freedom.

In a plane frame, consisting of a single sheet, every element of which is triangular, and in which the pieces form three systems of continuous lines, as at p. 11, if the frame contains e pieces connecting s points, s' of which are on the circumference of the frame and s_1 in the interior, then

$$3s - s' = e + 3 .$$

Hence

$$p = -(s-s') = -s_1 ,$$

a negative quantity, or such a frame is necessarily stiff; and if any of the points are in the interior of the frame, the frame has as many degrees of constraint as there are interior points—that is, the stresses in each piece will be functions of s_1 variables, and s_1 pieces may be removed from the frame without rendering it loose.

If there are n holes in the frame, so that s' points lie on the circumference of the frame or on those of the holes, and s_1 points lie in the interior, the degree of stiffness will be

$$-p = s_1 + 3n .$$

If a plane frame be a projection of a polyhedron of f faces, each of m sides, and enclosing a space n times connected, then

$$mf = 2e$$
$$e - s - f = 2n - 4$$
$$2s - e = p + 3 ,$$

whence

$$p = 5 - 4n + \left(1 - \frac{4}{m}\right) e .$$

If all the faces are quadrilaterals $m = 4$ and $p = 5 - 4n$, or a plane frame which is the projection of a closed polyhedron with quadrilateral faces, has one degree of freedom if the polyhedron is simply connected, as in the case of the projection of the solid bounded by six quadrilaterals, but if the polyhedron be doubly connected, the frame formed by its plane projection will have three degrees of stiffness. (See Diagram II.)

Theorem.—If every one of a system of points in a plane is in equilibrium under the action of tensions and pressures acting along the lines joining the points, then if we substitute for each point a small smooth ring through which smooth thin rods of indefinite length corresponding to the lines are compelled to pass, then, if to each rod be applied a couple in the plane, whose moment is equal to the product of the length of the rod between the points multiplied by the tension or pressure in the former case, and tends to turn the rod in the positive or the negative direction, according as the force was a tension or a pressure, then every one of the system of rings will be in equilibrium. For each ring is acted on by a system of forces equal to the tensions and pressures in the former case, each to each, the whole system being turned round a right angle, and therefore the equilibrium of each point is undisturbed.

Theorem.—In any system of points in equilibrium in a plane under the action of repulsions and attractions, the sum of the products of each attraction multiplied by the distance of the points between which it acts, is equal to the sum of the products of the repulsions multiplied each by the distance of the points between which it acts.

For since each point is in equilibrium under the action of a system of attractions and repulsions in one plane, it will remain in equilibrium if the system of forces is turned through a right angle in the positive direction. If this operation is performed on the systems of forces acting on all the points, then at the extremities of each line joining two points we have two equal forces at right angles to that line and acting in opposite directions, forming a couple whose magnitude is the product of the force between the points and their distance, and whose direction is positive if the force be repulsive, and negative if it be attractive. Now since every point is in equilibrium these two systems of couples are in equilibrium, or the sum of the positive couples is equal to that of the negative couples, which proves the theorem.

In a plane frame, loaded with weights in any manner, and supported by vertical thrusts, each weight must be regarded as attracted towards a horizontal base line, and each support of the frame as repelled from that line. Hence the following rule:—

Multiply each load by the height of the point at which it acts, and each tension by the length of the piece on which it acts, and add all these products together.

Then multiply the vertical pressures on the supports of the frame each by the height at which it acts, and each pressure by the length of the piece on which it acts, and add the products together. This sum will be equal to the former sum.

If the thrusts which support the frame are not vertical, their horizontal components must be treated as tensions or pressures borne by the foundations of the structure, or by the earth itself.

The importance of this theorem to the engineer arises from the circumstance that the strength of a piece is in general proportional to its section, so that if the strength of each piece is proportional to the stress which it has to bear, its weight will be proportional to the product of the stress multiplied by the length of the piece. Hence these sums of products give an estimate of the total quantity of material which must be used in sustaining tension and pressure respectively.

The following method of demonstrating this theorem does not require the consideration of couples, and is applicable to frames in three dimensions.

Let the system of points be caused to contract, always remaining similar to its original form, and with its pieces similarly situated, and let the same forces continue to act upon it during this operation, so that every point is always in equilibrium under the same system of forces, and therefore no work is done by the system of forces as a whole.

Let the contraction proceed till the system is reduced to a point. Then the work done by each tension is equal to the product of that tension by the distance through which it has acted, namely, the original distance between the points. Also the work spent in overcoming each pressure is the product of that pressure by the original distance of the points between which it acts; and since no work is gained or lost on the whole, the sum of the first set of products must be equal to the sum of the second set. In this demonstration it is not necessary to suppose the points all in one plane. This demonstration is mathematically equivalent to the following algebraical proof :—

Let the co-ordinates of the n different points of the system be $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$, $x_p\, y_p\, z_p$, &c., and let the force between any two points p, q, be P_{pq}, and their distance r_{pq}, and let it be reckoned positive when it is a pressure, and negative when it is a tension, then the equation of equilibrium of any point p with respect to forces parallel to x is

$$(x_p - x_1)\frac{P_{p1}}{r_{p1}} + (x_p - x_2)\frac{P_{p2}}{r_{p2}} + \&c. + (x_p - x_q)\frac{P_{pq}}{r_{pq}} + \&c. = 0\,,$$

or generally, giving t all values from 1 to n,

$$\Sigma_1^n{}_t \left\{(x_p - x_t)\frac{P_{pt}}{r_{pt}}\right\} = 0\,.$$

Multiply this equation by x_p. There are n such equations, so that if each is multiplied by its proper co-ordinate and the sum taken, we get

$$\Sigma_1^n{}_p\ \Sigma_1^n{}_t \left\{(x_p - x_t)^2\frac{P_{pt}}{r_{pt}}\right\} = 0\,,$$

and adding the corresponding equations in y and z, we get

$$\Sigma_1^n{}_p\ \Sigma_1^n{}_t \left(P_{pt}\ r_{pt}\right) = 0\,,$$

which is the algebraic expression of the theorem.

General Theory of Diagrams of Stress in Three Dimensions.

First Method of Representing Stress in a Body.

Definition.—A diagram of stress is a figure having such a relation to a body under the action of internal forces, that if a surface A, limited by a closed curve, is drawn in the body, and if the corresponding limited surface a be drawn in the diagram of stress, then the resultant of the actual internal forces on the positive side of the surface A in the body is equal and parallel to the resultant of a uniform normal pressure p acting on the positive side of the surface a in the diagram of stress.

Let x, y, z be the co-ordinates of any point in the body, ξ, η, ζ those of the corresponding point in the diagram of stress, then ξ, η, ζ are functions of x, y, z, the nature of which we have to ascertain, so that the internal forces in the body may be in equilibrium. For the present we suppose no external forces, such as gravity, to act on the particles of the body. We shall consider such forces afterwards.

Theorem 1.—If any closed surface is described in the body, and if the stress on any element of that surface is equal and parallel to the pressure on the corresponding element of surface in the diagram of stress, then the resultant stress on the whole closed surface will vanish; for the corresponding surface in the diagram of stress is a closed surface, and the resultant of a uniform normal pressure p on every element of a closed surface is zero by hydrostatics.

It does not, however, follow that the portion of the body within the closed surface is in equilibrium, for the stress on its surface may have a resultant moment.

Theorem 2.—To ensure equilibrium of every part of the body, it is necessary and sufficient that

$$\xi=\frac{dF}{dx} \qquad \eta=\frac{dF}{dy} \qquad \zeta=\frac{dF}{dz},$$

where F is any function of x y and z.

Let us consider the elementary area in the body $dy\,dz$. The stress acting on this area will be a force equal and parallel to the resultant of a pressure p acting on the corresponding element of area in the diagram of stress. Resolving this pressure in the directions of the co-ordinate axes, we find the three components of stress on $dy\,dz$, which we may call $p_{xx}\,dy\,dz$, $p_{xy}\,dy\,dz$, and $p_{xz}\,dy\,dz$, each equal to p multiplied by the area of the projection of the corresponding element of the diagram of stress on the three co-ordinate planes. Now, the projection on the plane yz, is

$$\left(\frac{d\eta}{dy}\frac{d\zeta}{dz}-\frac{d\eta}{dz}\frac{d\zeta}{dy}\right)dy\,dz\,.$$

Hence we find for the component of stress in the direction of x

$$p_{xx}=p\left(\frac{d\eta}{dy}\frac{d\zeta}{dz}-\frac{d\eta}{dz}\frac{d\zeta}{dy}\right),$$

which we may write for brevity at present

$$p_{xx}=p\mathrm{J}(\eta\,,\zeta;\;y\,,z)\,.$$

Similarly,

$$p_{xy}=p\mathrm{J}(\zeta\,,\xi\,;y\,,z) \qquad p_{xz}=p\mathrm{J}(\xi\,,\eta\,;\;y\,,z)\,.$$

In the same way, we may find the components of stress on the areas $dz\,dx$ and $dx\,dy$—

$$p_{yx}=p\mathrm{J}(\eta\,,\zeta;\;z\,,x) \qquad p_{yy}=p\mathrm{J}(\zeta\,,\xi;\;z\,,x) \qquad p_{yz}=p\mathrm{J}(\xi\,,\eta\,;\;z\,,x)$$

$$p_{zx}=p\mathrm{J}(\eta\,,\zeta;\;x\,,y) \qquad p_{zy}=p\mathrm{J}(\zeta\,,\xi;\;x\,,y) \qquad p_{zz}=p\mathrm{J}(\xi\,,\eta\,;\;x\,,y)\,.$$

Now, consider the equilibrium of the parallelopiped $dx\,dy\,dz$, with respect to the moment of the tangential stresses about its axes.

