# 14.

# De transformatione expressionis $\frac{\partial y}{\sqrt{[\pm(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]}}$ in formam simpliciorem $\frac{\partial x}{M\sqrt{[(1-xx)(1-\varkappa^2 xx)]}}$, adhibita substitutione $x=\frac{a+a'y+a''y^2}{1+b'y+b''y^2}$.

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Duplex in universum problema propositum solvendi genus cogitari potest. Alterum eo constat, ut expressio $\frac{\partial y}{\sqrt{[\pm(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]}}$, substituendo valorem ipsius $y$ in formam $\frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2 x^2)]}}$ transformetur. At quantitas $y$ irrationaliter per variabilem $x$ exprimitur, unde fit, ut expressio, facta substitutione sub radicali denominatoris oriunda, irrationales argumenti $x$ contineat functiones. Quae quidem res impedimento est, quominus a priori apta ad problematis solutionem methodus inveniatur. Quam ob causam ad alterum problema tractandi genus confugere praestat. Quod eo consistit, ut expressio $\frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2 x^2)]}}$, substituto valore ipsius $x$, constantibusque $a$, $a'$, $a''$, $b'$, $b''$ rite determinatis formam $\frac{\partial y}{\sqrt{[(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]}}$ induat. Sub oculos cadit, hanc viam esse priore multo planiorem, quippe quia difficultati illi, quae ex irrationali substitutione ortum ducet, hic nihil est loci. Quam ingrediamur.

Methodi, quibus utemur, eaedem sunt atque in *Fundamentis novis th. f. ellipt* a Cl. *Jacobi* adhibitae, ubi problema propositum indicatum invenis pag. 17.

Expressio $\frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2 x^2)]}}$, posito $x=\frac{a+a'y+a''y^2}{1+b'y+b''y^2}=\frac{U}{V}$ abit in sequentem

$$\frac{V\partial U-U\partial V}{M\sqrt{[(V-U)(V+U)(V-\varkappa U)(V+\varkappa U)]}},$$

in qua functio, quae sub radicali continetur, ad octavum usque ordinem adsurgit, unusquisque enim quatuor factorum $V-U$, $V+U$, $V-\varkappa U$, $V+\varkappa U$ secundi est ordinis. Quodsi igitur duo e quatuor factoribus modo dictis fierent quadratici, functio, quae sub radicale remaneret, quarti foret ordinis. Quod ut eveniat, duae requiruntur aequationes conditionales inter constantes indeterminatas, ita ut tres earum arbitrariae sunt, quae iuxta cum multiplicatore $M$ et modulum $\varkappa$ eo adhiberi possunt, ut functioni sub radicali forma $(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)$ concilietur.

Posito, factores $V-\varkappa U$, $V+\varkappa U$ fieri quadraticos, reputatisque relationibus,

$$(V-\varkappa U)\,\partial U-U\partial(V-\varkappa U) = V\partial U-U\partial V,$$
$$(V+\varkappa U)\,\partial U-U\partial(V+\varkappa U) = V\partial U-U\partial V$$

liquet, unamquamque functionem, quae unum ex factoribus $V-\varkappa U$, $V+\varkappa U$ bis metiatur, et expressionem $\frac{V\partial U-U\partial V}{\partial y}$ metiri. Est vero quantitas $\frac{V\partial U-U\partial V}{\partial y}$ secundi ordinis, ejusdemque est radix secunda e duobus factoribus quadraticis; ideoque secundum proprietatem, modo commemoratam, quotiens

$$\frac{V\partial U-U\partial V}{\partial y\sqrt{[(V-\varkappa U)(V+\varkappa U)]}}$$

aequalis fit quantitati cuidam constanti.

