On the Parallel Surfaces of Conicoids and Conics.

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1. If a straight line of given length moves with one extremity on a given surface, and remains normal to it in all positions, the locus of the other extremity is called a parallel of the given surface, or primitive.

Hence the parallel surface is the locus of the centres of spheres of constant radius which touch the primitive, or it is the envelope of such spheres or of spheres of constant radius which have their centres on the primitive. The idea of parallel curves in plane space is thus extended in a direct manner to space of three dimensions.

In the present paper, I propose to discuss the parallels of quadric surfaces or conicoids; but in several cases, where the results for a general surface of the order m are analogous to those corresponding to them for a conicoid, I have given general expressions, the proofs and developments of which I hope to give on some future occasion.

The parts of the theory already worked out are, as far as I know, contained in Dr. Salmon's "Geometry of Three Dimensions," 2nd Edition, p. 148, and in Professor Cayley's paper on the quartic surfaces $(*\between U, V, W)^2$, ("Quarterly Journal of Mathematics," vol. xi., p. 15 *et seq.*) I have also had occasion to refer to a paper by Professor Clebsch on the problem of normals and the surface of centres (Crelle-Borchardt, t. lxii., p. 64).†

I. *The Parallel of a Central Conicoid.*

2. The equation of a parallel of a central conicoid, represented by

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0$$

† Prof. Clebsch, in the memoir referred to, has considered the surface of centres in a generalized way. Parallel surfaces might be similarly treated. In fact, if $x^2+y^2+z^2+w^2=0$ represents a fixed quadric surface, equal quasi-spheres having a double contact with it along a plane are represented by

$$x^2+y^2+z^2+w^2-(Xx+Yy+Zz+Ww)^2=0;$$

and X, Y, Z, W are proportional to the coordinates of the quasi-centre.

Forming the condition that these quasi-spheres may touch

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}=0,$$

we get

$$\Sigma\frac{aX^2}{\theta+a}-1=0.$$

The discriminant with regard to θ must be made homogeneous. I have not found any important advantage in point of symmetry, and therefore confine myself to the special forms.

in rectangular coordinates, is obtained by equating to nothing the discriminant with respect to θ of the quartic equation

$$\frac{\theta x^2}{\theta+a}+\frac{\theta y^2}{\theta+b}+\frac{\theta z^2}{\theta+c}-\theta-k^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which may also be arranged thus:

$$\frac{ax^2}{\theta+a}+\frac{by^2}{\theta+b}+\frac{cz^2}{\theta+c}+\theta-x^2-y^2-z^2+k^2=0 \quad\ldots\ldots\ldots (2);$$

or again, if we put $\theta'+d'$ for θ, a' for $a+d'$, b' for $b+d'$, c' for $c+d'$, we have the equivalent and symmetrical form

$$\frac{x^2}{\theta'+a'}+\frac{y^2}{\theta'+b'}+\frac{z^2}{\theta'+c'}-\frac{k^2}{\theta'+d'}-1=0 \quad\ldots\ldots\ldots\ldots (3).$$

The forms which I principally use are (1) and (2). I call k, which is the value of the constant normal distance between the parallel and its primitive, the modulus.

The equation (1) is obtained by forming the discriminant with respect to α, β, γ of

$$\theta\left(\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}-1\right)+\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2-k^2\}=0.$$

The equation of the parallel is thus the condition that a sphere of the radius k shall touch the primitive, the coordinates of the centre being the current coordinates relative to the parallel. If, on the other hand, x, y, z are given constants, we have an expression connecting the squares of the six normal distances from the point whose coordinates are x, y, z to the quadric. Corresponding to these distances, there can be described through the point six parallel surfaces.

It is obvious, especially from the form (3), that $x^2, y^2, z^2, -k^2$ enter the equation in the same way; and since the origin is not generally on the locus, its order is 12.

A similar remark is true of surfaces generally. The order of the parallel surface is equal to twice the number of normals which can be drawn to the primitive from a given point; that is to say, for a general surface of the order m, the order of the parallel is $2(m^3-m^2+m)$.

Limiting Case $k=0$.

3. When the modulus is nothing, the quartic (1) contains θ as a factor. Hence (Salmon's "Higher Algebra," 2nd Edition, Art. 106, p. 86) the equation of the parallel is of the form

$$\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1\right)^2\Phi=0,$$

where $\Phi=0$ represents the imaginary developable which circumscribes the system of confocals

$$\frac{x^2}{\theta+a}+\frac{y^2}{\theta+b}+\frac{z^2}{\theta+c}-1=0 \quad\ldots\ldots\ldots\ldots (a).$$

This developable is generated by planes which touch the primitive and pass through the tangent lines of the imaginary circle at infinity, and the result is most readily extended to surfaces of higher order by the aid of this definition.

If X, Y, Z of the parallel (mod. k) correspond to x, y, z of the primitive, we have

$$X = x + \frac{k\dfrac{x}{a}}{\left(\Sigma\dfrac{x^2}{a^2}\right)^{\frac{1}{2}}}, \quad Y = y + \frac{k\dfrac{y}{b}}{\left(\Sigma\dfrac{x^2}{a^2}\right)^{\frac{1}{2}}}, \quad Z = z + \frac{k\dfrac{z}{c}}{\left(\Sigma\dfrac{x^2}{a^2}\right)^{\frac{1}{2}}}, \quad \Sigma\frac{x^2}{a} - 1 = 0,$$

from which we get (1) by means of

$$X - x = \theta\frac{x}{a}, \quad Y - y = \theta\frac{y}{b}, \quad Z - z = \theta\frac{z}{c}.$$

If $k=0$, we have generally $X=x$, $Y=y$, $Z=z$, except for points on the curve

$$\Sigma\frac{x^2}{a} - 1 = 0, \quad \Sigma\frac{x^2}{a^2} = 0.$$

The developable $\Phi=0$ is the parallel of this curve, mod. 0; and, generally, the corresponding developable is the parallel (mod. 0) of the curve

$$U = 0, \quad \Sigma\left(\frac{dU}{dx}\right)^2 = 0.$$

The surface $\Phi=0$ is of the 8th order, having the focal conics of the primitive and the circle at infinity for double lines, together with a cuspidal curve of the order 12. It will be observed, from the form of the equation, that the sections by planes parallel to the principal planes are plane parallels to an imaginary modulus of the focal conics in those planes. Projected orthogonally on them, the sections in question form systems of parallels, and the loci of their cusps, *i. e.* the corresponding projections of the edge of regression, are the evolutes of the focal conics ("Messenger of Mathematics," August and November, 1871).

In like manner, the section at infinity is the envelope of the conic

$$\frac{x^2}{\theta+a} + \frac{y^2}{\theta+b} + \frac{z^2}{\theta+c} = 0,$$

or four right lines, together with the nodal circle

$$(x^2+y^2+z^2)^2 = 0.$$

The general characteristics of the surface are those of a developable touching two given conicoids, or of the reciprocal of a quadriquartic curve.

Sections by the Principal Planes and by the Plane at Infinity.

4. In order to obtain the equation of the section of the parallel by the principal plane $z=0$, we may make $z^2=0$ in (1). The quartic then contains $\theta+c$ as a factor, and the resulting equation is of the form

$$\left\{\text{Disct. of } \frac{\theta x^2}{\theta+a}+\frac{\theta y^2}{\theta+b}-\theta-k^2\right\}\left\{\frac{cx^2}{c-a}+\frac{cy^2}{c-b}+c-k^2\right\}^2=0;$$

that is to say, we have the plane parallel (mod. k) of the principal section of the conicoid by the plane $z=0$ and the conic

$$\frac{cx^2}{c-a}+\frac{cy^2}{c-b}+c-k^2=0 \text{ .... } (4),$$

which is a nodal curve, since z enters the equation of the parallel surface as a square.

The sections by the other principal planes are of the same kind.

The section by the plane at infinity is obtained by making the terms in (1) independent of x, y, z vanish. In this case, the quartic may be homogeneously written

$$\frac{\eta\theta x^2}{\theta+a\eta}+\frac{\eta\theta y^2}{\theta+b\eta}+\frac{\eta\theta z^2}{\theta+c\eta}=0.$$

The resulting equation is therefore of the form

$$\{[(b+c)x^2+(c+a)y^2+(a+b)z^2]^2-4(x^2+y^2+z^2)(bcx^2+cay^2+abz^2)\}$$
$$\times(x^2+y^2+z^2)^2(bcx^2+cay^2+abz^2)^2=0.$$

It follows that the section at infinity contains the nodal conics

$$x^2+y^2+z^2=0, \quad bcx^2+cay^2+abz^2=0,$$

and besides the right lines determined by

$$(b-c)^{\frac{1}{2}}x+(c-a)^{\frac{1}{2}}y+(a-b)^{\frac{1}{2}}z=0 \text{ .... } (5).$$

It will be observed, that since k, a finite length, is evanescent at infinity, the section is the same as for $k=0$.

Similarly, for a surface of the order m, the section of the parallel at infinity contains the corresponding section of the primitive as a nodal curve, the circle at infinity multiple in the degree $m(m-1)^2$ and the $2m(m-1)$ common tangents.

5. We have now to consider the relations of the several nodal conics to the plane parallels with which they are associated.