The moments of the forces tending to turn this elementary parallelopiped about the axis of x are

$$dz\,dx\,p_{yz}\,.\,dy-dx\,dy\,p_{zy}\,.\,dz\,.$$

To ensure equilibrium as respects rotation about the axis of x, we must have

$$p_{yz}=p_{zy}\,.$$

Similarly, for the moments about the axes of y and z, we obtain the equations

$$p_{zx} = p_{xz} \quad \text{and} \quad p_{xy} = p_{yx} \,.$$

Now, let us assume for the present

$$\frac{d\xi}{dx} = A_1 , \quad \frac{d\xi}{dy} = B_3 + C_3 , \quad \frac{d\xi}{dz} = B_2 - C_2 ,$$

$$\frac{d\eta}{dx} = B_3 - C_3 , \quad \frac{d\eta}{dy} = A_2 , \quad \frac{d\eta}{dz} = B_1 + C_1 ,$$

$$\frac{d\zeta}{dx} = B_2 + C_2 , \quad \frac{d\zeta}{dy} = B_1 - C_1 , \quad \frac{d\zeta}{dz} = A_3 \,.$$

Then the equation $p_{yz} = p_{zy}$ becomes

$$p\left(\frac{d\xi}{dz}\frac{d\eta}{dx} - \frac{d\xi}{dx}\frac{d\eta}{dz}\right) = p\left(\frac{d\zeta}{dx}\frac{d\xi}{dy} - \frac{d\zeta}{dy}\frac{d\xi}{dx}\right)$$

or

$$(B_2 - C_2)(B_3 - C_3) - A_1(B_1 + C_1) = (B_2 + C_2)(B_3 + C_3) - A_1(B_1 - C_1)$$

$$0 = A_1C_1 + B_3C_2 + B_2C_3 \,.$$

Similarly, from the two other equations of equilibrium we should find

$$0 = A_2C_2 + B_1C_3 + B_3C_1$$

$$0 = A_3C_3 + B_2C_1 + B_1C_2 \,.$$

From these three equations it follows that

$$C_1 = 0 \qquad C_2 = 0 \qquad C_3 = 0 \,.$$

Hence

$$\frac{d\eta}{dz} = \frac{d\zeta}{dy} \;, \quad \frac{d\zeta}{dx} = \frac{d\xi}{dz} \;, \quad \frac{d\xi}{dy} = \frac{d\eta}{dx} \,,$$

and $\xi dx + \eta dy + \zeta dz$ is a complete differential of some function, F, of x, y and z, whence it follows that

$$\xi = \frac{dF}{dx} \;, \quad \eta = \frac{dF}{dy} \;, \quad \zeta = \frac{dF}{dz} \,.$$

F may be called the function of stress, because when it is known, the diagram of stress may be formed, and the components of stress calculated. The form of the function F is limited only by the conditions to be fulfilled at the bounding surface of the body.

The six components of stress expressed in terms of F are

$$p_{xx} = p\left(\frac{d^2F}{dy^2}\frac{d^2F}{dz^2} - \left(\frac{d^2F}{dydz}\right)^2\right), \; p_{yy} = p\left(\frac{d^2F}{dz^2}\frac{d^2F}{dx^2} - \left(\frac{d^2F}{dzdx}\right)^2\right), \; p_{zz} = p\left(\frac{d^2F}{dx^2}\frac{dF^2}{dy^2} - \left(\frac{d^2F}{dxdy}\right)^2\right),$$

$$p_{yz} = p\left(\frac{d^2F}{dzdx}\frac{d^2F}{dxdy} - \frac{d^2F}{dx^2}\frac{d^2F}{dydz}\right), \; p_{zx} = p\left(\frac{d^2F}{dxdy}\frac{d^2F}{dydz} - \frac{d^2F}{dy^2}\frac{d^2F}{dzdx}\right), \; p_{xy} = p\left(\frac{d^2F}{dydz}\frac{d^2F}{dzdx} - \frac{d^2F}{dz^2}\frac{d^2F}{dxdy}\right).$$

If $\frac{dF}{dz} = z$, F becomes AIRY's function of stress in two dimensions, and we have

$$p_{xx} = p\frac{d^2F}{dy^2}, \quad p_{yy} = p\frac{d^2F}{dx^2}, \quad p_{xy} = -p\frac{d^2F}{dxdy}.$$

The system of stress in three dimensions deduced in this way from any function, F, satisfies the equations of equilibrium of internal stress. It is not, however, a general solution of these equations, as may be easily seen by taking the case in which p_{xz} and p_{yz} are both zero at all points. In this case, since there is no tangential action in planes parallel to xy, the stresses p_{xx}, p_{xy} and p_{yy} in each stratum must separately fulfil the conditions of equilibrium,

$$\frac{d}{dx}p_{xx} + \frac{d}{dy}p_{xy} = 0, \qquad \frac{d}{dx}p_{xy} + \frac{d}{dy}p_{yy} = 0.$$

The complete solution of these equations is, as we have seen,

$$p_{xx} = \frac{d^2f}{dy^2}, \quad p_{xy} = -\frac{d^2f}{dx\,dy}, \quad p_{yy} = \frac{d^2f}{dx^2},$$

where f is any function of x and y, the form of which may be different for every different value of z, so that we may regard f as a perfectly general function of x y and z.

Again, if we consider a cylindrical portion of the body with its generating lines parallel to z, we shall see that there is no tangential action parallel to z between this cylinder and the rest of the body. Hence the longitudinal stress in this cylinder must be constant throughout its length, and is independent of the stress in any other part of the body.

Hence

$$p_{zz} = \phi(xy),$$

where ϕ is a function of x and y only, but may be any such function. But expressing the stresses in terms of F under the conditions $p_{xz}=0$, $p_{yz}=0$, we find that if F is a perfectly general function of x and y

$$\frac{d^2F}{dx\,dz} = 0 \quad \text{and} \quad \frac{d^2F}{dy\,dz} = 0,$$

whence it follows that $\frac{dF}{dx}$ and $\frac{dF}{dy}$ are functions of x and y only, and that $\frac{dF}{dz}$ is a function of z only. Hence

$$F = G + Z,$$

when G is a function of x and y only, and Z a function of z only, and the components of stress are

$$p_{xx} = p\frac{d^2G}{dy^2}\,\frac{d^2Z}{dz^2}, \quad p_{yy} = p\frac{d^2G}{dx^2}\,\frac{d^2Z}{dz^2}, \quad p_{zz} = p\left(\frac{d^2G}{dx^2}\,\frac{d^2G}{dy^2} - \overline{\frac{d^2G}{dx\,dy}}\right)^2$$

$$p_{yz} = 0, \qquad p_{zx} = 0, \qquad p_{xy} = -p\frac{d^2G}{dx\,dy}\,\frac{d^2Z}{dz^2}.$$

Here the function f which determines the stress in the strata parallel to xy is

$$f = p\,\mathrm{G}\frac{d^2\mathrm{Z}}{dz^2}.$$

Now, this function is not sufficiently general, for instead of being any function of x, y and z, it is the product of a function of x and y multiplied by a function of z.

Besides this, though the value of p_{zz} is, as it ought to be, a function of x and y only, it is not of the most general form, for it depends on G, the function which determines the stresses p_{xx}, p_{xy}, and p_{yy}, whereas the value of p_{zz} may be entirely independent of the values of these stresses. In fact, the equations give

$$p_{zz} = p\,\frac{p_{xx}\,p_{yy} - \overline{p_{xy}}^2}{\left(\frac{d^2\mathrm{Z}}{dz^2}\right)^2}.$$

This method, therefore, of representing stress in a body of three dimensions is a restricted solution of the equations of equilibrium.

On Reciprocal Diagrams in Three Dimensions.

Let us consider figures in two portions of space, which we shall call respectively the first and the second diagrams. Let the co-ordinates of any point in the first diagram be denoted by x, y, z, and those of the corresponding point in the second by ξ, η, ζ, measured in directions parallel to x, y, z respectively. Let F be a quantity varying from point to point of the first figure in any continuous manner; that is to say, if A, B are two points, and F_1, F_2 the values of F at those points; then, if B approaches A without limit, the value of F_2 approaches that of F_1 without limit. Let the co-ordinates (ξ, η, ζ) of a point in the second diagram be determined from x, y, z, those of the corresponding point in the first by the equations

$$\xi = \frac{d\mathrm{F}}{dx}\,, \quad \eta = \frac{d\mathrm{F}}{dy}\,, \quad \zeta = \frac{d\mathrm{F}}{dz} \quad . \quad . \quad . \quad . \quad (1).$$

This is equivalent to the statement, that the vector (ρ) of any point in the second diagram represents in direction and magnitude the rate of variation of F at the corresponding point of the first diagram.

Next, let us determine another function, ϕ, from the equation

$$x\xi + y\eta + z\zeta = \mathrm{F} + \phi \quad . \quad . \quad . \quad . \quad . \quad (2),$$

ϕ, as thus determined, will be a function of x, y, and z, since ξ, η, ζ are known in terms of these quantities. But, for the same reason, ϕ is a function of ξ, η, ζ. Differentiate ϕ with respect to ξ, considering x, y and z functions of ξ, η, ζ,

$$\frac{d\phi}{d\xi} = x + \xi\frac{dx}{d\xi} + \eta\frac{dy}{d\xi} + \zeta\frac{dz}{d\xi} - \frac{d\mathrm{F}}{d\xi}$$

Substituting the values of ξ, η, ζ from (1)

$$\frac{d\phi}{d\xi} = x + \frac{dF}{dx}\frac{dx}{d\xi} + \frac{dF}{dy}\frac{dy}{d\xi} + \frac{dF}{dz}\frac{dz}{d\xi} - \frac{dF}{d\xi}$$

$$= x + \frac{dF}{d\xi} - \frac{dF}{d\xi}$$

$$= x$$

Differentiating ϕ with respect to η and ζ, we get the three equations

$$x = \frac{d\phi}{d\xi} \qquad y = \frac{d\phi}{d\eta} \qquad z = \frac{d\phi}{d\zeta} \quad . \quad . \quad . \quad . \quad (3),$$

or the vector (r) of any point in the first diagram represents in direction and magnitude the rate of increase of ϕ at the corresponding point of the second diagram.

Hence the first diagram may be determined from the second by the same process that the second was determined from the first, and the two diagrams, each with its own function, are reciprocal to each other.

The relation (2) between the functions expresses that the sum of the functions for two corresponding points is equal to the product of the distances of these points from the origin multiplied by the cosine of the angle between the directions of these distances.

Both these functions must be of two dimensions in space. Let F′ be a linear function of xyz, which has the same value and rate of variation as F has at the point $x_0\, y_0\, z_0$

$$F' = F_0 + (x - x_0)\frac{dF_0}{dx} + (y - y_0)\frac{dF_0}{dy} + (z - z_0)\frac{dF_0}{dz} \quad . \quad . \quad (4).$$

The value of F′ at the origin is found by putting x, y and $z = 0$

$$F' = F_0 - x_0\xi - y_0\eta - z_0\zeta = -\phi \quad . \quad . \quad . \quad . \quad (5),$$

or the value of F′ at the origin is equal and opposite to the value of ϕ at the point ξ, η, ζ.

If the rate of variation of F is nowhere infinite, the co-ordinates $\xi\ \eta\ \zeta$ of the second diagram must be everywhere finite, and *vice versa*. Beyond the limits of the second diagram the values of x, y, z, in terms of ξ, η, ζ, must be impossible, and therefore the value of ϕ is also impossible. Within the limits of the second diagram, the function ϕ has an even number of values at every point, corresponding to an even number of points in the first diagram, which correspond to a single point in the second.