Quia unusquisque factorum $V-U$, $V+U$, $V-\varkappa U$, $V+\varkappa U$ est functio secundi ordinis ipsius elementi $y$, duo postrema adeo quadrata, ponere licet:

$$1)\quad V-\ U = A(y-\alpha)(y-\beta),$$
$$2)\quad V+\ U = B(y-\gamma)(y-\delta),$$
$$3)\quad V-\varkappa U = C(y+m)^2,$$
$$4)\quad V+\varkappa U = D(y+n)^2,$$

designantibus $A$, $B$, $C$, $D$ constantes, quarum una pro lubito assumi potest.

Posito et $y=\alpha$ et $y=\beta$ erit ex aequat. 1), $V=U$; ideoque secundum aeqq. 2) et 3) obtinetur:

$$\frac{1-\varkappa}{2}=\frac{C(\alpha+m)^2}{B(\alpha-\gamma)(\alpha-\delta)};\qquad \frac{1-\varkappa}{2}=\frac{C(\beta+m)^2}{B(\beta-\gamma)(\beta-\delta)},$$

unde

$$\frac{\alpha+m}{\beta+m}=\pm\frac{\sqrt{[(\alpha-\gamma)(\alpha-\delta)]}}{\sqrt{[(\beta-\gamma)(\beta-\delta)]}};$$

ergo, prout superius an inferius sumseris signum, erit:

$$m = m_1 = \frac{\beta\sqrt{[(\alpha-\gamma)(\alpha-\delta)]} - \alpha\sqrt{[(\beta-\gamma)(\beta-\delta)]}}{\sqrt{[(\beta-\gamma)(\beta-\delta)]} - \sqrt{[(\alpha-\gamma)(\alpha-\delta)]}},$$

$$m = m_2 = -\frac{\beta\sqrt{[(\alpha-\gamma)(\alpha-\delta)]} + \alpha\sqrt{[(\beta-\gamma)(\beta-\delta)]}}{\sqrt{[(\beta-\gamma)(\beta-\delta)]} + \sqrt{[(\alpha-\gamma)(\alpha-\delta)]}}.$$

Si in locum aequationis 3) adhibemus 4), videmus iisdem quantitatibus determinari $n$ atque $m$; at $m$ et $n$ aequales esse nequeunt; haberetur enim $\frac{V-\varkappa U}{V+\varkappa U} = \frac{C}{D}$ ideoque ipsa $x$ constanti aequalis; ergo alter valorum $m_1$, $m_2$ quantitati $m$, alter quantitati $m$ aequiparandus est.

Si posuisses $y=\gamma$, $y=\delta$, aequationesque 1), 3) et 4) adhibuisses ad valores ipsarum $m$ et $n$ eruendos, tum obtinuisses:

$$m_1 = -\frac{\gamma\sqrt{[(\alpha-\delta)(\beta-\delta)]} + \delta\sqrt{[(\alpha-\gamma)(\beta-\gamma)]}}{\sqrt{[(\alpha-\gamma)(\beta-\gamma)]} + \sqrt{[(\alpha-\delta)(\beta-\delta)]}},$$

$$m_2 = \frac{\delta\sqrt{[(\alpha-\gamma)(\beta-\gamma)]} - \gamma\sqrt{[(\alpha-\delta)(\beta-\delta)]}}{\sqrt{[(\alpha-\delta)(\beta-\delta)]} - \sqrt{[(\alpha-\gamma)(\beta-\gamma)]}}.$$

Quos valores cum supra inventis identicos esse, facile patet; si ex denominatoribus radicalia tollis, ex utrisque valoribus obtines:

$$\left.\begin{matrix} m_1 \\ m_2 \end{matrix}\right\} = \frac{-(\alpha\beta-\gamma\delta) \pm \sqrt{[(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)]}}{\alpha+\beta-\gamma-\delta}.$$

Est igitur, substitutis ipsarum $m$ et $n$ valoribus, $n=m_1$, $m=m_2$:

1) $V - U = A(y-\alpha)(y-\beta)$,

2) $V + U = B(y-\gamma)(y-\delta)$,

3) $$V-\varkappa U = C\left[y - \frac{\beta\sqrt{[(\alpha-\gamma)(\alpha-\delta)]} + \alpha\sqrt{[(\beta-\gamma)(\beta-\delta)]}}{\sqrt{[(\beta-\gamma)(\beta-\delta)]} + \sqrt{[(\alpha-\gamma)(\alpha-\delta)]}}\right]^2$$
$$= C\left[y + \frac{\delta\sqrt{[(\alpha-\gamma)(\beta-\gamma)]} - \gamma\sqrt{[(\alpha-\delta)(\beta-\delta)]}}{\sqrt{[(\alpha-\delta)(\beta-\delta)]} - \sqrt{[(\alpha-\gamma)(\beta-\gamma)]}}\right]^2,$$

4) $$V+\varkappa U = D\left[y + \frac{\beta\sqrt{[(\alpha-\gamma)(\alpha-\delta)]} - \alpha\sqrt{[(\beta-\gamma)(\beta-\delta)]}}{\sqrt{[(\beta-\gamma)(\beta-\delta)]} - \sqrt{[(\alpha-\gamma)(\alpha-\delta)]}}\right]^2$$
$$= D\left[y - \frac{\gamma\sqrt{[(\alpha-\delta)(\beta-\delta)]} + \delta\sqrt{[(\alpha-\gamma)(\beta-\gamma)]}}{\sqrt{[(\alpha-\delta)(\beta-\delta)]} + \sqrt{[(\alpha-\gamma)(\beta-\gamma)]}}\right]^2.$$

Posito $y=\alpha$, quo casu $V=U$, obtinetur ex aeqq. 3) et 4):

$$a. \quad \frac{1-\varkappa}{1+\varkappa} = \frac{C[\sqrt{((\beta-\gamma)(\beta-\delta))} - \sqrt{((\alpha-\gamma)(\alpha-\delta))}]^2}{D[\sqrt{((\beta-\gamma)(\beta-\delta))} + \sqrt{((\alpha-\gamma)(\alpha-\delta))}]^2}.$$

Facto porro $y=\gamma$, quo casu $V=-U$, ex iisdem aeqq. 3) et 4) sequitur:

$$b. \quad \frac{1+\varkappa}{1-\varkappa} = \frac{C}{D}\cdot\left[\frac{\sqrt{[(\alpha-\gamma)(\beta-\gamma)]} + \sqrt{[(\alpha-\delta)(\beta-\delta)]}}{\sqrt{[(\alpha-\gamma)(\alpha-\delta)]} - \sqrt{[(\alpha-\delta)(\beta-\delta)]}}\right]^2.$$

Aequationibus (*a.*) et (*b.*) in se ductis, fit

$$\frac{C^2}{D^2} = \frac{[\sqrt{((\alpha-\gamma)(\beta-\gamma))} - \sqrt{((\alpha-\delta)(\beta-\delta))}]^2\,[\sqrt{((\beta-\gamma)(\beta-\delta))} + \sqrt{((\alpha-\gamma)(\alpha-\delta))}]^2}{[\sqrt{((\alpha-\gamma)(\beta-\gamma))} - \sqrt{((\alpha-\delta)(\beta-\delta))}]^2\,[\sqrt{((\beta-\gamma)(\beta-\delta))} + \sqrt{((\alpha-\gamma)(\alpha-\delta))}]^2};$$

**unde, cum e constantibus $C, D$ altera ex arbitrio accipi possit, statuere licet:**

$$C=[\sqrt{((\alpha-\gamma)(\beta-\gamma))}-\sqrt{((\alpha-\delta)(\beta-\delta))}][\sqrt{((\beta-\gamma)(\beta-\delta))}+\sqrt{((\alpha-\gamma)(\alpha-\delta))}],$$

$$D=[\sqrt{((\alpha-\gamma)(\beta-\gamma))}+\sqrt{((\alpha-\delta)(\beta-\delta))}][\sqrt{((\beta-\gamma)(\beta-\delta))}-\sqrt{((\alpha-\gamma)(\alpha-\delta))}].$$