The plane parallel of the section of the primitive by the plane $x=0$ meets the axis of y in the two double points

$$x=0, \quad y^2=\frac{(c-k^2)(b-c)}{c}, \quad z=0;$$

but these points lie on the conic (4).

In like manner, the same conic passes through the two double points on the axis of x of the plane parallel of the section of the primitive by the principal plane $y=0$.

This determines the conic which has, with the focal conic in its plane, common asymptotes, their points of contact being two of the intersections of the lines (5).

Again (Salmon's "Geometry of Three Dimensions," 2nd Edition,

p. 402), when we make $z^2=0$ in (2), and take the discriminant relative to θ, we in effect find the envelope of the conics

$$\frac{ax^2}{\theta+a}+\frac{by^2}{\theta+b}+\theta-x^2-y^2+k^2=0, \quad z=0 \quad \dots\dots\dots\dots (6),$$

which envelope will touch a conic of the system corresponding to a given value θ, where it meets the consecutive conic

$$\frac{ax^2}{(\theta+a)^2}+\frac{by^2}{(\theta+b)^2}-1=0 \quad \dots\dots\dots\dots\dots\dots (7).$$

For the plane $z=0$, we make $\theta=-c$. The first conic is the double conic of the parallel in that plane; the second conic is the corresponding cuspidal conic of the surface of centres (Clebsch, Crelle-Borchardt, t. lxii., p. 64; Salmon's "Geometry of Three Dimensions," pp. 143 and 402). Hence the double conic of the parallel touches the plane parallel, which makes up with it the complete section by the plane $z=0$ at the points where it meets the corresponding cuspidal conic of the surface of centres.

The coordinates of the points are

$$x=\pm\left(\frac{bk^2-c^2}{b-a}\right)^{\frac{1}{2}}\frac{c-a}{c}, \quad y=\pm\left(\frac{ak^2-c^2}{a-b}\right)^{\frac{1}{2}}\frac{c-b}{c};$$

and the common tangents therefore are

$$\left(\frac{bk^2-c^2}{b-a}\right)^{\frac{1}{2}}x+\left(\frac{ak^2-c^2}{a-b}\right)^{\frac{1}{2}}y+c-k^2=0.$$

Similarly, the nodal conic at infinity

$$bcx^2+cay^2+abz^2=0$$

is touched by the lines (5) at points on the surface of centres, and the circle

$$x^2+y^2+z^2=0$$

is touched by the same lines where it meets

$$ax^2+by^2+cz^2=0,$$

the corresponding cuspidal conic of the surface of centres.

6. I investigate here the general intersections of the conic (4) with the associated plane parallel, because they will recur in the sequel.

If we write u for $bx^2+ay^2-ab-k^2(a+b)$, v for $x^2+y^2-a-b-c-k^2$, and $-c$ for θ, the equations (6) and (7) may be replaced by

$$c^2v-cu+2c^3-abk^2=0 \quad \dots\dots\dots\dots\dots\dots\dots (8),$$

$$c^2(c^2-ab)(v+k^2+a+b+c)+c^2(a+b-2c)\{u+ab+k^2(a+b)\}$$
$$-k^2(c-a)^2(c-b)^2=0 \dots (9),$$

the second equation representing the differential of (4) with respect to c, after dividing by the same quantity.

The discriminant of $\dfrac{\theta x^2}{\theta+a}+\dfrac{\theta y^2}{\theta+b}-\theta-k^2=0,$

or $$\theta^3-(v+c)\,\theta^2-u\theta+abk^2=0,$$

is $-27a^2b^2k^4+4u^3+4abk^2(v+c)^2+u^2(v+c)^2+18abk^2u(v+c)=0$ ...(10).

If u be eliminated from (8), (9), it will be found that, after division by the factor $c(a+b)-(c^2+ab)$, the result is

$$c^2v+3c^3+abk^2=0.$$

When therefore $c^2v+2c^3-abk^2$ is substituted for cu in (10), the result will be divisible by $(c^2v+3c^3+abk^2)^2$.

Knowing this, we are able to see that the result of the elimination of cu is, when multiplied by c^3,

$$(c^2v+3c^3+abk^2)^2\,\{c^{\frac{1}{2}}v+2(c^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}\,\{c^{\frac{1}{2}}v+2(c^{\frac{3}{2}}-a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}=0.$$

We have also

$$(cu+c^3+2abk^2)^2\,\{cu+a^{\frac{1}{2}}b^{\frac{1}{2}}k\,(2c^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}$$
$$\times\{cu+a^{\frac{1}{2}}b^{\frac{1}{2}}k\,(2c^{\frac{3}{2}}-a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}=0.$$

Equating to nothing the corresponding factors—that is to say, the two first, the two second, and the two third—we obtain in sets of four the intersections in question.

The Complementary Nodal Lines.

7. Independently of the five nodal conics in the principal planes and the plane at infinity, there is a nodal curve of the order 16, which we shall find to break up into two groups of right lines corresponding to positive and negative values of k. I call these the complementary nodal lines.

The equation (1), when expanded, is

$$\theta^4+(k^2+\Sigma a-\Sigma x^2)\,\theta^3+\{k^2\Sigma a+\Sigma bc-\Sigma(b+c)\,x^2\}\,\theta^2$$
$$+\{k^2\Sigma bc+abc-\Sigma(bcx^2)\}\,\theta+abck^2=0\,;$$

where Σa denotes $a+b+c$ &c.

We may write it $\theta^4+B\theta^3+C\theta^2+D\theta+E=0$ .... (11),

and we must bear in mind that E is constant, and B, C, D are quadric functions.

The conditions that this equation may have two pairs of equal roots, are the equalities

$$\frac{4C-B^2}{8}=\frac{4EC-D^2}{8E}=\frac{D}{B}=\frac{EB}{D}=\frac{8E+BD}{4C}\text{ ....(12).}$$

These give seven different equations, which, however, in the present case, are reducible to the two pairs of equations,

$D-E^{\frac{1}{2}}B=0$......(13), $D+E^{\frac{1}{2}}B=0$......(15),

$4C-B^2-8E^{\frac{1}{2}}=0$......(14); $4C-B^2+8E^{\frac{1}{2}}=0$......(16),

showing that the nodal curve in question breaks up into two lines of the order 8 corresponding to positive and negative values of k.

It will be convenient to call these by anticipation the two nodal groups.

8. When we make $E=0=k$ in (11), divide by θ and take the discriminant of the residual equation, the result represents, as we know, the imaginary developable circumscribing the system of confocals (a).

The equation is then of the form

$$27\,(abc-\Sigma\, bcx^2)^2+4\,[\Sigma\, bc-\Sigma\,(b+c)\,x^2]^3$$
$$+4\,(abc-\Sigma\, bcx^2)(\Sigma a-\Sigma x^2)^3-(\Sigma\, a-\Sigma x^2)^2\,[\Sigma\, bc-\Sigma\,(b+c)\,x^2]^2$$
$$-18\,(\Sigma\, a-\Sigma\, x^2)[\Sigma\, bc-\Sigma\,(b+c)\,x^2](abc-\Sigma bcx^2) = 0 \ldots (17).$$

The intersection of this with the primitive is given by the surface

$$[\Sigma\, bc-\Sigma\,(b+c)\,x^2]^2\,\{4\,[\Sigma\, bc-\Sigma\,(b+c)\,x^2]-(\Sigma a-\Sigma x^2)^2\} = 0 \;..(b).$$

The first factor relates to the line of contact

$$U=\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0,$$
$$\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2+\left(\frac{dU}{dz}\right)^2=0.$$

The second factor corresponds to (14) and (16).

I find it of consequence, with respect to the general theory, to remark that the complementary nodal groups when $k=0$ are represented by the intersection (taken twice) of the conicoid with the developable (17) as distinguished from the line of contact. This intersection is known to consist of eight rectilinear generators of the conicoid. The cuspidal curve, which will be presently considered, is represented in the same limiting case by the cuspidal edge of the developable and the line of contact taken three times. Thus a portion of the cuspidal curve is replaced by two united nodal curves, as in plane space we find cusps superseded by pairs of united double points.

9. The equations (13), (14) and (15), (16) may be written, if we attend to the corresponding signs,

$$D\mp E^{\frac{1}{2}}B=0,\;\; 4\,(C\mp 6E^{\frac{1}{2}})-(B-4E^{\frac{1}{4}})(B+4E^{\frac{1}{4}})=0 \;\;\ldots\ldots(c),$$

or in full,

$$(bc-k\sqrt{abc})x^2+(ca-k\sqrt{abc})\,y^2+(ab-k\sqrt{abc})z^2$$
$$+k^3\sqrt{abc}-k^2\,\Sigma bc^2+k\sqrt{abc}\,\Sigma a-abc=0\ldots(18),$$
$$(\Sigma x^2)^2+2\Sigma\,(b+c-a-k^2)x^2+k^4-2r^2\Sigma a+8k\sqrt{abc}$$
$$-\Sigma a^{\frac{1}{2}}(\Sigma a^{\frac{1}{2}}-2a^{\frac{1}{2}})(\Sigma a^{\frac{1}{2}}-2b^{\frac{1}{2}})(\Sigma a^{\frac{1}{2}}-2c^{\frac{1}{2}})=0\ldots(19).$$

The circle at infinity is a double line on the surface represented by the second equation and the intersections of $\Sigma x^2=0$, $\Sigma bcx^2=0$ count twice on each oft he groups. They are in fact sextuple points on the total nodal curve.