To find these points in the first diagram, let ρ be the vector of a given point in the second diagram, and let surfaces be drawn in the first diagram for which F is constant, and let points be found in each of these surfaces at which the tangent plane is perpendicular to ρ, these points will form one or more curves, which must be either closed or infinite, and the points on these curves correspond to the points in the second diagram which lie in the direction of the vector ρ. If p be the perpendicular from a point in the first diagram on a plane through the origin perpendicular to ρ, then all those points on these curves at which $\frac{dF}{dp} = \rho$ correspond to the given point in the second diagram. Now, since this point is within the second diagram, there are values of ρ both greater and less than the given one; and therefore $\frac{dF}{dp}$ is neither an absolute maximum nor an absolute minimum value. Hence there are in general an even number of points on the curve or curves which correspond to the given point. Some of these points may coincide, but at least two of them must be different, unless the given point is at the limit of the second diagram.

Let us now consider the two reciprocal diagrams with their functions, and ascertain in what the geometrical nature of their reciprocity consists.

(1.) Let the first diagram be simply the point P_1, (x_1, y_1, z_1), at which $F = F_1$, then in the other diagram

$$\phi = x_1\xi + y_1\eta + z_1\zeta - F_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

or a point in one diagram is reciprocal to a space in the other, in which the function ϕ is a linear function of the co-ordinates.

(2.) Let the first diagram contain a second point P_2, (x_2, y_2, z_2) at which $F = F_2$, then we must combine equation (6) with

$$\phi = x_2\xi + y_2\eta + z_2\zeta - F_2 \quad . \quad . \quad . \quad . \quad . \quad (7),$$

whence eliminating ϕ,

$$(x_1 - x_2)\xi + (y_1 - y_2)\eta + (z_1 - z_2)\zeta = F_1 - F_2 .$$

If r_{12} is the length of the line drawn from the first point P_1 to the second P_2; and if $l_{12}\, m_{12}\, n_{12}$ are its direction cosines, this equation becomes

$$l_{12}\xi + m_{12}\eta + n_{12}\zeta = \frac{F_2 - F_1}{r_{12}},$$

or the reciprocal of the two points P_1 and P_2 is a plane, perpendicular to the line joining them, and such that the perpendicular from the origin on the plane multiplied by the length of the line P_1P_2 is equal to the excess of F_2 over F_1.

(3.) Let there be a third point P_3 in the first diagram, whose co-ordinates are $x_3\, y_3\, z_3$ and for which $F = F_3$; then we must combine with equations (6) and (7)

$$\phi = x_3\xi + y_3\eta + z_3\zeta - F_3 \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The reciprocal of the three points $P_1\, P_2\, P_3$ is a straight line perpendicular to the plane of the three points, and such that the perpendicular on this line from the origin represents, in direction and magnitude, the rate of most rapid increase of F in the plane $P_1\, P_2\, P_3$, F being a linear function of the co-ordinates whose values at the three points are those given.

(4.) Let there be a fourth point P_4 for which $F = F_4$.

The reciprocal of the four points is a single point, and the line drawn from the origin to this point represents, in direction and magnitude, the rate of greatest increase of F, supposing F such a linear function of xyz that its values at the four points are those given. The value of ϕ at this point is that of F at the origin.

Let us next suppose that the value of F is continuous, that is, that F does not vary by a finite quantity when the co-ordinates vary by infinitesimal quantities, but that the form of the function F is discontinuous, being a different linear function of xyz in different parts of space, bounded by definite surfaces.

The bounding surfaces of these parts of space must be composed of planes. For let the linear functions of xyz in contiguous portions of space be

$$F_1 = \alpha_1 x + \beta_1 y + \gamma_1 z - \phi_1$$
$$F_2 = \alpha_2 x + \beta_2 y + \gamma_2 z - \phi_2 ,$$

then at the bounding surface, where $F_1 = F_2$

$$(\alpha_1 - \alpha_2)x + (\beta_1 - \beta_2)\gamma + (\gamma_1 - \gamma_2)z = \phi_1 - \phi_2 \quad . \quad . \quad . \quad (9),$$

and this is the equation of a plane.

Hence the portion of space in which any particular form of the value of F holds good must be a polyhedron or cell bounded by plane faces, and therefore having straight edges meeting in a number of points or summits.

Every face is the boundary of two cells, every edge belongs to three or more cells, and to two faces of each cell.

Every summit belongs to at least four cells, to at least three faces of each cell, and to two edges of each face.

The whole space occupied by the diagram is divided into cells in two different ways, so that every point in it belongs to two different cells, and has two values of F and its derivatives.

The reciprocal diagram is made up of cells in the same way, and the reciprocity of the two diagrams may be thus stated:—

1. Every summit in one diagram corresponds to a cell in the other.

The radius vector of the summit represents the rate of increase of the function within the cell, both in direction and magnitude.

The value of the function at the summit is equal and opposite to the value which the function in the cell would have if it were continued under the same algebraical form to the origin.

2. Every edge in the one diagram corresponds to a plane face in the other, which is the face of contact of the two cells corresponding to the two extremities of the edge.

The edge in the one diagram is perpendicular to the face in the other.

The distance of the plane from the origin represents the rate of increase of the function along the edge.

3. Every face in the one diagram corresponds to an edge in which as many cells meet as there are angles in the face, that is, at least three. Every face must belong to two, and only two cells, because the edge to which it corresponds has two, and only two extremities.

4. Every cell in the one diagram corresponds to a summit in the other. Every face of the cell corresponds and is perpendicular to an edge having an extremity in the summit. Since every cell must have four or more faces, every summit must have four or more edges meeting there.

Every edge of the cell corresponds to a face having an angle in the summit. Since every cell has at least six edges, every summit must be the point of concourse of at least six faces, which are the boundaries of cells.

Every summit of the cell corresponds to a cell having a solid angle at the summit. Since every cell has at least four summits, every summit must be the meeting place of at least four cells.

Mechanical Reciprocity of the Diagrams.

If along each of the edges meeting in a summit forces are applied proportional to the areas of the corresponding faces of the cell in the reciprocal diagram, and in a direction which is always inward with respect to the cell, then these forces will be in equilibrium at the summit.

This is the "Polyhedron of Forces," and may be proved by hydrostatics.

If the faces of the cell form a single closed surface which does not intersect itself, it is easy to understand what is meant by the inside and outside of the cell; but if the surface intersects itself, it is better to speak of the positive and negative sides of the surface. A cell, or portion of a cell, bounded by a closed surface, of which the positive side is inward, may be called a positive cell. If the surface intersects itself, and encloses another portion of space with its negative side inward, that portion of space forms a negative cell. If any portion of space is surrounded by n sheets of the surface of the same cell with their positive side inward, and by m sheets with their negative side inward, the space enclosed in this way must be reckoned $n - m$ times.

In passing to a contiguous cell, we must suppose that its face in contact with the first cell has its positive surface on the opposite side from that of the first cell. In this way, by making the positive side of the surface continuous throughout each cell, and by changing it when we pass to the next cell, we may settle the positive and negative side of every face of every cell, the sign of every face depending on which of the two cells it is considered for the moment to belong to.

If we now suppose forces of tension or pressure applied along every edge of the first diagram, so that the force on each extremity of the edge is in the direction of the positive normal to the corresponding face of the cell corresponding to that extremity, and proportional to the area of the face, then these pressures and tensions along the edges will keep every point of the diagram in equilibrium.

Another way of determining the nature of the force along any edge of the first diagram, is as follows:—

Round any edge of the first diagram draw a closed curve, embracing it and no other edge. However small the curve is, it will enter each of the cells which meet in the edge. Hence the reciprocal of this closed curve will be a plane polygon whose angles are the points reciprocal to these cells taken in order. The area of this polygon represents, both in direction and magnitude, the whole force acting through the closed curve, that is, in this case the stress along the edge. If, therefore, in going round the angles of the polygon, we travel in the same direction of rotation in space as in going round the closed curve, the stress along the edge will be a pressure; but if the direction is opposite, the stress will be a tension.

This method of expressing stresses in three dimensions comprehends all cases in which RANKINE's reciprocal figures are possible, and is applicable to certain cases of continuous stress. That it is not applicable to all such cases is easily seen by the example of p (18).

On Reciprocal Diagrams in Two Dimensions.

If we make F a function of x and y only, all the properties already deduced for figures in three dimensions will be true in two; but we may form a more distinct geometrical conception of the theory by substituting cz for F and $c\zeta$ for ϕ. We have then for the equations of relation between the two diagrams

$$\left.\begin{array}{ll} \xi = c\dfrac{dz}{dx} & \eta = c\dfrac{dz}{dy} \\ x = c\dfrac{d\zeta}{d\xi} & y = c\dfrac{d\zeta}{d\eta} \\ x\xi + y\eta = cz + c\zeta\,. & \end{array}\right\} \quad . \quad . \quad (10).$$

These equations are equivalent to the following definitions :—

Let z in the first diagram be given as a function of x and y, z will lie on a surface of some kind. Let x_0, y_0 be particular values of x and y, and let z_0 be the corresponding value of z. Draw a tangent plane to the surface at the point x_0, y_0, z_0, and from the point $\xi = 0, \eta = 0, \zeta = -c$; in the second diagram draw a normal to this tangent plane. It will cut the plane $\zeta = 0$ at the point ξ, η corresponding to xy, and the value of ζ is equal and opposite to the segment of the axis of z cut off by the tangent plane. The two surfaces may be defined as reciprocally polar (in the ordinary sense) with respect to the paraboloid of revolution

$$x^2 + y^2 = 2cz \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

and the diagrams are the projections on the planes of xy and $\xi\eta$ of points and lines on these surfaces.

If one of the surfaces is a plane-faced polyhedron, the other will also be a plane-faced polyhedron, every face in the one corresponding to a point in the other, and every edge in the one corresponding to the line joining the points corresponding to the faces bounded by the edge. In the projected diagrams every line is perpendicular to the corresponding line, and lines which meet in a point in one figure form a closed polygon in the other.

These are the conditions of reciprocity mentioned at p. 8, and it now appears that if either of the diagrams is a projection of a plane-faced polyhedron, the other diagram can be drawn. If the first diagram cannot be a projection of a plane-faced polyhedron, let it be a projection of a polyhedron whose faces are polygons not in one plane. These faces must be conceived to be filled up by surfaces, which are either curved or made up of different plane portions. In the first case the polygon will correspond not to a point, but to a finite portion of a surface; in the second, it will correspond to several points, so that the lines, which correspond to the edges of such a polygon, will terminate in several points, and not in one, as is necessary for reciprocity.