Divisa autem aequatione (*a.*) per aequationem (*b.*), nanciscimur

$$\left(\frac{1-\varkappa}{1+\varkappa}\right)^2=\frac{[\sqrt{((\alpha-\gamma)(\beta-\delta))}-\sqrt{((\alpha-\delta)(\beta-\gamma))}]^2}{[\sqrt{((\alpha-\gamma))(\beta-\delta)}+\sqrt{((\alpha-\delta)(\beta-\gamma))}]^2},$$

unde sequitur

$$\varkappa=\frac{\sqrt{[(\alpha-\delta)(\beta-\gamma)]}}{\sqrt{[(\alpha-\gamma)(\beta-\delta)]}}.$$

Cum sit

$$(\alpha-\gamma)(\beta-\delta)=(\alpha-\beta)(\gamma-\delta)+(\alpha-\delta)(\beta-\gamma),$$

e valore ipsius $\varkappa$ invento sequitur etiam:

$$\sqrt{(1-\varkappa^2)}=\sqrt{\left(\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)}\right)}.$$

Ad valores constantium $A$ et $B$ determinandos ponatur

$$y=-m=\frac{\alpha\sqrt{[(\beta-\gamma)(\beta-\delta)]}+\beta\sqrt{[(\alpha-\gamma)(\alpha-\delta)]}}{\sqrt{[(\beta-\gamma)(\beta-\delta)]}+\sqrt{[(\alpha-\gamma)(\alpha-\delta)]}},$$

quo casu $V=\varkappa U$; quo facto ex aequationibus 1) et 4) erit

$$\frac{\varkappa-1}{2\varkappa}=\frac{-A[\sqrt{((\beta-\gamma)(\beta-\delta))}-\sqrt{((\alpha-\gamma)(\alpha-\delta))}]}{4\sqrt{((\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta))}[\sqrt{((\alpha-\gamma)(\beta-\gamma))}+\sqrt{((\alpha-\delta)(\beta-\delta))}]},$$

unde

$$A=-2(\gamma-\delta)\sqrt{((\alpha-\gamma)(\beta-\delta))}.$$

Simili modo ex aequationibus 3), 2) et 4) obtinetur

$$B=-2(\alpha-\beta)\sqrt{((\alpha-\gamma)(\beta-\delta))}.$$

Ope aequationis

$$(V-\varkappa U)\partial(V+\varkappa U)-(V+\varkappa U)\partial(V-\varkappa U)=2\varkappa(V\partial U-U\partial V)$$

fit

$$\frac{V\partial U-U\partial V}{\partial y}=\frac{1}{\varkappa}C.D.(m-n)(m+y)(n+y),$$

ergo

$$\frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2x^2)]}}=\frac{C.D(m-n)(m+y)(n+y)\partial y}{\varkappa M\sqrt{[A.B.C.D(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)(m+y)^2(n+y)^2]}}$$

$$=\frac{(m-n)\partial y}{\varkappa M\sqrt{\left(\frac{A.B}{C.D}(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)\right)}}.$$

Est vero, omnibus factis reductionibus,

$$m-n=\frac{2\sqrt{[(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)]}}{\gamma+\delta-(\alpha+\beta)},$$

est porro

$$\sqrt{\left(\frac{A.B}{C.D}\right)}=\frac{2\sqrt{[(\alpha-\gamma)(\beta-\delta)]}}{(\gamma+\delta)-(\alpha+\beta)},$$

est denique $\frac{m-n}{\varkappa\sqrt{\left(\frac{A.B}{C.D}\right)}}=\sqrt{((\alpha-\gamma)(\beta-\delta))}$: ergo $M=\sqrt{((\alpha-\gamma)(\beta-\delta))}$.