For the plane $z=0$, the equation $D-E^{\frac{1}{2}}B = 0$ becomes

$$bcx^2+cay^2-abc-k^2\Sigma bc-k\sqrt{abc}\,(x^2+y^2-\Sigma a-k^2) = 0;$$

or, according to the former notation,

$$cu-k\sqrt{abc}\,v-abk^2 = 0.$$

Substituting in the same way in $B^2-4C+8E^{\frac{1}{2}} = 0$, we get

$$v^2+4\,(u+cv+c^2)+8k\sqrt{abc} = 0;$$

or, eliminating u, $\qquad \{c^{\frac{1}{2}}v+2\,(c^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}^2 = 0,$

and similarly, $\qquad \{cu+a^{\frac{1}{2}}b^{\frac{1}{2}}k\,(2c^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}^2 = 0.$

The nodal groups therefore meet the principal planes in the points of simple intersection of the corresponding nodal conics and plane parallels (§ 6). These points are eight in number, and may be separated into two sets of four, each set belonging to one of the nodal groups, and each point being a double point thereon. The points are triple on the total nodal curve.

The actual coordinates are given by

$$c\,(a-b)\,x^2+(c-a)(a^{\frac{1}{2}}c^{\frac{1}{2}}-b^{\frac{1}{2}}k)^2 = 0,$$
$$c\,(b-a)\,y^2+(c-b)(b^{\frac{1}{2}}c^{\frac{1}{2}}-a^{\frac{1}{2}}k)^2 = 0,$$

and the same with k of the opposite sign.

When $k=0$, the systems of equations give the umbilics real and unreal of the primitive.

The results are, of course, analogous for the other principal planes.

By the foregoing conclusions, and the consideration of the reciprocal surface, I was led to infer that the complementary nodal curve must consist of sixteen right lines. I shall now show that this is really the case.

10. If we eliminate z^2 from

$$\kappa x^2+\lambda y^2+\mu z^2+\nu = 0,$$
$$(x^2+y^2+z^2)^2+2\,(Kx^2+Ly^2+Mz^2)+N = 0,$$

we have

$$\{(\mu-\kappa)\,x^2+(\mu-\lambda)\,y^2\}^2 + 2\,\{[\kappa\,(\nu-\mu M)+\mu^2K-\mu\nu]\,x^2+[\lambda\,(\nu-\mu M)+\mu^2L-\mu\nu]\,y^2\} + (\nu-\mu M)^2-\mu^2\,(M^2-N) = 0.$$

Identifying the given pair of equations with (18) and (19),

$$\begin{aligned}\nu-\mu M &= k^3\sqrt{abc}-k^2\Sigma ab+k\sqrt{abc}\,\Sigma a-abc \\ &\qquad -(a+b-c-k^2)(ab-k\sqrt{abc}) \\ &= -(a+b)(\sqrt{ab}-k\sqrt{c})^2,\end{aligned}$$

$$\begin{aligned}M^2-N &= (a+b-c-k^2)^2 \\ &\qquad -(k^4-2k^2\Sigma a+8k\sqrt{abc}+a^2+b^2+c^2-2ab-2bc-2ca) \\ &= 4\,(\sqrt{ab}-k\sqrt{c})^2;\end{aligned}$$

therefore $(\nu-\mu M)^2-\mu^2(M^2-N) = (a-b)^2(\sqrt{ab}-k\sqrt{c})^4$.

Again
$$\mu^2K-\mu\nu = (b+c-a-k^2)\,ab\,(\sqrt{ab}-k\sqrt{c})^2 + (\sqrt{ab}-k\sqrt{c})^2\{c\sqrt{ab}-k\sqrt{c}\,(a+b)+k^2\sqrt{ab}\}\sqrt{ab}$$
$$= (\sqrt{ab}-k\sqrt{c})^2\{ab^2-a^2b+2abc-k\sqrt{abc}\,(a+b)\},$$
$$\kappa(\nu-\mu M) = -(a+b)(\sqrt{ab}-k\sqrt{c})^2(bc-k\sqrt{abc});$$

therefore $\kappa(\nu-\mu M)+\mu^2K-\mu\nu = b\,(a-b)(c-a)(\sqrt{ab}-k\sqrt{c})^2$.

Similarly the coefficient of y^2 is
$$a\,(a-b)(b-c)(\sqrt{ab}-k\sqrt{c})^2.$$

Hence we have
$$\{b\,(c-a)\,x^2-a\,(b-c)\,y^2\}^2 + 2\,\{b\,(c-a)\,x^2+a\,(b-c)\,y^2\}(a-b)(\sqrt{ab}-k\sqrt{c})^2 + (a-b)^2(\sqrt{ab}-k\sqrt{c})^4 = 0,$$

or
$$\{b\,(c-a)\,x^2+a\,(b-c)\,y^2+(a-b)(\sqrt{ab}-k\sqrt{c})^2\}^2 - 4ab\,(c-a)(b-c)\,x^2y^2 = 0;$$

giving finally
$$\left.\begin{array}{l}\sqrt{b\,(c-a)}\,x+\sqrt{a\,(b-c)}\,y+\sqrt{b-a}\,(\sqrt{ab}-k\sqrt{c}) = 0. \\ \text{Similarly we get} \\ \sqrt{c\,(a-b)}\,y+\sqrt{b\,(c-a)}\,z+\sqrt{c-b}\,(\sqrt{bc}-k\sqrt{a}) = 0, \\ \sqrt{a\,(b-c)}\,z+\sqrt{c\,(a-b)}\,x+\sqrt{a-c}\,(\sqrt{ca}-k\sqrt{b}) = 0.\end{array}\right\}\ldots(20).$$

Each of these equations represents four planes containing one group of eight right lines. The planes are, of course, tangents to the conicoid represented by (18), which may be written thus :

$$\frac{\sqrt{bc}\,x^2}{(\sqrt{ab}-k\sqrt{c})(\sqrt{ca}-k\sqrt{b})}+\frac{\sqrt{ca}\,y^2}{(\sqrt{ab}-k\sqrt{c})(\sqrt{bc}-k\sqrt{a})} + \frac{\sqrt{ab}\,z^2}{(\sqrt{ca}-k\sqrt{b})(\sqrt{bc}-k\sqrt{a})}-1 = 0 \;\ldots.\; (21);$$

and the right lines are the eight generators of the conicoid passing through the circle at infinity.

We can consequently substitute for the equation (19) a form corresponding to second factor of (*b*), but derived from the above equation.

The six traces of the planes (20) on the plane at infinity are the six lines joining the points of intersection of
$$\Sigma x^2=0,\quad \Sigma\,(bcx^2) = 0.$$

The planes consequently touch the conicoid (21) at its twelve umbilics, and the nodal group consists of the eight imaginary generators on which the umbilics lie by threes.

If now we eliminate k from

$$\sqrt{b\,(c-a)}\,x - \sqrt{a\,(b-c)}\,y + \sqrt{b-a}\,(\sqrt{ab} - k\sqrt{c}) = 0,$$

$$\sqrt{c\,(b-a)}\,x - \sqrt{a\,(c-b)}\,z + \sqrt{c-a}\,(\sqrt{ca} - k\sqrt{b}) = 0,$$

we get

$$\frac{a^{\frac{1}{2}}x}{\sqrt{(a-b)(a-c)}} + \frac{b^{\frac{1}{2}}y}{\sqrt{(b-a)(b-c)}} + \frac{c^{\frac{1}{2}}z}{\sqrt{(c-a)(c-b)}} - 1 = 0 \;\ldots\; (22),$$

representing eight planes independent of k which contain the whole of the sixteen right lines. In selecting the forms of the equations from which we eliminate, we must take care that both the planes contain the same line of the group.

The order of the total nodal curve is 26.

For a surface of the order m and otherwise general, the order of the nodal curve is $2m^6 - 4m^5 + 6m^4 - 6m^3 - 14m^2 + 17m$.

I may remark that each section of the parallel surface by a plane (22) consists of a pair of nodal right lines twice over and a curve of the 8th degree, with eight double points and twelve cusps. In fact, considering one of the pair of nodal right lines, we have seen that it meets a nodal conic and a nodal right line in each of the three principal planes. Moreover, the pair of nodal right lines in the plane meet another pair and an intersection of two nodal conics at infinity. This accounts for sixteen intersections of the plane with the nodal curve. There are eight more, six of them on nodal right lines and two on the nodal conics at infinity. These give the nodes of the section. The cuspidal curve (§ 11) meets the plane in 24 points, but 12 of these are on the nodal right lines and are due to stationary points; the remaining 12 give the cusps of the section.

The section by any plane containing two nodal right lines is of course similarly compound.