Second Method of representing Stress in a Body.

Let a, b be any two consecutive points in the first diagram, distant s, and α, β the corresponding points in the second, distant σ, then if the direction cosines of the line $a\,b$ are l, m, n and those of $\alpha\,\beta$, λ, μ, ν

$$\left.\begin{aligned} \sigma\lambda &= sl\frac{d\xi}{dx} + sm\frac{d\xi}{dy} + sn\frac{d\xi}{dz} \\ \sigma\mu &= sl\frac{d\eta}{dx} + sm\frac{d\eta}{dy} + sn\frac{d\eta}{dz} \\ \sigma\nu &= sl\frac{d\zeta}{dx} + sm\frac{d\zeta}{dy} + sn\frac{d\zeta}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad (12).$$

Hence

$$\frac{\sigma}{s}\left(l\lambda + m\mu + n\nu\right) = l^2\frac{d\xi}{dx} + m^2\frac{d\eta}{dy} + n^2\frac{d\zeta}{dz} + mn\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) + nl\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) + lm\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) \quad (13).$$

If we put $l\lambda + m\mu + n\nu = \cos\epsilon$, where ϵ is the angle between s and σ, and if we take three sets of values of $l\,m\,n$, corresponding to three directions at right angles to each other, we find

$$\frac{\sigma_1}{s_1}\cos\epsilon_1 + \frac{\sigma_2}{s_2}\cos\epsilon_2 + \frac{\sigma_3}{s_3}\cos\epsilon_3 = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2} \quad (14).$$

Hence this quantity depends only on the position of the point, and not on the directions of $s_1\ s_2\ s_3$ or of $x\ y\ z$, let us call it Δ^2F.

Now, let us take an element of area perpendicular to s, and let us suppose that the stress on this element is compounded of a normal pressure $= p\Delta^2F$, and a tension parallel to σ and equal to $p\,\frac{\sigma}{s}$.

By the rules for the composition of stress, we have for the components of the force on this element, in terms of the six components of stress,

$$\left.\begin{aligned} X &= lp_{xx} + mp_{xy} + np_{xz} = p\left(l\Delta^2F - \lambda\frac{\sigma}{s}\right) \\ Y &= lp_{xy} + mp_{yy} + np_{yz} = p\left(m\Delta^2F - \mu\frac{\sigma}{s}\right) \\ Z &= lp_{xz} + mp_{yz} + np_{zz} = p\left(n\Delta^2F - \nu\frac{\sigma}{s}\right) \end{aligned}\right\} \quad . \quad . \quad (15).$$

Hence,

$$\left.\begin{aligned} p_{xx} &= p\left(\Delta^2F - \frac{d\xi}{dx}\right) = p\left(\Delta^2F - \frac{d^2F}{dx^2}\right) = p\left(\frac{d^2F}{dy^2} + \frac{d^2F}{dz^2}\right) \\ p_{xx} &= p\left(\frac{d^2F}{dy^2} + \frac{d^2F}{dz^2}\right), p_{yy} = p\left(\frac{d^2F}{dz^2} + \frac{d^2F}{dx^2}\right), p_{zz} = p\left(\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2}\right) \\ p_{yz} &= -p\frac{d^2F}{dxdy}, \quad p_{zx} = -p\frac{d^2F}{dzdx}, \quad p_{xy}p = -\frac{d^2F}{dxdy} \end{aligned}\right\} \quad (16).$$

By substituting these values in the equations of equilibrium

$$\frac{dp_{xx}}{dx} + \frac{dp_{xy}}{dy} + \frac{dp_{xz}}{dz} = 0, \ \&c. \quad . \quad . \quad . \quad . \quad (17),$$

it is manifest that they are fulfilled for any value of F.

The most general solution of these equations of equilibrium is contained in the values

$$\left.\begin{aligned} p_{xx} &= \frac{d^2B}{dz^2} + \frac{d^2C}{dy^2} \quad p_{yy} = \frac{d^2C}{dx^2} + \frac{d^2A}{dz^2} \quad p_{zz} = \frac{d^2A}{dy^2} + \frac{d^2B}{dx^2} \\ p_{yz} &= -\frac{d^2A}{dydz} \quad p_{zx} = -\frac{d^2B}{dzdx} \quad p_{xy} = -\frac{d^2C}{dxdy} \end{aligned}\right\} \quad (18).$$

By making $A = B = C = pF$ we get a case which, though restricted in its generality, has remarkable properties with respect to diagrams of stress. We have seen that a distribution of stress according to the definition above (16), is consistent with itself, and will keep a body in equilibrium. Since the stresses are linear functions of F, any two systems of stress can be compounded by adding their respective functions, a process not applicable to the first method of representation by areas.

Let us ascertain what kind of stress is represented in this way in the case of the system of cells already considered.

Since F in each cell is a linear function of x, y, z, there can be no stress at any point within it. Let us take a and b two contiguous points in different cells, then α and β will be the points at a finite distance to which these cells are reciprocal, and $\Delta^2 F = \frac{\alpha\beta}{ab}$, which becomes infinite when ab vanishes.

If a and b are in the surface bounding the cells, α and β coincide. Hence there is a stress in this surface, uniform in all directions in the plane of the surface, and such that the stress across unit of length drawn on the surface is proportional to the distance between the points which are reciprocal to the two cells bounded by the surface, and this stress is a tension or a pressure according as the two points are similarly or oppositely situated to the two cells.

The kind of equilibrium corresponding to this case is therefore that of a system of liquid films, each having a tension like that of a soap bubble, depending on the nature of the fluid of which it is composed. If all the films are composed of the same fluid, their tensions must be equal, and all the edges of the reciprocal diagram must be equal.

On AIRY'S *Function of Stress.*

Mr AIRY, in a paper "On the Strains in the Interior of Beams,"* was, I believe, the first to point out that, in any body in equilibrium under the action of internal stress in two dimensions, the three components of the stress in any two rectangular directions are the three second derivatives, with respect to these directions, of a certain function of the position of a point in the body.

This important simplification of the theory of the equilibrium of stress in two dimensions does not depend on any theory of elasticity, or on the mode in which stress arises in the body, but solely on the two conditions of equilibrium of an element of a body acted on only by internal stress

$$\frac{d}{dx}p_{xx} + \frac{d}{dy}p_{xy} = 0 \quad \text{and} \quad \frac{d}{dx}p_{xy} + \frac{d}{dy}p_{yy} = 0 \qquad . \quad . \qquad (19),$$

whence it follows that

$$p_{xx} = \frac{d^2F}{dy^2} \quad p_{xy} = -\frac{d^2F}{dxdy} \quad \text{and} \quad p_{yy} = \frac{d^2F}{dx^2} \qquad . \quad . \quad . \qquad (20),$$

where F is a function of x and y, the form of which is (as far as these equations are concerned) perfectly arbitrary, and the value of which at any point is independent of the choice of axes of co-ordinates. Since the stresses depend on the second derivatives of F, any linear function of x and y may be added to F without affecting the value of the stresses deduced from F. Also, since the stresses are linear functions of F, any two systems of stress may be mechanically compounded by adding the corresponding values of F.

The importance of AIRY'S function in the theory of stress becomes even more

* Phil. Trans. 1863.

manifest when we deduce from it the diagram of stress, the co-ordinates of whose points are

$$\xi = \frac{dF}{dy} \quad \text{and} \quad \eta = -\frac{dF}{dx} \qquad (21).$$

For if s be the length of any curve in the original figure, and σ that of the corresponding curve in the diagram of stress, and if Xds, Yds are the components of the whole stress acting on the element ds towards the right hand of the curve s

$$\left.\begin{aligned} Xds &= p_{xx}\frac{dy}{ds}ds = \frac{d^2F}{dy^2}\frac{dy}{ds}ds = \frac{d\xi}{dy}\frac{dy}{dx}ds = \frac{d\xi}{d\sigma}d\sigma \\ \text{and} \qquad Yds &= -p_{yy}\frac{dx}{ds}ds = -\frac{d^2F}{dx^2}\frac{dx}{ds}ds = \frac{dy}{dx}\frac{dx}{ds}ds = \frac{dy}{d\sigma}d\sigma \end{aligned}\right\} \qquad (22).$$

Hence the stress on the right hand side of the element ds of the original curve is represented, both in direction and magnitude, by the corresponding element $d\sigma$ of the curve in the diagram of stress, and, by composition, the resultant stress on any finite arc of the first curve s is represented in direction and magnitude by the straight line drawn from the beginning to the end of the corresponding curve σ.

If P_1, P_2 are the principal stresses at any point, and if P_1 is inclined a to the axis of x, then the component stresses are

$$\left.\begin{aligned} p_{xx} &= P_1 \cos^2 a + P_2 \sin^2 a \\ p_{xy} &= (P_1 - P_2)\sin a \cos a \\ p_{yy} &= P_1 \sin^2 a + P_2 \cos^2 a \end{aligned}\right\} \qquad (23).$$

Hence

$$\left.\begin{aligned} \tan 2a &= \frac{p_{xy}}{p_{xx} - p_{yy}} = \frac{\dfrac{d^2F}{dxdy}}{\dfrac{d^2F}{dx^2} - \dfrac{d^2F}{dy^2}} \\ P_1 + P_2 &= p_{xx} + p_{yy} = \frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} \\ P_1P_2 &= p_{xx}p_{yy} - p_{xy}^2 = \frac{d^2F}{dx^2}\frac{d^2F}{dy^2} - \overline{\frac{d^2F}{dxdy}}^2 \end{aligned}\right\} \qquad (24).$$

Consider the area bounded by a closed curve s, and let us determine the surface integral of the sum of the principal stresses over the area within the curve. The integral is

$$\iint (P_1 + P_2)\,dxdy = \iint \left(\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2}\right)dxdy \qquad (25).$$

By a well-known theorem, corresponding in two dimensions to that of GREEN in three dimensions, the latter expression becomes, when once integrated,

$$\int \left(\frac{dF}{dy}\frac{dx}{ds} - \frac{dF}{dx}\frac{dy}{ds}\right)ds \qquad (26),$$

or

$$\int\left(\xi\frac{dx}{ds} + \eta\frac{dy}{ds}\right)ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

These line integrals are to be taken round the closed curve s. If we take a point in the curve s as origin in the original body, and the corresponding point in σ as origin in the diagram of stress, then ξ and η are the components of the whole stress on the right hand of the curve from the origin to a given point. If ρ denote the line joining the origin with the point $\xi\eta$, then ρ will represent in direction and magnitude the whole stress on the arc σ.