Quibus omnibus collectis, sequitur, fieri

$$\frac{\partial x}{M\sqrt{((1-x^2)(1-\varkappa^2 x^2))}}=\frac{\partial y}{\sqrt{((y-\alpha)(y-\beta)(y-\gamma)(y-\delta))}},$$

posito

$$\frac{1-x}{1+x}=\frac{(\gamma-\delta)(y-\alpha)(y-\beta)}{(\alpha-\beta)(y-\gamma)(y-\delta)},\quad \varkappa=\frac{\sqrt{((\alpha-\delta)(\beta-\gamma))}}{\sqrt{((\alpha-\gamma)(\beta-\delta))}},\quad M=\sqrt{((\alpha-\gamma)(\beta-\delta))}.$$

E formulis antecedentibus derivantur aliae, quantitatibus $\alpha$, $\beta$, $\gamma$, $\delta$ inter se permutatis.

Si quantitates $\alpha$, $\beta$, $\gamma$, $\delta$ sunt reales, atque ita comparatae, ut $\alpha>\beta>\gamma>\delta$ sit, tum singulis casibus, quibus elementum $y$ inter limites $\alpha\ldots\pm\infty\ldots\delta$; $\gamma$ et $\beta$; $\gamma$ et $\delta$; $\beta$ et $\alpha$ continetur, respondent substitutiones, quas tabula I., quae sequitur, exhibet; e quibus, eodem remedio, quod in „Fundamentis novis theoriae functionum ellipticarum" pag. 11 indicatum est, facile formulae, quae transformandae expressioni

$$\frac{\partial y}{\sqrt{[\pm(y-\alpha)(y-\beta)(y-\gamma)]}}$$

inserviunt, derivari possunt, quaeque in Tabula II. proponuntur.

## *Tabula I.*

**A.** $$\frac{\partial y}{\sqrt{[+(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]}}=\frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2 x^2)]}};$$

$$\varkappa=\frac{\sqrt{[(\alpha-\delta)(\beta-\gamma)]}}{\sqrt{[(\alpha-\gamma)(\beta-\delta)]}};\qquad M=\sqrt{[(\alpha-\gamma)(\beta-\delta)]}.$$

**I.** Limites $\alpha\ldots\pm\infty\ldots\delta$; $\quad\frac{1-x}{1+x}=\frac{(\alpha-\beta)(y-\gamma)(y-\delta)}{(\gamma-\delta)(y-\alpha)(y-\beta)}$.

**II.** Limites $\gamma\ldots\ldots\ldots\beta$; $\quad\frac{1-x}{1+x}=\frac{(\gamma-\delta)(\alpha-y)(\beta-y)}{(\alpha-\beta)(y-\gamma)(y-\delta)}$.

**B.** $$\frac{\partial y}{\sqrt{[-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]}}=\frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2 x^2)]}};$$

$$\varkappa=\frac{\sqrt{[(\alpha-\beta)(\gamma-\delta)]}}{\sqrt{[(\alpha-\gamma)(\beta-\delta)]}};\qquad M=\sqrt{[(\alpha-\gamma)(\beta-\delta)]}.$$

**I.** Limites $\delta\ldots\gamma$; $\quad\frac{1-x}{1+x}=\frac{(\alpha-\delta)(\beta-y)(\gamma-y)}{(\beta-\gamma)(y-\delta)(\alpha-y)}$.

**II.** Limites $\beta\ldots\alpha$; $\quad\frac{1-x}{1+x}=\frac{(\beta-\gamma)(y-\delta)(\alpha-y)}{(\alpha-\delta)(y-\gamma)(y-\beta)}$.