The planes (22) touch the circle at infinity, where it is met by the primitive conicoid, and touch the conicoid at the eight points

$$x = \frac{a^{\frac{3}{2}}}{\sqrt{(a-b)(a-c)}}, \quad y = \frac{b^{\frac{3}{2}}}{\sqrt{(b-a)(b-c)}}, \quad z = \frac{c^{\frac{3}{2}}}{\sqrt{(c-a)(c-b)}},$$

which are remarkable as being points through which no confocal different from the primitive conicoid can be drawn. The planes intersect in these points as triple, and in two points on each of the intersections of the principal planes and the line at infinity as quadruple. The eight points above mentioned coincide with their principal centres of curvature, and therefore lie on the surface of centres—in fact, on the cuspidal conics.

The Cuspidal Curve.

11. The equations of the cuspidal curve, corresponding to the case of three equal roots θ, are obtained by equating to zero the two fundamental invariants of (1); that is to say, we have

$$12\text{E}-3\text{BD}+\text{C}^2=0 \quad\dots\dots\dots\dots\dots\dots\dots\dots (23),$$
$$72\text{EC}+9\text{BCD}-27\text{D}^2-27\text{EB}^2-2\text{C}^3=0 \quad\dots\dots\dots\dots (24).$$

The curve therefore is of the order 24, and forms the complete intersection of a quartic and a sextic surface.

From the above we get, by elimination of the term BCD,

$$108\text{EC}-27\text{D}^2-27\text{EB}^2+\text{C}^3=0 \quad\dots\dots\dots\dots\dots\dots (25);$$

and again, by means of (23),

$$\left.\begin{array}{l}(\text{C}-6\text{E}^{\frac{1}{2}})^3-27(\text{D}-\text{E}^{\frac{1}{2}}\text{B})^2=0 \\ (\text{C}+6\text{E}^{\frac{1}{2}})^3-27(\text{D}+\text{E}^{\frac{1}{2}}\text{B})^2=0\end{array}\right\} \quad\dots\dots\dots\dots\dots\dots (26),$$

or $\quad (\text{C}-6\text{E}^{\frac{1}{2}})^3-(\text{C}+6\text{E}^{\frac{1}{2}})^3-27\{(\text{D}-\text{E}^{\frac{1}{2}}\text{B})^2-(\text{D}+\text{E}^{\frac{1}{2}}\text{B})^2\}=0,$

$$(\text{C}-6\text{E}^{\frac{1}{2}})^3+(\text{C}+6\text{E}^{\frac{1}{2}})^3-27\{(\text{D}-\text{E}^{\frac{1}{2}}\text{B})^2+(\text{D}+\text{E}^{\frac{1}{2}}\text{B})^2\}=0,$$

which are equivalent to (23) and (25).

The cuspidal curve meets the plane at infinity in the points of intersection of $\Sigma(b+c)x^2=0$ with $\Sigma x^2=0$ and $\Sigma(bcx^2)=0$; that is to say, in the eight points of contact of the tangents common to the nodal conics at infinity, which are also the points where the locus of the intersection of three rectangular tangent lines to the primitive meets the primitive and the circle. Each of these points counts as three intersections with the cuspidal curve. They are, in fact, stationary points on the curve.

12. To find the intersections of the cuspidal curve with the principal plane $z=0$, we may employ the equations (26), note being taken of the extraneous curve, counting six times, where C=0 meets the plane at infinity.

Making $z^2=0$, we may write those equations, according to the former notation,

$$(u+cv+c^2-6k\sqrt{abc})^3+27(cu-k^2ab+k\sqrt{abc}\,.\,v)^2=0,$$
$$(u+cv+c^2+6k\sqrt{abc})^3+27(cu-k^2ab-k\sqrt{abc}\,.\,v)^2=0.$$

But for the cusps of the plane parallel represented by the discriminant of
$$\theta^3-(v+c)\theta^2-u\theta+abk^2=0,$$
that is to say, of the plane parallel which forms part of the section of the parallel surface by the plane $z=0$, we have

$$v=-3(abck^2)^{\frac{1}{3}}-c, \quad u=-3(abk^2)^{\frac{2}{3}},$$

values which satisfy the equations of the cuspidal curve. Hence, as might be geometrically anticipated, the twelve cusps of the plane parallel are on the cuspidal curve.

Again, we have seen that the points where the double conic in the plane $z=0$ touches the associated plane parallel are determined by

$$c^2v+3c^3+abk^2=0, \quad cu+c^3+2abk^2=0.$$

The values of v and u thence derived also satisfy the equations of the cuspidal curve. Moreover, from the form of these equations, it appears that the points in question count three times as intersections with the cuspidal curve. They are, in fact, stationary points.

13. Since the elimination of θ from (2), and its first and second differentials with respect to θ, gives the equations of the cuspidal curve of the parallel, and since the like elimination from those differentials gives the surface of centres, we see, what is also apparent geometrically, that the cuspidal curve lies upon the surface of centres. This seems to be generally true. A system of parallels may be regarded as a system of surfaces having a common surface of centres. There will be two branches of the cuspidal curve corresponding to the two sheets of the surface of centres. The general order for a surface of the order m is $12m(m-1)$.

Stationary Points due to four equal roots θ.

14. We have next to consider the points corresponding to the case when (1) has four equal roots.

These points are stationary points of the cuspidal curve, and are in one way obtained by combining (2) with its first, second, and third differentials with regard to θ.

We have then to eliminate θ and obtain the equations of three surfaces whose 32 nett intersections are the points in question.

Thus we may infer that these points are, in fact, the intersections of the cuspidal curve of the parallel with the cuspidal curve of the surface of centres, *i.e.* its eight cuspidal conics not in the principal planes or the plane at infinity.

In fact, the equations (c) and (26) have been so written that we see at once that the 32 points determined by

$$\left.\begin{aligned} B &= 4\omega\, E^{\frac{1}{4}} \\ C &= 6\omega^2 E^{\frac{1}{2}} \\ D &= 4\omega^3 E^{\frac{3}{4}} \end{aligned}\right\}, \quad \omega^4=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

are points on the complementary nodal lines and on the cuspidal curve.

The above equations are directly obtained by making (1) a perfect biquadratic or fourth power.

Further, the system of conditions being

$$\Sigma \frac{ax^2}{\theta+a}+\theta-\Sigma x^2+k^2=0,$$

$$\Sigma \frac{ax^2}{(\theta+a)^2}-1=0, \quad \Sigma \frac{ax^2}{(\theta+a)^3}=0, \quad \Sigma \frac{ax^2}{(\theta+a)^4}=0,$$

the three last may also be written

$$\Sigma \frac{ax^2}{(\theta+a)^4}=0, \quad \Sigma \frac{a^2x^2}{(\theta+a)^4}=0, \quad \Sigma \frac{a^3x^2}{(\theta+a)^4}-1=0.$$

From these we get

$$\frac{x^2}{(\theta+a)^4}=\frac{1}{a(b-a)(c-a)}, \qquad \frac{y^2}{(\theta+b)^4}=\frac{1}{b(a-b)(c-b)},$$

$$\frac{z^2}{(\theta+c)^4}=\frac{1}{c(a-c)(b-c)};$$

therefore, by the second of the given system of equations,

$$\frac{a^{\frac12}x}{\sqrt{(b-a)(c-a)}}+\frac{b^{\frac12}y}{\sqrt{(a-b)(c-b)}}+\frac{c^{\frac12}z}{\sqrt{(a-c)(b-c)}}-1=0 \;\ldots(22).$$

The points in question lie therefore in the eight planes (22), in which also lie the cuspidal conics of the surface of centres not in the principal planes or the plane at infinity.

15. Actually to determine the points, we may revert to the system (27), which may be written

$$\begin{aligned}
x^2-k^2 \quad -\Sigma a+4(abck^2)^{\frac14}\omega+ \quad y^2+ \quad z^2&=0,\\
bcx^2-k^2\Sigma bc-abc+4(abck^2)^{\frac34}\omega^3+ \quad cay^2+ \quad abz^2&=0,\\
(b+c)x^2-k^2\Sigma a-\Sigma bc+6(abck^2)^{\frac12}\omega^2+(c+a)y^2+(a+b)z^2&=0,
\end{aligned}$$

giving

$$\begin{aligned}
&\{x^2-k^2-\Sigma a+4(abck^2)^{\frac14}\omega\}\,a^2(c-b)\\
&\quad+\{bcx^2-k^2\Sigma bc-abc+4(abck^2)^{\frac34}\omega^3\}\,(c-b)\\
&\quad\quad-\{(b+c)\,x^2-k^2\Sigma a-\Sigma bc+6(abck^2)^{\frac12}\omega^2\}\,a(c-b)=0.
\end{aligned}$$