The line integral may now be interpreted as the work done on a point which travels once round the closed curve s, and is everywhere acted on by a force represented in direction and magnitude by ρ. We may express this quantity in terms of the stress at every point of the curve, instead of the resultant stress on the whole arc, as follows :—

For integrating (27) by parts it becomes,

$$-\int\left(x\frac{d\xi}{ds} + y\frac{dy}{ds}\right)ds = -\int(\mathrm{X}x + \mathrm{Y}y)\,ds \quad . \quad . \quad (28),$$

or if $\mathrm{R}ds$ is the actual stress on ds, and r is the radius vector of ds, and if R makes with r an angle ϵ, we obtain the result

$$\iint(\mathrm{P}_1 + \mathrm{P}_2)dxdy = -\int \mathrm{R}r\cos\epsilon . ds \, . \quad . \quad . \quad . \quad . \quad (29).$$

This line integral, therefore, which depends only on the stress acting on the closed curve s, is equal to the surface integral of the sum of the principal stresses taken over the whole area within the curve.

If there is no stress on the curve s acting from without, then the surface integral vanishes. This is the extension to the case of continuous stress of the theorem, given at p. 13, that the algebraic sum of all the tensions multiplied each by the length of the piece in which it acts is zero for a system in equilibrium. In the case of a frame, the stress in each piece is longitudinal, and the whole pressure or tension of the piece is equal to the longitudinal stress multiplied by the section, so that the integral $\iint(\mathrm{P}_1 + \mathrm{P}_2)dxdy$ for each piece is its tension multiplied by its length.

If the closed curve s is a small circle, the corresponding curve σ will be an ellipse, and the stress on any diameter of the circle will be represented in direction and magnitude by the corresponding diameter of the ellipse. Hence, the principal axes of the ellipse represent in direction and magnitude the principal stresses at the centre of the circle.

Let us next consider the surface integral of the product of the principal stresses at every point taken over the area within the closed curve s.

$$\iint \mathrm{P}_1\mathrm{P}_2dxdy = \iint\left(\frac{d^2\mathrm{F}}{dx^2}\frac{d^2\mathrm{F}}{dy^2} - \overline{\frac{d^2\mathrm{F}}{dxdy}}\Big|^2\right)dxdy \quad . \quad . \quad . \quad (30),$$

$$= \iint\left(\frac{d\xi}{dx}\frac{d\eta}{dy} - \frac{d\xi}{dy}\frac{d\eta}{dx}\right)dxdy\,,$$

or by transformation of variables

$$=\iint d\xi d\eta \, .$$

Hence the surface integral of the product of the principal stresses within the curve is equal to the area of the corresponding curve σ in the diagram of stress, and therefore depends entirely on the external stress on the curve s. This is seen from the construction of the curve σ in the diagram of stress, since each element $d\sigma$ represents the stress on the corresponding element ds of the original curve.

If ρ represents in direction and magnitude the resultant of the stress on the curve s from the origin to a point which moves round the curve, then the area traced out by ρ is equal to the surface-integral required. If Xds and Yds are the components of the stress on the element ds, and l the whole length of the closed curve s, then the surface integral is equal to either of the quantities.

$$\int_0^l \mathrm{Y} \int_0^s \mathrm{X} ds \, . \, ds \, , \quad \text{or} \; -\int_0^l \mathrm{X} \int_0^s \mathrm{Y} ds \, . \, ds \, .$$

In a frame the stress in each piece is entirely longitudinal, so that the product of the principal stresses is zero, and therefore nothing is contributed to the surface integral except at the points where the pieces meet or cross each other. To find the value of the integral for any one of these points, draw a closed curve surrounding it and no other point, and therefore cutting all the pieces which meet in that point in order. The corresponding figure in the diagram of stress will be a polygon, whose sides represent in magnitude and direction the tensions in the several pieces *taken in order*. The area of this polygon, therefore, represents the value of $\iint \mathrm{P}_1 \mathrm{P}_2 dxdy$ for the point of concourse, and is to be considered positive or negative, according as the tracing point travels round it in the positive or the negative cyclical direction.

Hence the following theorem, which is applicable to all plane frames, whether a diagram of forces can be drawn or not.

For each point of concourse or of intersection construct a polygon, by drawing in succession lines parallel and proportional to the forces acting on the point in the several pieces which meet in that point, taking the pieces in cyclical order round the point. The area of this polygon is to be taken positive or negative, according as it lies on the left or the right of the tracing point.

If, then, a closed curve be drawn surrounding the entire frame, and a polygon be drawn by drawing in succession lines parallel and proportional to all the external forces which act on the frame in the order in which their lines of direction meet the closed curve, then the area of this polygon is equal to the algebraic sum of the areas of the polygons corresponding to the various points of the frame.

In this theorem a polygon is to be drawn for every point, whether the lines of the frame meet or intersect, whether they are really jointed together, or whether two pieces simply cross each other without mechanical connection. In the latter case the polygon is a parallelogram, whose sides are parallel and proportional to the stresses in the two pieces, and it is positive or negative according as these stresses are of the same or of opposite signs.

If three or more pieces intersect, it is manifestly the same whether they intersect at one point or not, so that we have the following theorem :—

The area of a polygon of an even number of sides, whose opposite sides are equal and parallel, is equal to the sum of the areas of all the different parallelograms which can be formed with their sides parallel and equal to those of the polygon.

This is easily shown by dividing the polygon into the different parallelograms.

On the Equilibrium of Stress in a Solid Body.

Let PQR be the longitudinal, and STU the tangential components of stress, as indicated in the following table of stresses and strains, taken from THOMSON and TAIT'S "Natural Philosophy," p. 511, § 669 :—

Components of the		Planes, of which Relative Motion, or across which Force, is reckoned.	Direction of Relative Motion or of Force.
Strain.	Stress.		
e	P	yz	x
f	Q	zx	y
g	R	xy	z
a	S	$\left\{\begin{matrix} yx \\ zx \end{matrix}\right.$	y z
b	T	$\left\{\begin{matrix} zy \\ xy \end{matrix}\right.$	z x
c	U	$\left\{\begin{matrix} xz \\ yz \end{matrix}\right.$	x y

Then the equations of equilibrium of an element of the body are, by § 697 of that work,

$$\left.\begin{aligned} \frac{dP}{dx} + \frac{dU}{dy} + \frac{dT}{dz} + X = 0 \\ \frac{dU}{dx} + \frac{dQ}{dy} + \frac{dS}{dz} + Y = 0 \\ \frac{dT}{dx} + \frac{dS}{dy} + \frac{dR}{dz} + Z - 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1),$$

If we assume three functions ABC, such that

and put

$$\left.\begin{aligned} S &= -\frac{d^2A}{dydz} & T &= -\frac{d^2B}{dzdx} & U &= -\frac{d^2C}{dxdy} \\ X &= \frac{dV}{dx} & Y &= \frac{dV}{dy} & Z &= \frac{dV}{dz} \end{aligned}\right\} \quad . \quad (2),$$

then a sufficiently general solution of the equations of equilibrium is given by putting

$$\left.\begin{aligned} P &= \frac{d^2B}{dz^2} + \frac{d^2C}{dy^2} - V \\ Q &= \frac{d^2C}{dx^2} + \frac{d^2A}{dz^2} - V \\ R &= \frac{d^2A}{dy^2} + \frac{d^2B}{dx^2} - V \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3),$$

I am not aware of any method of finding other relations between the components of stress without making further assumptions. The most natural assumption to make is that the stress arises from elasticity in the body. I shall confine myself to the case of an isotropic body, such that it can be deprived of all stress and strain by a removal of the applied forces. In this case, if $\alpha\,\beta\,\gamma$ are the components of displacement, and n the co-efficient of rigidity, the equations of tangential elasticity are, by equation (6) §§ 670 and 694 of THOMSON and TAIT,

$$a = \frac{d\beta}{dz} + \frac{d\gamma}{dy} = \frac{1}{n}S = -\frac{1}{n}\frac{d^2A}{dydz} \quad . \quad . \quad . \quad . \quad . \quad (4),$$

with similar equations for b and c. A sufficiently general solution of these equations is given by putting

$$\left.\begin{aligned} \alpha &= \frac{1}{2n}\frac{d}{dx}\Big(A - B - C\Big) \\ \beta &= \frac{1}{2n}\frac{d}{dy}\Big(B - C - A\Big) \\ \gamma &= \frac{1}{2n}\frac{d}{dz}\Big(C - A - B\Big) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5).$$

The equations of longitudinal elasticity are of the form given in § 693,

$$P = \left(k + \frac{4}{3}n\right)\frac{d\alpha}{dx} + \left(k - \frac{2}{3}n\right)\left(\frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) \quad . \quad . \quad . \quad (6),$$

where k is the co-efficient of cubical elasticity, with similar equations for Q and R. Substituting for P, α, β and γ in equation (6) their values from (3) and (5),

$$2n\left(\frac{d^2B}{dz^2} + \frac{d^2C}{dy^2} - V\right) = \left(k + \frac{4}{3}n\right)\left(\frac{d^2A}{dx^2} - \frac{d^2B}{dx^2} - \frac{d^2C}{dx^2}\right) + \left(k - \frac{2}{3}n\right)\left(\frac{d^2B}{dy^2} - \frac{d^2C}{dy^2} - \frac{d^2A}{dy^2} + \frac{d^2C}{dz^2} - \frac{d^2A}{dz^2} - \frac{d^2B}{dz^2}\right).$$

If we put

$$\frac{d^2A}{dx^2}+\frac{d^2B}{dy^2}+\frac{d^2C}{dz^2}=p, \text{ and } \frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}=\Delta^2\ ,$$

this equation becomes

$$\left(k+\frac{4}{3}n\right)\left(\Delta^2A+\Delta^2B+\Delta^2C\right)-\left(k+\frac{1}{3}n\right)2p-2nV=2n\Delta^2A, \quad . \quad (7).$$

We have also two other equations differing from this only in having B and C instead of A on the right hand side. Hence equating the three expressions on the right hand side we find

$$\Delta^2A=\Delta^2B=\Delta^2C=D^2\text{, say,} \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$(3k+n)2p=(3k+2n)3D^2-2nV, \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and

$$P+Q+R=\frac{9k}{2}\frac{D^2-2V}{3k+n}=3k\frac{p-3V}{3k+2n} \quad . \quad . \quad . \quad . \quad . \quad (10).$$

These equations are useful when we wish to determine the stress rather than the strain in a body. For instance, if the co-efficients of elasticity, k and n, are increased in the same ratio to any extent, the displacements of the body are proportionally diminished, but the stresses remain the same, and, though their distribution depends essentially on the elasticity of the various parts of the body, the values of the internal forces do not contain the co-efficients of elasticity as factors.