## Tabula II.

**A.** $$\frac{\partial y}{\sqrt{[+(y-\alpha)(y-\beta)(y-\gamma)]}}=\frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2 x^2)]}};$$

$$\varkappa=\frac{\sqrt{(\beta-\gamma)}}{\sqrt{(\alpha-\gamma)}};\qquad M=\sqrt{(\alpha-\gamma)}.$$

I. Limites $\alpha \dots\dots +\infty$; $\quad \frac{1-x}{1+x}=\frac{(\alpha-\beta)(y-\gamma)}{(y-\alpha)(y-\beta)}$.

II. Limites $\gamma \dots\dots \beta$; $\quad \frac{1-x}{1+x}=\frac{(\alpha-y)(\beta-y)}{(\alpha-\beta)(y-\gamma)}$.

**B.** $$\frac{\partial y}{\sqrt{[-(y-\alpha)(y-\beta)(y-\gamma)]}}=\frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2 x^2)]}};$$

$$\varkappa=\frac{\sqrt{(\alpha-\beta)}}{\sqrt{(\alpha-\gamma)}};\qquad M=\sqrt{(\alpha-\gamma)}.$$

I. Limites $-\infty \dots\dots \gamma$; $\quad \frac{1-x}{1+x}=\frac{(\beta-y)(\gamma-y)}{(\beta-\gamma)(\alpha-y)}$.

II. Limites $\beta \dots\dots \alpha$; $\quad \frac{1-x}{1+x}=\frac{(\beta-\gamma)(\alpha-y)}{(y-\beta)(y-\gamma)}$.

Modulum $\varkappa$ unitate minorem esse, in substitutionibus Tab. I. *B*. et Tab. II. *A*., *B*. jam ipso intuitu liquet; valorem $\frac{\sqrt{[(\alpha-\delta)(\beta-\gamma)]}}{\sqrt{[(\alpha-\gamma)(\beta-\delta)]}}$ quoque unitate minorem esse ex forma, qua exhiberi potest, sequenti

$$\frac{\sqrt{[(\alpha-\delta)(\beta-\gamma)]}}{\sqrt{[(\alpha-\gamma)(\beta-\delta)]}}=\frac{\sqrt{[(\alpha-\delta)(\beta-\gamma)]}}{\sqrt{[(\alpha-\delta)(\beta-\gamma)+(\alpha-\beta)(\gamma-\delta)]}}$$

elucet.

In formulis propositis, transeunte $y$ ab altero limite ad alterum $x$ ab $-1$ ad $+1$ transit. —

Formulae, quae in Tabula I. *A*. exhibitae sunt, tum quoque substitutionem realem suggerunt, quum omnes quantitates $\alpha$, $\beta$, $\gamma$, $\delta$ sunt imaginariae.

Ponatur nimirum, designantibus $n$, $q$ quantitates positivas, $\alpha=m+n\sqrt{-1}$; $\beta=m-n\sqrt{-1}$; $\gamma=p+q\sqrt{-1}$; $\delta=p-q\sqrt{-1}$; designantibus $m$, $n$, $p$ et $q$ quantitates reales; quo facto expressio proposita haecce erit

$$\frac{\partial y}{\sqrt{[((y-m)^2+n^2)((y-p)^2+q^2)]}}.$$

Substitutionis formulam, hunc ad casum spectantem, sine ullo negotio ex formula paulo ante laudata (Tab. I. *A*.) derivabis substituendo ipsarum quantitatum $\alpha$, $\beta$, $\gamma$, $\delta$ valores mutandoque $\varkappa$ in $\frac{1}{\varkappa}$ atque $x$ in $\varkappa x$; quibus peractis erit, ubi simul loco $\frac{M}{\varkappa}$ ponis $M$:

$$\frac{\partial y}{\sqrt{[((y-m)^2+n^2)((y-p)^2+q^2)]}} = \frac{\partial x}{M\sqrt{[(1-x^2)(1-\varkappa^2 x^2)]}},$$

$$\varkappa = \frac{\sqrt{[(m-p)^2+(n-q)^2]}}{\sqrt{[(m-p)^2+(n+q)^2]}}; \qquad M = \sqrt{[m-p)^2+(n+q)^2]};$$

$$a. \quad \frac{1-\varkappa x}{1+\varkappa x} = \frac{n[(y-p)^2+q^2]}{q[(y-m)^2+n^2]}.$$