The coefficient of k^2 is $-bc(c-b)$, and the value of $a^2\Sigma a+abc-a\Sigma bc$ is a^3. Hence we get

$$a^{\frac12}x=\pm\frac{\{a-(abck^2)^{\frac14}\omega\}^2}{\sqrt{(a-b)(a-c)}},$$

$$b^{\frac12}y=\pm\frac{\{b-(abck^2)^{\frac14}\omega\}^2}{\sqrt{(b-a)(b-c)}},$$

$$c^{\frac12}y=\pm\frac{\{c-(abck^2)^{\frac14}\omega\}^2}{\sqrt{(c-a)(c-b)}}.$$

Those points lie upon the cuspidal conics of the surface of centres which are determined by the cones

$$a^{\frac{1}{4}}(b-a)^{\frac{1}{4}}(c-a)^{\frac{1}{4}}x^{\frac{1}{2}}-b^{\frac{1}{4}}(a-b)^{\frac{1}{4}}(c-b)^{\frac{1}{4}}y^{\frac{1}{2}}-a+b=0,$$
$$b^{\frac{1}{4}}(a-b)^{\frac{1}{4}}(c-b)^{\frac{1}{4}}y^{\frac{1}{2}}-c^{\frac{1}{4}}(a-c)^{\frac{1}{4}}(b-c)^{\frac{1}{4}}z^{\frac{1}{2}}-b+c=0,$$
$$c^{\frac{1}{4}}(a-c)^{\frac{1}{4}}(b-c)^{\frac{1}{4}}z^{\frac{1}{2}}-a^{\frac{1}{4}}(b-a)^{\frac{1}{4}}(c-a)^{\frac{1}{4}}x^{\frac{1}{2}}-c+a=0.$$

The right lines (13), (14) then intersect the cuspidal conics in sixteen points on two spheres. The points are determined by

$$B=\pm 4E^{\frac{1}{4}},$$
$$D=\pm 4E^{\frac{3}{4}},$$
$$C=\quad 6E^{\frac{1}{2}}.$$

The right lines (15), (16) intersect the cuspidal conics in the sixteen points

$$B=\pm 4E^{\frac{1}{4}}\sqrt{-1},$$
$$D=\pm 4E^{\frac{3}{4}}\sqrt{-1},$$
$$C=-6E^{\frac{1}{2}},$$

also on two spheres. Each sphere passes through one point in each of the eight planes (22), and each plane contains four points which are determined by the cuspidal conics of the surface of centres.

When $k=0$, the thirty-two points fall together in eight points which are the intersections of the primitive, the locus of the intersection of three rectangular tangent lines, and the locus of the intersection of three rectangular tangent planes.

II. *The Parallel of a Paraboloid.*

16. This case requires a separate discussion, though some detail may be dispensed with.

If the primitive is represented by

$$\frac{x^2}{a}+\frac{y^2}{b}-2z=0,$$

the equation of the parallel is obtained by equating to zero the discriminant of

$$\frac{\theta x^2}{\theta+a}+\frac{\theta y^2}{\theta+b}-\theta(2z+\theta)-k^2=0 \quad\text{.................}(28),$$

or

$$\frac{ax^2}{\theta+a}+\frac{by^2}{\theta+b}+\theta(2z+\theta)-x^2-y^2+k^2=0.$$

As before, the equation of the parallel represents also the relation which exists between the squares of the five normal distances from a point to the primitive; and, from the manner in which x^2, y^2, $-k^2$ enter, the order of the parallel is 10.

When $k=0$, we have the paraboloid itself twice over and the developable circumscribing the confocal system

$$\frac{x^2}{\theta+a}+\frac{y^2}{\theta+b}-(2z+\theta)=0.$$

The characteristics of this developable are those of a developable circumscribing two conicoids which touch, or the reciprocal of a quadri-quartic curve with a double point. The cuspidal curve is therefore of the order 6. There are two nodal conics, and the line of simple intersection with the primitive is of the order 4. The remark in § 8 holds good in this case also.

Sections by the Principal Planes and the Plane at Infinity.

17. Making $y^2=0$ in (28), we have a factor $\theta+b$, and the section is given by

$$\left\{ \text{Disct. of } \frac{\theta x^2}{\theta+a}-(2b+\theta)-k^2 \right\} \left\{ \frac{bx^2}{b-a}+b\,(2z-b)-k^2 \right\}^2=0,$$

and contains the plane parallel (mod. k) of the parabola which is the section of the primitive by the plane $y=0$, together with a nodal parabola.

The case is similar with respect to the section by $x=0$.

As to the section by the plane at infinity, we may make $k=0$. The section is then determined by

$$(x^2+y^2+z^2)\,(x^2+y^2)^2\left(\frac{x^2}{a}+\frac{y^2}{b}\right)^2=0.$$

The lines $\dfrac{x^2}{a}+\dfrac{y^2}{b}=0$ are nodal lines.

The plane at infinity touches along the lines $x^2+y^2=0$; but they appear not to be nodal.

18. If we write u for $x^2-2az-k^2$, and v for $-2z-a-b$, and proceed as in § 6, we get for the intersections of the nodal conic in the plane $y=0$ with the associated plane parallel

$$(b^2v+3b^3+ak^2)^2\,\{b^{\frac{1}{2}}v+2\,(b^{\frac{3}{2}}+a^{\frac{1}{2}}k)\}\,\{b^{\frac{1}{2}}v+2\,(b^{\frac{3}{2}}-a^{\frac{1}{2}}k)\}=0,$$
$$b^2v-bu+2b^3-ak^2=0.$$

These expressions give the finite intersections.

These points are singular in the same manner as in the general case; that is to say, the complementary nodal lines which will be next discussed pass in pairs through the four finite simple intersections. The cuspidal curve passes through the two finite points of contact, each of which counts as three intersections.

Two of the points of contact are at infinity, and each of these also counts as three intersections with the cuspidal curve.

The Complementary Nodal Lines.

19. We may still make use of equation (11), but B is now to be taken as of the first order in x, y, z. Consequently the equations (14) and

(16) are only of the second degree, and the nodal curve divides into two lines of the order 4, each corresponding to one value of $\sqrt{k^2}$.

In fact, for the primitive $\frac{x^2}{a}+\frac{y^2}{b}-2z=0$ we have

$$D-E^{\frac{1}{2}}B \equiv bx^2+ay^2-(\sqrt{ab}-k)\{2\sqrt{ab}\,z-k(a+b)\}=0 \quad\ldots(29),$$

$$4C-B^2-8E^{\frac{1}{2}} \equiv x^2+y^2+\left(z-\frac{a+b}{2}\right)^2-(\sqrt{ab}-k)^2=0 \quad\ldots\ldots\ldots(30).$$

The sphere is one of those parallel (modulus k) to the sphere which passes through the intersection of the primitive with the developable circumscribing the confocal system.

Eliminating y^2 from (29) and (30), we get

$$(b-a)x^2-a\left(z-\frac{a+b}{2}\right)^2-2(\sqrt{ab}-k)\sqrt{ab}\left(z-\frac{a+b}{2}\right)-b(\sqrt{ab}-k)^2=0,$$

or $$(b-a)x^2-\left\{\sqrt{a}\left(z-\frac{a+b}{2}\right)+\sqrt{b}\,(\sqrt{ab}-k)\right\}^2=0,$$
representing two planes.

Similarly we get
$$(a-b)y^2-\left\{\sqrt{b}\left(z-\frac{a+b}{2}\right)+\sqrt{a}\,(\sqrt{ab}-k)\right\}^2=0, \quad\ldots\ldots(31).$$
representing other two planes.

Hence the nodal line in question consists of four right lines, imaginary generators of the paraboloid (29). The other nodal group also consists of four similar right lines.

The intersections of $\Sigma x^2=0$, $\frac{x^2}{a}+\frac{y^2}{b}=0$ at infinity are intersections of pairs of lines of the nodal groups. The total order of the nodal curve is 14.

The Cuspidal Curve.

20. The surfaces (23) and (24) intersect in more than the cuspidal curve of the parallel. This curve is of the order 18. In fact, the surfaces $12E-3BD+C^2=0$, $96EC+3BCD-27D^2-57EB^2=0$, respectively of the orders 4 and 5, intersect in the cuspidal curve, and in the curve where $C=0$ meets the plane at infinity; *i.e.*, in the lines $x^2+y^2=0$.

This cuspidal curve also lies on the surface of centres, and it will be observed that its order is twice the order of that surface, as in the general case.

The curve meets each of the principal planes $x=0$, $y=0$, as has been stated in two finite points counting as three, and also passes through the six cusps of the corresponding plane parallel of a parabola. Also in each of these planes the curve passes through the point where the axis of z meets the plane at infinity, each passage counting as six intersections with those planes. The nature of these intersections is most clearly perceived by reference to the general case where the corresponding section of the primitive is an ellipse.

We now see, from the form of the above equations, that the curve in question meets the plane at infinity in the point, or rather adjacently to the point where the axis of z meets the plane at infinity, and also where $x^2+y^2+z^2=0$ is touched by the lines $x^2+y^2=0$.

21. We have then, at the point where the axis of z meets the plane at infinity, a singular point of high complexity. From the analogy of the general case, I infer that the singularity arises from the union of two stationary points in each of the planes $x=0$, $y=0$, and of four stationary points in the plane at infinity. We shall see subsequently that this estimate of the singularities is consistent with the general characteristics.

In the case of the plane parallel of a parabola, we have two branches having with one another a contact of the second order at infinity; *i.e.*, the corresponding singularity counts as three adjacent, but not coincident, double points. There is evidently a corresponding singularity here, but it is more difficult to conceive clearly, and involves more, probably, than the mutual contact of two sheets of the surface.

Stationary Points due to four equal roots θ.

22. Interpreting the equations (27) in accordance with the forms of B, C, D, E for a paraboloid primitive, we see that there are sixteen such points.