There are two cases in which the functions may be treated as functions of two variables.

The first is when there is no stress, or a constant pressure in the direction of z, as in the case of a stratum originally of uniform thickness, in the direction of z, the thickness being small compared with the other dimensions of the body, and with the rate of variation of strain.

The second is when there is no strain, or a uniform longitudinal strain in the direction of z, as in the case of a prismatic body whose length in the direction of z is very great, the forces on the sides being functions of x and y only.

In both of these cases S = 0 and T = 0, so that we may write

$$P=\frac{d^2C}{dy^2}-V \qquad U=-\frac{d^2C}{dxdy} \qquad Q=\frac{d^2C}{dx^2}-V \quad . \quad . \quad (11).$$

This method of expressing the stresses in two dimensions was first given by the Astronomer Royal, in the "Philosophical Transactions" for 1863. We shall write F instead of C, and call it AIRY's Function of Stress in Two Dimensions.

Let us assume two functions, G and H, such that

$$F=\frac{d^2G}{dxdy} \text{ and } V=\frac{d^2H}{dxdy} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

then by THOMSON and TAIT, § 694, if α is the displacement in the direction of x

$$2n(\sigma+1)\frac{d\alpha}{dx} = \mathrm{P} - \sigma(\mathrm{Q}+\mathrm{R}) \quad . \quad . \quad . \quad . \qquad (13).$$

CASE I.—If $\mathrm{R}=0$ this becomes

$$2n(\sigma+1)\frac{d\alpha}{dx} = \frac{d^2}{dxdy}\left\{\frac{d^2\mathrm{G}}{dy^2} - \sigma\frac{d^2\mathrm{G}}{dx^2} + (\sigma-1)\mathrm{H}\right\}.$$

Integrating with respect to x we find the following equation for α—

$$2n(\sigma+1)\alpha = \frac{d}{dy}\left\{\frac{d^2\mathrm{G}}{dy^2} - \sigma\frac{d^2\mathrm{G}}{dx^2} + (\sigma-1)\mathrm{H}\right\} + \mathrm{Y} \quad . \qquad (14),$$

where Y is a function of y only. Similarly for the displacement β in the direction of y,

$$2n(\sigma+1)\beta = \frac{d}{dx}\left\{\frac{d^2\mathrm{G}}{dx^2} - \sigma\frac{d^2\mathrm{G}}{dy^2} + (\sigma-1)\mathrm{H}\right\} + \mathrm{X} \quad . \quad . \qquad (15),$$

where X is a function of x only. Now the shearing stress U depends on the shearing strain and the rigidity, or

$$\mathrm{U} = n\left(\frac{d\alpha}{dy} + \frac{d\beta}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16).$$

Multiplying both sides of this equation by $2(\sigma+1)$ and substituting from (11), (14), and (15),

$$-2(\sigma+1)\frac{d^4\mathrm{G}}{dx^2dy^2} = \frac{d^4\mathrm{G}}{dy^4} - 2\sigma\frac{d^4\mathrm{G}}{dx^2dy^2} + \frac{d^4\mathrm{G}}{dx^4} + (\sigma-1)\left(\frac{d^2\mathrm{H}}{dx^2} + \frac{d^2\mathrm{H}}{dy^2}\right) + \frac{d\mathrm{X}}{dx} + \frac{d\mathrm{Y}}{dy} \quad (17).$$

Hence

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)^2\mathrm{G} + \frac{d\mathrm{X}}{dx} + \frac{d\mathrm{Y}}{dy} = (1-\sigma)\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\mathrm{H} \quad . \quad . \qquad (18),$$

an equation which must be fulfilled by G when the body is originally without strain.

CASE II.—In the second case, in which there is no strain in the direction of z, we have

$$\frac{d\mathrm{Y}}{dz} = \mathrm{R} - \sigma(\mathrm{P}+\mathrm{Q}) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19).$$

Substituting for R in (13), and dividing by $\sigma+1$,

$$2n\frac{d\alpha}{dx} = (1-\sigma)\mathrm{P} - \sigma\mathrm{Q}$$

$$= \frac{d^2}{dxdy}\left\{(1-\sigma)\frac{d^2\mathrm{G}}{dy^2} - \sigma\frac{d^2\mathrm{G}}{dx^2} + \sigma\mathrm{H}\right\} \quad . \quad . \quad . \qquad (20),$$

with a similar equation for β. Proceeding as in the former case, we find

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)^2\mathrm{G} + \frac{d\mathrm{X}}{dx} + \frac{d\mathrm{Y}}{dy} = \frac{\sigma}{1-\sigma}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\mathrm{H} \quad . \quad . \qquad (21).$$

This equation is identical with that of the first case, with the exception of the coefficient of the part due to H, which depends on the density of the body, and the value of σ, the ratio of lateral expansion to longitudinal compression.

Hence, if the external forces are given in the two cases of no stress and no strain in the direction of z, and if the density of the body or the intensity of the force acting on its substance is in the ratio of σ to $(1-\sigma)^2$ in the two cases, the internal forces will be the same in every part, and will be independent of the actual values of the coefficients of elasticity, provided the strains are small. The solutions of the cases treated by Mr AIRY, as given in his paper, do not exactly fulfil the conditions deduced from the theory of elasticity. In fact, the consideration of elastic strain is not explicitly introduced into the investigation. Nevertheless, his results are statically possible, and exceedingly near to the truth in the cases of ordinary beams.

As an illustration of the theory of AIRY'S Function, let us take the case of

$$F = \frac{1}{2p} r^{2p} \cos 2p\theta \quad . \quad . \quad . \quad . \quad . \quad (22).$$

In this case we have for the co-ordinates of the point in the diagram corresponding to (xy)

$$\xi = \frac{dF}{dx} = r^{2p-1} \cos (2p-1)\theta \qquad \eta = -r^{2p-1} \sin (2p-1)\theta \quad . \quad . \quad (23),$$

and for the components of stress

$$\left.\begin{aligned} p_{xx} &= \frac{d\eta}{dy} = -(2p-1)r^{2p-2} \cos (2p-2)\theta = -\frac{d\xi}{dx} = -p_{yy} \\ p_{xy} &= \frac{d^2F}{dx\,dy} = (2p-1)r^{2p-2} \sin (2p-2)\theta \end{aligned}\right\} \quad . \quad (24).$$

If we make

$$G = \frac{1}{p} r^{p} \cos p\theta \quad \text{and} \quad H = \frac{1}{p} r^{p} \sin p\theta \quad . \quad . \quad . \quad (25),$$

then

$$\left.\begin{aligned} (2p-1)\left(\overline{\frac{dG}{dx}}\bigg|^2 - \overline{\frac{dG}{dy}}\bigg|^2\right) p_{yy} &= -p_{xx} \\ (2p-1)\, 2\frac{dG}{dx}\frac{dG}{dy} &= p_{xy} \end{aligned}\right\} \quad . \quad (26).$$

Hence the curves for which G and H respectively are constant will be lines of principal stress, and the stress at any point will be inversely as the square of the distance between the consecutive curves G or H.

If we make

$$\xi = \rho \cos \phi \quad \text{and} \quad \eta = \rho \sin \phi$$

then we must have

$$\left.\begin{aligned} \rho &= r^{2p-1} \quad \text{and} \quad \phi = (2p-1)\theta \end{aligned}\right\} \quad (27).$$

If we put

$$q \text{ for } \frac{p}{2p-1} \quad \text{then} \quad \frac{1}{p} + \frac{1}{q} = 2 \quad \text{and} \quad (2p-1)(2q-1) = 1,$$

so that if f, g, h in the diagram of stress correspond to F, G, H in the original figure, we have

$$f = \frac{1}{2q}\rho^{2q}\cos 2q\phi \qquad g = \frac{1}{q}\rho^{q}\cos q\phi \qquad h = \frac{1}{q}\rho^{q}\sin q\phi \quad . \quad (28).$$

Case of a Uniform Horizontal Beam.

As an example of the application of the condition that the stresses must be such as are consistent with an initial condition of no strain, let us take the case of a uniform rectangular beam of indefinite length placed horizontally with a load $= h$ per unit of length placed on its upper surface, the weight of the beam being k per unit of length. Let us suppose the beam to be supported by vertical forces and couples in a vertical plane applied at the ends; but let us consider only the middle portion of the beam, where the conditions applicable to the ends have no sensible effect. Let the horizontal distance x be reckoned from the vertical plane where there is no shearing force, and let the planes where there is no moment of bending be at distances $\pm a_0$ from the origin. Let y be reckoned from the lower edge of the beam, and let b be the depth of the beam. Then, if $U = -\frac{d^2F}{dxdy}$ is the shearing stress, the total vertical shearing force through a vertical section at distance x is

$$\int_0^b U dy = \left(\frac{dF}{dx}\right)_{y=0} - \left(\frac{dF}{dx}\right)_{y=b},$$

and this must be equal and opposite to the weight of the beam and load from 0 to x, which is evidently $(h+k)x$.

Hence

$$\frac{dF}{dx} = -(h+k)x\phi(y) \quad \text{where} \quad \phi(b) - \phi(0) = 1 \quad . \quad . \quad . \quad (29).$$

From this we find the vertical stress

$$Q = \frac{d^2F}{dx^2} + \frac{k}{b}y = -(h+k)\phi(y) + \frac{k}{b}y \, .$$

The vertical stress is therefore a function of y only. It must vanish at the lower side of the beam, where $y = 0$, and it must be $-h$ on the upper side of the beam, where $y = b$. The shearing stress U must vanish at both sides of the beam, or $\phi'(y) = 0$, when $y = 0$, and when $y = b$.