Inquiramus, quosnam valores $x$ induat, dum argumentum $y$ inde ab altero limite $-\infty$ ad alterum $+\infty$ transit. Valor ipsius $x$, hisce limitibus respondens, unus idemque est et quidem aequalis quantitati

$$\frac{q-n}{\varkappa(q+n)} = \frac{(q-n)\sqrt{[(m-p)^2+(n+q)^2]}}{(q+n)\sqrt{[(m-p)^2+(n-q)^2]}} = \frac{\sqrt{[(q-n)^2(m-p)^2+(q^2-n^2)^2]}}{\sqrt{[(q+n)^2(m-p)^2+(q^2-n^2)^2]}},$$

quae, ut e posteriore forma adparet, unitate absolute minor est.

Transeunte $y$ ab $-\infty$ usque ad valorem

$$y = \frac{p^2+q^2-(m^2+n^2)-\sqrt{[((m-p)^2+(n-q))^2((m-p)^2+(n+q)^2)]}}{2(p-m)}$$

variabilis $x$ a valore $\frac{q-n}{\varkappa(q+n)}$ ad maximum valorem $+1$ adsurgit; ex quo ad minimum decrescit, eumque attingit, facto

$$y = \frac{p^2+q^2-(m^2+n^2)+\sqrt{[((m-p)^2+(n-q)^2)((m-p)^2+(n+q)^2)]}}{2(p-m)}$$

ex quo minimo valore, crescente $y$ ad $+\infty$, ad valorem primitivum redit. Ex antecedentibus igitur apparet, cum transeunte $y$ a $-\infty$ ad $+\infty$ ipsam $x$ intervallum inter $-1$ et $+1$ positum bis permeare videamus, esse

$$\int_{-\infty}^{+\infty} \frac{\partial y}{\sqrt{[((y-m)^2+n^2))((y-p)^2+q^2)]}} = 2\int_{-1}^{+1} \frac{\partial x}{M\sqrt{[1-x^2)(1-\varkappa^2 x^2)]}}$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{\partial\varphi}{\sqrt{[((m-p)^2+(n+q)^2)\cos^2\varphi+4nq\sin^2\varphi]}}.$$

Quibus absolutis, facile erit ostensu, unde pendeat prosper successus prioris, quod supra commemoravimus, substitutionis irrationalis.

Resoluta enim aequatione $x = \frac{a+a'y+a''y^2}{1+b'y+b''y^2}$ obtinetur

$$y = \frac{P\pm\sqrt{R}}{Q}$$

designantibus $P$ et $Q$ functiones rationales ipsius $x$ lineares, $R$ autem functionem secundi ordinis. Quo substituto valore expressio

$\frac{\partial y}{\sqrt{[(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]}}$ induit formam

$$A.\quad \frac{\left[\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)\sqrt{R}\pm\left(\tfrac{1}{2}Q\frac{\partial R}{\partial x}-R\frac{\partial Q}{\partial x}\right)\right]\partial x}{\sqrt{[R(P\pm\sqrt{R}-\alpha Q)(P\pm\sqrt{R}-\beta Q)(P\pm\sqrt{R}-\gamma Q)(P\pm\sqrt{R}-\delta Q)]}}$$

ex qua propter irrationales, ibi comprehensas functiones, a priori haud liquet, quamnam viam, ad solutionem problematis idoneam, ingredi debeamus. Adhibita autem una ex formulis substitutionis, quas supra dedimus, e. g. illa, quae limitibus $\beta$ et $\gamma$ respondet, statim ad liquidum perducitur res.