We have, by substitution in these equations,

$$2z+a+b-4\,(abk^2)^{\frac{1}{4}}\,\omega=0,$$

$$x^2+y^2-2\,(a+b)\,z-abk^2+6\,(abk^2)^{\frac{1}{2}}\,\omega^2=0,$$

$$bx^2+cy^2-2abz-(a+b)\,k^2+4\,(abk^2)^{\frac{3}{4}}\,\omega^3=0,$$

$$\omega^4=1;$$

giving

$$x^2=\frac{\{a-(abk^2)^{\frac{1}{4}}\,\omega\}^4}{a\,(b-a)},\quad y^2=\frac{\{b-(abk^2)^{\frac{1}{4}}\,\omega\}^4}{b\,(a-b)},$$

$$z=2\,(abk^2)^{\frac{1}{4}}\,\omega-\frac{a+b}{2}.$$

These values satisfy the equation

$$\frac{a^{\frac{1}{2}}x}{\sqrt{b-a}}+\frac{b^{\frac{1}{2}}y}{\sqrt{a-b}}-z+\frac{a+b}{2}=0 \quad\text{.............. (32)},$$

representing four planes. We also get these equations by eliminating k from (31).

If now we determine for the surface of centres the cuspidal conics not in the principal planes or the plane at infinity, by means of

$$\frac{ax^2}{(\theta+a)^2}+\frac{by^2}{(\theta+b)^2}-2z-2\theta=0,$$
$$\frac{ax^2}{(\theta+a)^3}+\frac{by^2}{(\theta+b)^3}+1\qquad=0,$$
$$\frac{ax^2}{(\theta+a)^4}+\frac{by^2}{(\theta+b)^4}\qquad=0,$$

we get
$$\frac{by^2}{(\theta+b)^4}-\frac{1}{(a-b)}=0,$$
$$\frac{ax^2}{(\theta+a)^4}-\frac{1}{(b-a)}=0,$$
$$4\theta+2z+a+b=0;$$

and the cuspidal conics are in the four planes (32). They are also determined by the cylinders

$$b^{\frac{1}{4}}(a-b)^{\frac{1}{4}}y^{\frac{1}{2}}-a^{\frac{1}{4}}(b-a)^{\frac{1}{4}}x^{\frac{1}{2}}+a-b=0,$$
$$b^{\frac{1}{4}}(a-b)^{\frac{1}{4}}y^{\frac{1}{2}}+\frac{z}{2}+\frac{a+b}{4}-b=0,$$
$$a^{\frac{1}{4}}(b-a)^{\frac{1}{4}}x^{\frac{1}{2}}+\frac{z}{2}+\frac{a+b}{4}-a=0.$$

III. *The Parallel of a Central Conicoid of Revolution.*

23. If in the fundamental equation (1) we put $a=b$, the biquadratic contains the factor $\theta+a$, and we have, by taking the discriminant with regard to θ, the extraneous planes $(x^2+y^2)^2=0$, and the parallel of a central conicoid of revolution.

The nett equation of the parallel is thus reduced to the degree 8, and is the discriminant of a cubic equation. From the form

$$\frac{x^2+y^2}{\theta+a}+\frac{z^2}{\theta+c}-1-\frac{k^2}{\theta}=0,$$

it is evident that the parallel is represented by writing x^2+y^2 for x^2 in $\phi(x^2, z^2, -k^2)=0$, the equation of the plane parallel of the conic $\frac{x^2}{a}+\frac{z^2}{c}-1=0$. The surface is therefore generated by the revolution of the plane parallel of a conic about one of its principal axes. The general nature and form of the surface is easily conceived. It is instructive, however, with reference to the general characteristics, to consider the nodal and cuspidal curves.

A section by a plane perpendicular to the axis will consist generally of four circles. For $z=0$, however, we have a double circle

$$\frac{c(x^2+y^2)}{c-a}+c-k^2=0,$$

due, of course, to the revolution of the corresponding pair of double points of the revolving parallel. There will also be a pair of nodal conics at infinity; namely, the circle at infinity and a conic having a double contact with it.

Hence we have a nodal curve of the order 6. We shall also have six cuspidal circles due to the corresponding pairs of cusps of the plane parallel. The order of the cuspidal curve is therefore 12. The double points of the revolving plane parallel situate on the axis give rise to a pair of conical points or cnicnodes.

IV. *The Parallel of a Paraboloid of Revolution.*

24. The order of the parallel is reduced to 6. The finite double point of the parallel on the axis generates a cnicnode. The three corresponding pairs of cusps generate three cuspidal circles. We have, besides, to estimate the singularity generated by the revolution of the three united double points of plane parallel at an infinite distance on or adjacent to the axis. The result seems to be that at infinity the lines $x^2+y^2=0$ are cuspidal. The revolution of two symmetrically placed adjacent double points at infinity gives rise to the lines $x^2+y^2=0$ as nodal on the surface. The addition of a double point on the axis of the generating curve again gives these lines on the surface. The three united double points cause the lines to be cuspidal (art. 30).

V. *The Parallel in space of a Central Conic.*

25. The parallel surface of a curve is a tubular surface. For a curve of the order m and rank r, the order of the tubular surface is, in general, $2(r+m)$, and the class $2r$. When the primitive is an ellipse, the parallel is called the elliptic ring. Using the term "ring" in a free sense, we may similarly speak of the hyperbolic ring, and the parabolic ring.

For the equation of the parallel of the central conic $\frac{x^2}{a}+\frac{y^2}{b}-1=0$, $z=0$, we must take the discriminant, with regard to θ, of

$$\frac{x^2}{\theta+a}+\frac{y^2}{\theta+b}+\frac{z^2-k^2}{\theta}-1=0 \quad\text{..................}\quad (33),$$

obtained by putting $c=0$ in (1). The primitive is thus treated as a limiting case of a surface, as conics are included in a system of confocal conicoids. The equation (33) can, however, be obtained directly. (Cayley, Quarterly Journal of Mathematics, xi. p. 19.)

The equation of the parallel surface is therefore obtained by writing z^2-k^2 for $-k^2$ in the equation of the plane parallel of the primitive.

As long as $z^2 < k^2$, the section of the surface by a plane parallel to that of the primitive, is a plane parallel of the primitive modulus $\sqrt{k^2-z^2}$. The planes $z^2-k^2=0$ touch the surface along conics which are the same as the primitive in form and magnitude. Outside these planes the sections are imaginary. The order of the surface is evidently 8.

If $y=0$, $\theta+b$ becomes a factor of the cubic equation, and the discriminant is of the form

$$\left\{(x^2+z^2-k^2-a)^2+4a(z^2-k^2)\right\}\left\{\frac{x^2}{a-b}-\frac{z^2-k^2}{b}-1\right\}^2=0.$$

If $x=0$, we have

$$\left\{(y^2+z^2-k^2-b)^2+4b(z^2-k^2)\right\}\left\{\frac{y^2}{b-a}-\frac{z^2-k^2}{a}-1\right\}^2=0.$$

For the plane at infinity, the section is determined by the discriminant of

$$\frac{x^2}{\theta+a}+\frac{y^2}{\theta+b}+\frac{z^2}{\theta}=0;$$

or $$\{[bx^2+ay^2+(a+b)z^2]^2-4abz^2(x^2+y^2+z^2)\}(x^2+y^2+z^2)^2=0.$$

We have found three nodal conics, and two torsal conics, indicating nodes on the reciprocal surface.

The equation (33) gives for the cuspidal curve

$$\theta^3+3B\theta^2+3C\theta+D=0,$$
$$C-B^2=0,\quad BD-C^2=0,\quad D-BC=0.$$

Its order is 12.

There are two cnicnodes at infinity where the primitive meets the plane at infinity.

V′.

If, however, the primitive is a circle, the parallel surface is the common tore or anchor ring, generated by the revolution of a circle about an axis in its plane. The surface has as nodal conic the circle at infinity, and four cnicnodes; viz., two on the axis, and two at the points where the primitive meets the plane at infinity, that is to say, on the nodal circle. The order is 4. A good deal has been written on this surface and the general tore.

VI. *The Parallel in space of a Parabola.*

26. The equation of this parallel is obtained in the same manner as that of a central conic by equating to nothing the discriminant of

$$\frac{\theta x^2}{\theta+a}+y^2-(2z+\theta)\theta-k^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (34),$$

the result of writing $b=0$ in (28). We may, however, derive the form independently, as in the last case.

As long as $y^2 < k^2$, the section by a plane parallel to the plane of the primitive represented by $\frac{x^2}{a} - 2z = 0$, is a parallel of the primitive.

The planes $y^2 - k^2 = 0$ touch along two parabolas equal and similar to the primitive.

When $x=0$, $\theta + a$ is a factor of (34), and the discriminant is of the form
$$(z^2+y^2-k^2)\{y^2+a(2z-a)-k^2\}^2 = 0,$$
showing a nodal parabola.

The order of the surface is evidently 6, and we have a cuspidal curve of the order 6.