The simplest form of $\phi(y)$ which will satisfy these conditions is

$$\phi(y) = \frac{1}{b^3}(3by^2 - 2y^3) \, .$$

Hence we find the following expression for the function of stress by integrating (29) with respect to x,

$$F = \frac{h+k}{2b^3}(a^2 - x^2)(3by^2 - 2y^3) + Y \quad . \quad . \quad . \quad . \quad (30),$$

where a is a constant introduced in integration, and depends on the manner in which the beam is supported. From this we obtain the values of the vertical, horizontal, and shearing stresses,

$$Q = \frac{d^2F}{dx^2} + \frac{k}{b}y = \frac{k}{b}y - \frac{h+k}{b^3}(3by^2 - 2y^3) \quad . \quad . \quad . \quad . \quad (31),$$

$$P = \frac{d^2F}{dy^2} = 3\frac{h+k}{b^3}(a^2 - x^2)(b - 2y) + \frac{d^2Y}{dy^2} \quad . \quad . \quad . \quad . \quad (32),$$

$$U = -\frac{d^2F}{dxdy} = 6\frac{h+k}{b^3}xy(b-y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$

The values of Q and of U, the vertical and the shearing stresses, as given by these equations, are perfectly definite in terms of h and k, the load and the weight of the beam per unit of length. The value of P, the horizontal stress, however, contains an arbitrary function Y, which we propose to find from the condition that the beam was originally unstrained. We therefore determine α and β, the horizontal and vertical displacement of any point (x, y), by the method indicated by equations (13), (14), (15)

$$2n(\sigma+1)\alpha = \frac{h+k}{b^3}\left\{(3a^2x - x^3)(b-2y) - \sigma x(3by^2 - 2y^3)\right\} - \sigma\frac{k}{b}xy + x\frac{d^2Y}{dy^2} + Y' \quad (34),$$

$$2n(\sigma+1)\beta = -\frac{h+k}{b^3}\left\{\left(by^3 - \frac{1}{2}y^4\right) + 3\sigma(a^2 - x^2)(by - y^2)\right\} + \frac{1}{2}\frac{k}{b}y^2 - \sigma\frac{dY}{dy} + X' \quad (35),$$

where X' is a function of x only, and Y' of y only. Deducing from these displacements the shearing strain, and comparing it with the value of the shearing stress, U, we find the equation

$$\frac{h+k}{b^3}\left\{6a^2x - 2x^3 + 12x(by - y^2)\right\} + \sigma\frac{k}{b}x = x\frac{d^3Y}{dy^3} + \frac{dX'}{dx} + \frac{dY'}{dy} \quad . \quad (36).$$

Hence

$$\frac{d^3Y}{dy^3} = 12\frac{h+k}{b^3}(by - y^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

$$\frac{dX'}{dx} = 2\frac{h+k}{b^3}(3a^2x - 2x^3) + \sigma\frac{k}{bx}, \quad \frac{dY'}{dy} = 0 \quad . \quad . \quad (38).$$

If the total longitudinal stress across any vertical section of the beam is zero, the value of $\frac{dF}{dy}$ must be the same when $y = 0$ and when $y = b$. From this condition we find the value of P by equation (32)

$$P = \frac{h+k}{b^3}\left\{3(a^2 - x^2) + 2y^2 - 2by - b^2\right\}(b - 2y) \quad . \quad . \quad . \quad (39).$$

The moment of bending at any vertical section of the beam is

$$\int_0^b Py\,dy = (h+k)\left(\frac{1}{2}(x^2 - a^2) + \frac{1}{5}b^2\right) \quad . \quad . \quad . \quad . \quad (40).$$

This becomes zero when $x = \pm a_0$ where

$$a_0^2 = a^2 - \frac{2}{5}b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (41).$$

If we wish to compare this case with that of a beam of finite length supported at both ends and loaded uniformly, we must make the moment of bending zero at the supports, and the length of the beam between the supports must therefore be $2a_0$. Substituting a_0 for a in the value of P, we find

$$P = \frac{h+k}{b^3}\left(3a_0^2 - 3x^2 + 2y^2 - 2by + \frac{1}{5}b^2\right)(b-2y) \quad . \quad . \quad . \quad (42).$$

If we suppose the beam to be cut off just beyond the supports, and supported by an intense pressure over a small area, we introduce conditions into the problem which are not fulfilled by this solution, and the investigation of which requires the use of FOURIER's series. In order that our result may be true, we must suppose the beam to extend to a considerable distance beyond the supports on either side, and the vertical forces to be applied by means of frames clamped to the ends of the beam, as in Diagram V*a*, so that the stresses arising from the discontinuity at the extremities are insensible in the part of the beam between the supports.

This expression differs from that given by Mr AIRY only in the terms in the longitudinal stress P depending on the function Y, which was introduced in order to fulfil the condition that, when no force is applied, the beam is unstrained. The effect of these terms is a maximum when $y = \cdot 12788\, b$, and is then equal to $(h + k)\cdot 314$, or less than a third of the pressure of the beam and its load on a flat horizontal surface when laid upon it so as to produce a uniform vertical pressure $h + k$.

EXPLANATION OF THE DIAGRAMS (Plates I. II. III.).

Diagrams I.*a* and I.*b* illustrate the necessity of the condition of the possibility of reciprocal diagrams, that each line must be a side of two, and only two, polygons. Diagram I.*a* is a skeleton of a frame such, that if the force along any one piece be given, the force along any other piece may be determined. But the piece N forms a side of four triangles, NFH, NGI, NJL, and NKM, so that if there could be a reciprocal diagram, the line corresponding to N would have four extremities, which is impossible. In this case we can draw a diagram of forces in which the forces H, I, J, and K are each represented by two parallel lines.

Diagrams II.*a* and II.*b* illustrate the case of a frame consisting of thirty-two pieces, meeting four and four in sixteen points, and forming sixteen quadrilaterals. Diagram II.*a* may be considered as a plane projection of a polyhedron of double continuity, which we may describe as a quadrilateral frame consisting of four quadrilateral rods, of which the ends are bevelled so as to fit exactly. The projection of this frame, considered as a plane frame, has three degrees of stiffness, so that three of the forces may be arbitrarily assumed.

In the reciprocal diagram II.*b* the lines are drawn by the method given at p. 7, so that each line is perpendicular to the corresponding line in the other figure. To make the corresponding lines parallel we have only to turn one of the figures round a right angle.

Diagrams III.*a* and III.*b* illustrate the principle as applied to a bridge designed by Professor F. Jenkin. The loads Q_1 Q_2, &c., are placed on the upper series of joints, and R_1 R_2, &c., on the lower series. The diagram III.*b* gives the stresses due to both sets of loads, the vertical lines of loads being different for the two series.

Diagrams IV.*a* and IV.*b* illustrate the application of Airy's Function to the construction of diagrams of continuous stress.

IV.*a* represents a cylinder exposed to pressure in a vertical and horizontal direction, and to tension in directions inclined 45° to these. The lines marked *a, b, c*, &c., are lines of pressure, and those marked *o, p, q*, are lines of tension. In this case the lines of pressure and tension are rectangular hyperbolas, the pressure is always equal to the tension, and varies inversely as the square of the distance between consecutive curves, or, what is the same thing, directly as the square of the distance from the centre.

IV.*b* represents the reciprocal diagram corresponding to the upper quadrant of the former one. The stress on any line in the first diagram is represented in magnitude and direction by the corresponding line in the second diagram, the correspondence being ascertained by that of the corresponding systems of lines *a, b, c*, &c., and *o, p, q*, &c.

We may also consider IV.*b* as a sector of a cylinder of 270°, exposed to pressure along the lines *a, b, c*, and to tension along *o, p, q*, the magnitude of the stress being in this case $r^{-\frac{2}{3}}$. The upper quadrant of IV.*a* is in this case the reciprocal figure. This figure illustrates the tendency of any strained body to be ruptured at a re-entering angle, for it is plain that at the angle the stress becomes indefinitely great.

In diagram IV.*a*—

$$F = \frac{1}{4} r^4 \cos 4\theta \qquad G = \frac{1}{2} r^2 \cos 2\theta \qquad H = \frac{1}{2} r^2 \sin 2\theta .$$

In diagram IV.*b*—

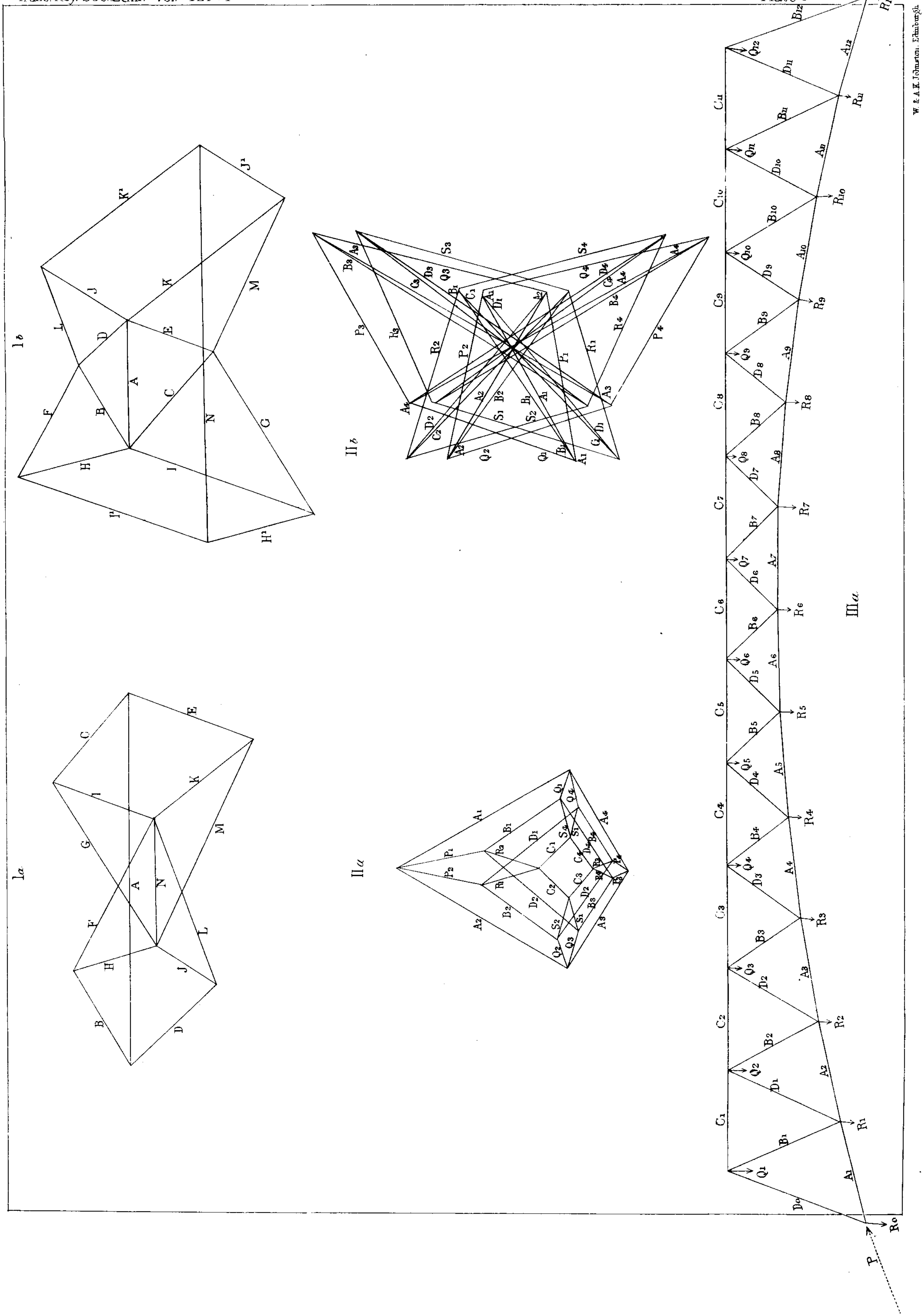
$$f = \frac{3}{4} \rho^{\frac{4}{3}} \cos \frac{4}{3}\phi \qquad g = \frac{3}{2} \rho^{\frac{2}{3}} \cos \frac{2}{3}\phi \qquad h = \frac{3}{2} \rho^{\frac{2}{3}} \sin \frac{2}{3}\phi .$$

Diagrams V.*a* and V.*b* illustrate Airy's theory of stress in beams.