Valor variabilis $y$, ex formula commemorata fluens, hic est

$$y=\frac{x(\alpha\gamma-\beta\delta)-(\alpha\delta-\beta\gamma)\pm\sqrt{[(\alpha-\beta)(\gamma-\delta)((\alpha-\gamma)(\beta-\delta)-(\alpha-\delta)(\beta-\gamma)x^2)]}}{x(\alpha-\beta+\gamma-\delta)-(\alpha-\beta)+\gamma-\delta}$$
$$=\frac{P\pm\sqrt{R}}{Q},$$

cujus ope nanciscimur aequationes, quae sequuntur, memorabiles:

$$B.\quad \left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)\sqrt{R}\pm\left(\tfrac{1}{2}Q\frac{\partial R}{\partial x}-R\frac{\partial Q}{\partial x}\right)$$
$$=\pm(\alpha-\beta)(\gamma-\delta)[x(\alpha-\beta-\gamma+\delta)(\alpha-\delta)(\beta-\gamma)-(\alpha-\gamma)(\beta-\delta)(\alpha-\beta+\gamma-\delta)$$
$$\mp(\alpha+\beta-\gamma-\delta)\sqrt{R}],$$

$$C.\quad (y-\alpha)(y-\beta)$$
$$=\frac{(\alpha-\beta)}{Q^2}[x^2(\alpha-\delta)(\beta-\gamma)(\alpha-\beta-\gamma+\delta)+x(\alpha-\beta)((\alpha-\gamma)(\gamma-\beta)+(\alpha-\delta)(\delta-\beta))$$
$$+(\alpha-\gamma)(\beta-\delta)(\alpha-\beta+\gamma-\delta)\pm(1-x)(\alpha+\beta-\gamma-\delta)\sqrt{R})$$
$$=\mp\frac{(1-x)}{(\gamma-\delta)Q^2}\left[\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)\sqrt{R}\pm\left(\tfrac{1}{2}Q\frac{\partial R}{\partial x}-R\frac{\partial Q}{\partial x}\right)\right],$$

$$D.\quad (y-\gamma)(y-\delta)$$
$$=-\frac{(\gamma-\delta)}{Q^2}[x^2(\alpha-\delta)(\beta-\gamma)(\alpha-\beta-\gamma+\delta)-x(\gamma-\delta)((\alpha-\gamma)(\alpha-\delta)+(\beta-\gamma)(\beta-\delta))$$
$$-(\alpha-\gamma)(\beta-\delta)(\alpha-\beta+\gamma-\delta)\mp(1+x)(\alpha+\beta-\gamma-\delta)\sqrt{R}]$$
$$=\mp\frac{1+x}{(\alpha-\beta)Q^2}\left[\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)\sqrt{R}\pm\left(\tfrac{1}{2}Q\frac{\partial R}{\partial x}-R\frac{\partial Q}{\partial x}\right)\right],$$

sive ponendo loco $y$ valorem $\frac{P\pm\sqrt{R}}{R}$

$$E.\quad (P\pm\sqrt{R}-\alpha Q)(P\pm\sqrt{R}-\beta Q)$$
$$=\mp\frac{(1-x)}{\gamma-\delta}\left[\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)\sqrt{R}\pm\left(\tfrac{1}{2}Q\frac{\partial R}{\partial x}-R\frac{\partial Q}{\partial x}\right)\right],$$

$$(P\pm\sqrt{R}-\gamma Q)(P\pm\sqrt{R}-\delta Q)$$
$$=\mp\frac{(1+x)}{\alpha-\beta}\left[\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)\sqrt{R}\pm\left(\tfrac{1}{2}Q\frac{\partial R}{\partial x}-R\frac{\partial Q}{\partial x}\right)\right],$$

ita ut quatuor ultimi factores denominatoris in expressione (*A.*) in duplex productum discerpi possunt, quorum utrumque aequat functionem, cujus alter factor est functio linearis ipsius variabilis quantitatis $x$, alter vero aequalis est numeratori expressionis (*A.*). Substitutis expressionibus (*E.*) in (*A.*), transformatio provenit,

$$\frac{\partial y}{\sqrt{[(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]}}=\frac{\partial x}{\sqrt{(1-x^2)}\sqrt{[(\alpha-\gamma)(\beta-\delta)-(\alpha-\delta)(\beta-\gamma)x^2]}},$$

quae cum supra proposita convenit.

Regiomonti m. Oct. a. 1835.