There are two adjacent cnicnodes at infinity where the primitive meets the plane at infinity.*

VII. *The Parallel of a Cone of the second order.*

27. Let $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ be the equation of the cone. For the equation of the parallel we must equate to nothing the discriminant of
$$\frac{\theta x^2}{\theta+a} + \frac{\theta y^2}{\theta+b} + \frac{\theta z^2}{\theta+c} - k^2 = 0.$$
The surface is a developable of the order 8, with four double conics and a cuspidal curve of the order 12; the form is similar to that of the developable generated by a plane which touches a conic at a constant inclination to its plane. If, indeed, we write ζ for $\frac{k^2}{\theta}$, A for $\frac{k^2}{a}$, B for $\frac{k^2}{b}$, C for $\frac{k^2}{c}$, X^2 for $\frac{x^2}{a}$, &c., the equation becomes
$$\frac{X^2}{\zeta+A} + \frac{Y^2}{\zeta+B} + \frac{Z^2}{\zeta+C} - 1 = 0;$$
so that the parallel surface is a homographic deformation of the developable circumscribing a system of confocal conicoids, and has the same general characteristics.

I need not dwell on the cases of cylinders. Their double and cuspidal edges correspond to the nodes and cusps of the corresponding plane parallel.

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* A model of this surface, constructed under the direction of Prof. Henrici, is in the possession of the Society.

The Reciprocal Surfaces and General Characteristics.

(I.) *Primitive a Central Conicoid.*

28. The equation of the reciprocal surface may be formed by writing $\frac{1}{\rho}+k$ for $\frac{1}{\rho}$ in the reciprocal of the primitive equation with respect to a concentric sphere of radius unity.

Then the reciprocal of $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0$ being

$$aX^2+bY^2+cZ^2-1=0,$$

we get
$$(aX^2+bY^2+cZ^2)-(1+k\rho)^2=0,$$

where $\rho=\sqrt{X^2+Y^2+Z^2}$, for the equation sought.

This is of the form

$$\{(a-k^2)X^2+(b-k^2)Y^2+(c-k^2)Z^2-1\}^2-4k^2(X^2+Y^2+Z^2)=0 \;\ldots\; (35).$$

I use the above method because it is applicable to surfaces generally. For the particular case (and many others to which the substitution $\frac{1}{\rho}+k$ for $\frac{1}{\rho}$ is not applicable), Prof. Cayley has given an equally simple process. We have only to form the reciprocal of (1), and take the envelope of that reciprocal relative to θ as parameter. ("Quarterly Journal of Mathematics," vol. xi., pp. 15—25.)

It is not within the scope of the present paper to go into much detail with regard to the reciprocal surfaces, and develope the meaning of the several forms considered as tangential equations of the parallels. I confine myself to a few points bearing on the characteristics.

As quartic surfaces with nodal curves of the second order, the reciprocal surfaces have received much attention from various geometers. I refer to Prof. Cayley's "Sketch of Recent Researches upon Quartic and Quintic Surfaces," "Proceedings," No. 32, p. 186.

Several more recent researches on the subject have, however, now appeared. I find some conclusions as to the case of a quartic surface with two intersecting nodal lines in a paper by Dr. Zeuthen, entitled "Recherche des singularités qui ont rapport à une droite multiple d'une surface," "Math. Ann.," b. iv., p. 18. A memoir by Herr Rudolf Sturm, entitled "Ueber die Flächen mit einer endlichen Zahl von (einfachen) Geraden, vorzugsweise die der vierten und fünften Ordnung" ("Math. Ann.," b. iv., p. 245), contains valuable details relative to quartic surfaces with a nodal conic and cnicnodes. The author determines the most important of the singularities which become point singularities on the reciprocal surface. An earlier acquaintance with this memoir would have materially assisted me in the study of parallels.

For a general surface of the order m, the reciprocal of the parallel is of the order $2m(m-1)^2$, or twice the class of the primitive as in plane curves.

The equation (35) gives for the nodal conic at infinity

$$(a-k^2)X^2+(b-k^2)Y^2+(c-k)Z^2=0,$$

the reciprocal of which represents the corresponding node-couple torse of the parallel surface (Cayley).

The *pinch-points* or cuspidal points on the nodal curve are determined as the intersections at infinity of

$$aX^2+bY^2+cZ^2=0,$$
$$X^2+Y^2+Z^2=0.$$

(II.) *Primitive a Paraboloid.*

29. The reciprocal equation may be written

$$(aX^2+bY^2-2Z)^2-4k^2Z^2(X^2+Y^2+Z^2)=0 \quad \text{.........} \quad (36),$$

obtained either directly or in the manner pointed out in the last article.

The nodal curve now consists of two intersecting right lines. The four pinch-points fall together in the point $(X=0, Y=0, Z=0)$, being determined by

$$aX^2+bY^2 \qquad =0,$$
$$X^2+Y^2+Z^2=0,$$
$$Z=0.$$

The singularity at the origin consists of four pinch-points and an adjacent cnicnode or conical point.

In fact, if we substitute $X'\cos\omega-Y'\sin\omega$ for X, and $X'\sin\omega+Y'\cos\omega$ for Y, and then make $Y'=0$, we get for a general section by a plane through the axis of Z

$$\{(a\cos^2\omega+b\sin^2\omega)X'^2-2Z\}^2-4k^2Z^2(X'^2+Z^2)=0.$$

This represents the reciprocal of the plane parallel of a parabola, and corresponds to a tangent cone (cylinder) to the parallel surface from a point at infinity. The section therefore is a unicursal quartic curve having three adjacent double points at the origin; that is to say, two branches have there a contact of the second order, and the axis of X′ is the tangent. Now two of these adjacent double points are due to the intersection of the nodal right lines at the origin. The third double point indicates a cnicnode adjacent thereto.

(III.) *Primitive a Central Conicoid of Revolution.*

30. The equation of the reciprocal surface is

$$\{(a-k^2)(X^2+Y^2)+(c-k^2)Z^2-1\}^2-4k^2(X^2+Y^2+Z^2)=0.$$

It may also be written

$$\left\{(a-k^2)(X^2+Y^2)+(c-k^2)Z^2-\frac{a+k^2}{a-k^2}\right\}^2 - \frac{4k^2}{a-k^2}\left\{(a-c)Z^2+\frac{a}{a-k^2}\right\}=0,$$

showing that we have a nodal conic at infinity and two torsal conics lying on the same surface of the second order as the nodal conic. The pinch-points at infinity are given by

$$a(X^2+Y^2)+cZ^2=0,$$
$$X^2+Y^2+Z^2=0.$$

The two torsal conics correspond to the two cnicnodes on the axis of the parallel surface. We have also two cnicnodes or two points in the nature of cnicnodes; for if a quartic surface is represented by

$$U^2-L^2MN=0,$$

where L, M, N are linear, but U is of the second order, we have in general, besides the nodal conic, two cnicnodes

$$U=0,\quad M=0,\quad N=0,$$

and it may so happen that these points also lie on the conic, their coordinates satisfying $L=0$. This is the case in the present instance.

The fact that the original parallel and the reciprocal surface are both surfaces of revolution assists us to see the necessity of a singularity of this sort; for, taking the equation of the generating curve as

$$\{(a-k^2)X^2+(c-k^2)Z^2-1\}^2-4k^2(X^2+Z^2)=0,$$

we know that the curve is a quartic, binodal at infinity and symmetrical about the axis of z, which it cuts at right angles. There are then, from geometrical considerations, no proper cnicnodes, which, if any, would be due to double points on the axis. Yet, since the reciprocal is of the order 8, we must have a singularity with the effect of two cnicnodes in reducing the order of the reciprocal.

(IV.) *Primitive a Paraboloid of Revolution.*

31. The reciprocal equation is

$$\{a(X^2+Y^2)-2Z\}^2-4k^2Z^2(X^2+Y^2+Z^2)=0.$$

The pinch-point system

$$X^2+Y^2=0,$$
$$X^2+Y^2+Z^2=0,$$
$$Z=0,$$

shows that the multiple lines have pinch-points all along them. In other words, the lines $X^2+Y^2=0$, $Z=0$ are cuspidal. In fact, the equation may be written in the form

$$\left\{a(X^2+Y^2)-2Z-\frac{2k^2}{a}Z^2\right\}^2-4k^2Z^3\left\{\left(\frac{k^2}{a^2}+1\right)Z+\frac{2}{a}\right\}=0.$$

The effect, then, of the revolution of the three adjacent double points at the origin in the generating plane curve is to cause the nodal lines to be cuspidal (§ 24).

(V.) *Primitive a Central Conic.*

The equation of the reciprocal surface may be written

$$\{(a-k^2)X^2+(b-k^2)Y^2-k^2Z^2-1\}^2-4k^2(X^2+Y^2+Z^2)=0.$$

This is obtained by making $c=0$ in (35).

The equation may also be written in the form

$$\{(a-k^2)X^2+(b-k^2)Y^2-k^2Z^2+1\}^2-4(aX^2+bY^2)=0,$$

showing that, in addition to the nodal conic at infinity, there are two torsal circles lying on the same quadric surface as the nodal conic. These correspond to the two cnicnodes at infinity of the parallel surface. The form also shows that there are two cnicnodes.

(V′.) *Primitive a Circle.*

In this case, the reciprocal surface has the same characteristics as the parallel; that is to say, there is a nodal conic at infinity, and there are four cnicnodes, two on the axis and two on the nodal conic.