V.*a* is the beam supported at C and D by means of bent pieces clamped to the ends of the beam at A and B, at such a distance from C and D, that the part of the beam between C and D is free from the local effects of the pressures of the clamps at A and B. The beam is divided into six strata by horizontal dotted lines, marked 1, 2, 3, 4, 5, 6, and into sixteen vertical slices by vertical lines marked *a*, *b*, *c*, &c.

The corresponding lines in the diagram V.*b* are marked with corresponding figures and letters. The stress across any line joining any two points in V.*a* is represented in magnitude by the line in V.*b*, joining corresponding points, and is perpendicular to it in direction.

These illustrations of the application of the graphic method to cases of continuous stress, are intended rather to show the mathematical meaning of the method, than as practical aids to the engineer. In calculating the stresses in frames, the graphic method is really useful, and is less liable to accidental errors than the method of trigonometrical calculation. In cases of continuous stress, however, the symbolical method of calculation is still the best, although, as I have endeavoured to show in this paper, analytical methods may be explained, illustrated, and extended by considerations derived from the graphic method.



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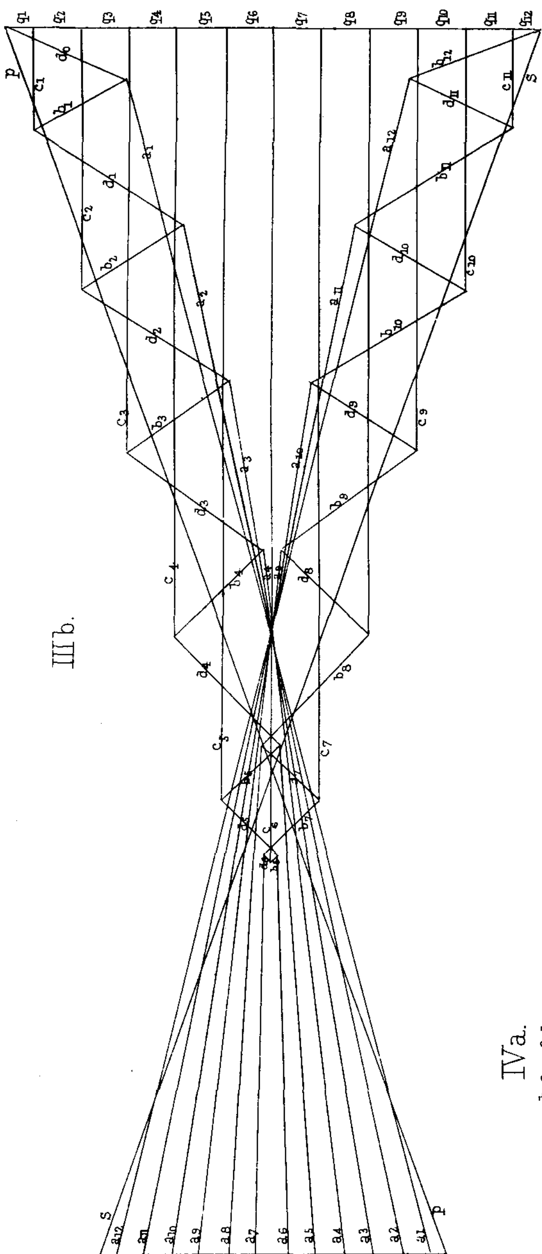
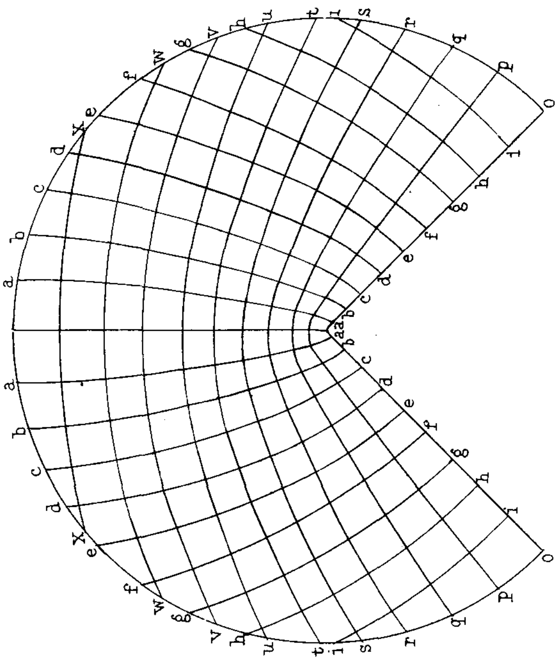
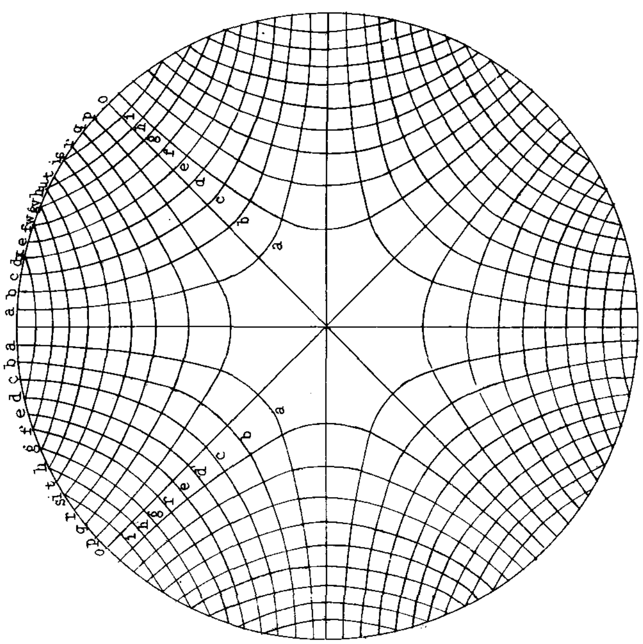
III b.



IV b.



IV a.



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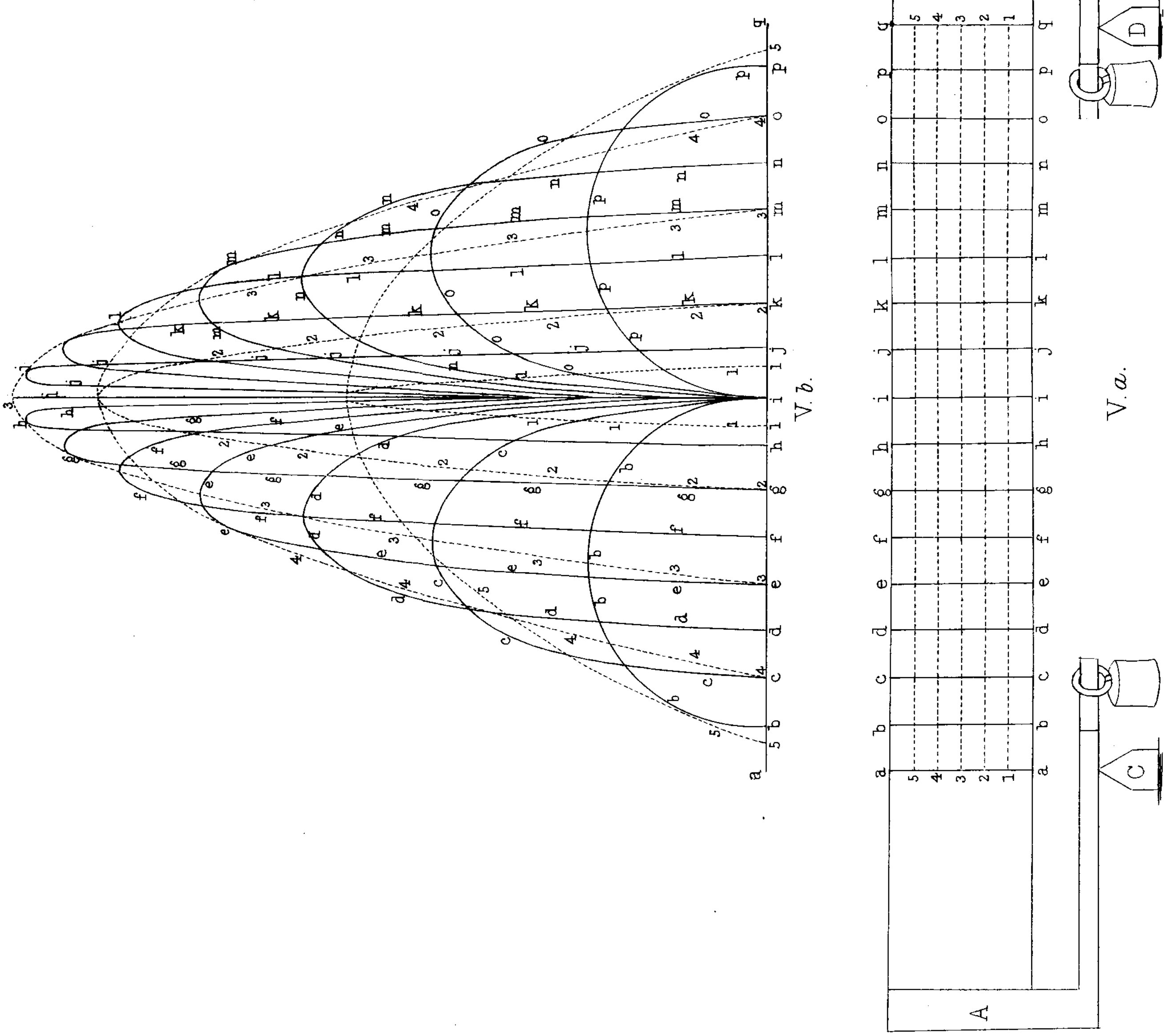

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