The surface is generated by the revolution of a central conic about an axis in its plane and parallel to a principal axis. It is therefore a particular case of the general tore.

(VI.) *Primitive a Parabola.*

The reciprocal equation is

$$(aX^2-2Z)^2-4k^2Z^2(X^2+Y^2+Z^2)=0,$$

obtained by making $b=0$ in (36).

In this case, the nodal lines are coincident (tacnodal). The singularity beyond this is equivalent to four pinch-points and three points in the nature of cnicnodes.

We have seen, in a similar case, that there is a point in the nature of a cnicnode at the origin.

From the nature of the substitution $\phi(X, Y, Z, 1\pm k\rho)=0$ for $\phi(X, Y, Z)=0$, it follows that in general, if there is a cnicnode on

$\phi(X, Y, Z) = 0$, there will be two corresponding cnicnodes on the related surface. In the present case, the parabolic cylinder $aX^2-2ZW=0$ having a node at infinity, $X=0$, $Z=0$, $W=0$, the related surface has the corresponding singularities $X=0$, $Z=0$, $1\pm k\rho=0$.

The equation may, in fact, be written in the form

$$a^2X^4-4aX^2Z-4Z^2\{k^2(X^2+Y^2+Z^2)-1\} = 0,$$

showing that the points $X=0$, $Z=0$, $1\pm k\rho=0$ are triple points on the surface, though not triple on the nodal lines.

(VII.) *Primitive a Cone of the Second Order.*

33. The reciprocal curve is a spherical conic determined by

$$X^2+Y^2+Z^2-\frac{1}{k^2}=0,$$

$$aX^2+bY^2+cZ^2=0,$$

corresponding to the primitive

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0.$$

I do not dwell on this and the remaining cases.

34. We are now in a position to consider the general characteristics of the several parallels. For this purpose I make use of Dr. Salmon's notation. (See his chapter on reciprocal surfaces; also Cayley, T. R. S., 1869, Part I., "On Reciprocal Surfaces," and "Corrections and Additions" thereto, R. S. Proceedings, 1871; also Zeuthen, "Mathematische Annalen," Band IV., Heft i., 1871.)

I must content myself here with a list of the singularities to which my results refer, and a few formulæ. Although this is repetition, the list may be useful for reference, when the original sources are not at hand.

Characteristics of a Surface.

n, order of surface.	n', class of surface.
a, order of tangent cone.	a', class of section by any plane.
δ, number of its double edges.	δ', number of its double tangents.
κ, number of its cuspidal edges.	κ', number of its inflexions.
b, order of nodal curve.	b', class of node-couple torse.
f, number of its actual double points.	f', number of its actual double planes.
k, number of its apparent double points.	k', number of its apparent double planes.
t, number of its triple points.	t', number of its triple planes.
q, its class.	q', its order.

ρ, number of points where nodal curve meets curve of contact of tangent cone.	ρ', order of node-couple curve.
j, number of pinch-points.	j', number of pinch-planes.
c, order of cuspidal curve.	c', class of spinode torse.
h, number of its apparent double points.	h', number of its apparent double planes.
r, its class.	r', its order.
σ, number of points where cuspidal curve meets curve of contact of tangent cone.	σ', order of spinode curve.
θ, not defined, but on the cuspidal curve.	θ', not defined.
χ, number of close points.	χ', number of close planes.
β, number of intersections of nodal and cuspidal curves, stationary points on the latter curve.	β', number of common planes of node-couple torse and spinode torse, stationary planes of spinode torse.
γ, number of like intersections, stationary points on nodal curve.	γ', number of common planes, stationary planes of node-couple torse.
i, number of like intersections, not stationary.	i', number of common planes, not stationary planes of either torse.
B, number of binodes.	B$'$, number of bitropes of surface.
C, number of cnicnodes.	C$'$, number of cnictropes.
ω, number of off points.	ω', number of off planes.

With regard to these, we have the following fundamental equations:

$$
\begin{aligned}
n(n-1) &= a+2b+3c,\\
a(a-1) &= n+2\delta'+3\kappa',\\
c-\kappa' &= 3(n-a),\\
b(b-1) &= q+2(k+f)+3\gamma+6t,\\
c(c-1) &= r+2h+3\beta,\\
a(n-2) &= \kappa-\mathrm{B}+\rho+2\sigma+3\omega,\\
b(n-2) &= \rho+2\beta+3\gamma+3t,\\
c(n-2) &= 2\sigma+4\beta+\gamma+\theta+\omega,\\
a(n-2)(n-3) &= 2(\delta-\mathrm{C}-3\omega)+3(ac-3\sigma-\chi-3\omega)+2(ab-2\rho-j),\\
b(n-2)(n-3) &= 4k+(ab-2\rho-j)+3(bc-3\beta-2\gamma-i),\\
c(n-2)(n-3) &= 6h+(ac-3\sigma-\chi-3\omega)+2(bc-3\beta-2\gamma-i);
\end{aligned}
$$

and the same with accented letters.

35. In the absence of a general theory as to the effect of multiple points existing on the nodal and cuspidal curves, I propose to give the usual characteristics of the several classes of surfaces to which the parallels belong. The specialities of these will then be remarked upon.

There is a certain number of characteristics which may be taken as known. Thus we have those of a general section and of a general tangent cone; or what is the same thing, we know the order of the surface and its reciprocal, and the orders of its nodal and cuspidal curves and of those of its reciprocal. These characteristics form a class by themselves, being independent of isolated point and plane singularities.

Following Dr. Zeuthen, we have $i = i' = 0$.

In the table marked (A), under the numbers I., II., &c., I refer to the sections concerning the parallels to which the corresponding columns relate, and give the characteristics of the reciprocal of a quartic surface.

I., with a nodal conic;

II., with a nodal conic and a cnicnode;

III., with a nodal conic and two cnicnodes;

IV., with a cuspidal conic;

V. = III., with a nodal conic and two cnicnodes;

(V′.), with a nodal conic and four cnicnodes;

VI., with a nodal conic and three cnicnodes;

and further I give

VII., the characteristics of the reciprocal of a quadriquartic curve or those of the developable circumscribing two conicoids.

The characteristics I.—VI. may, of course, be regarded as those of the corresponding quartic surfaces, the meanings of corresponding letters being interchanged.

The deficiency expressed by Prof. Cayley's formula,

$$\tfrac{1}{6}(n-1)(n-2)(n-3)-(n-3)(b+c)+\tfrac{1}{2}(q+r)+2t+\tfrac{3}{2}\beta+\tfrac{5}{2}\gamma+i-\tfrac{1}{2}\theta,$$

is nothing in each case, VII. excepted.

	I.	II.	III. =V.	IV.	V′.	VI.	VII.
n	12	10	8	6	4	6	8
a	8	8	8	6	8	8	4
δ	8	8	8	1	8	8	6
κ	12	12	12	8	12	12	
b	26	14	6		2	2	8
k	200	52	12				24
t	40	12					
q	10	6	6		2	2	8
ρ	36	20	12		4	4	16
j					4	2	
c	24	18	12	8		6	12
h	180	96	38	12		6	38
r	36	30	20	32		12	8
σ	16	16	12	8		8	4
β	52	28	12			2	16
χ							
θ				16			
C			2	1	4	2	
n'	4	4	4	4	4	4	
a'	8	8	8	6	8	8	
δ'	4	5	6		8	7	
κ'	12	12	12	8	12	12	
b'	2	2	2		2	2	
k'							
t'							
q'	2	2	2		2	2	
ρ'	4	4	4		4	4	
j'	4	4	4		4	4	
c'				2			
h'							
r'				2			
σ'				2			
β'							
χ'				2			
θ'							
C′		1	2		4	5	

Observations.

36. I. The cuspidal curve being the complete intersection of a quartic and a sextic surface, the number of apparent double points $h=180$. The class $q=10$ of the nodal curve is due to the five nodal conics; the class of the residue is therefore zero, indicating that it consists of 16

right lines. In general, these correspond to the 16 right lines discovered by Prof. Clebsch on the surface of the fourth degree with a nodal conic.

A plane through one of these lines will cut the surface in the line itself and a cubic with a double point. There will be in the section four double points; two of them are due to the nodal conic; the remaining two are points of contact of the plane. Hence the reciprocals of these lines will be double lines on the reciprocal surface. The five nodal conics correspond to Kummer's five bitangent cones on the quartic surface.

In the special case of a parallel, we have seen that the system becomes two groups of eight generators of a conicoid. Four points of quadruple intersection, being also points of intersection of the nodal conics at infinity, become sextuple points of the nodal curve. These points stand for 16 triple points and 12 apparent double points, or each for four triple points and three apparent double points. There are also eight triple points in each of the three principal planes.

$\beta=52$ is made up of four points in each of the principal planes and eight points at infinity, together with the 32 stationary points in the eight special planes.

It might be supposed that the sextuple points on the nodal curve would count for a greater number of triple points than 16. If, however, we consider the case of four planes intersecting in a point, which will be a sextuple point of the nodal curve, the planes being regarded as a single surface, we see that in this case also the sextuple point will represent four triple points and three apparent double points; for the number of apparent double points for six non-intersecting lines is 15. But the number of possible triple points is the number of ways in which four planes can be taken three and three together, *i. e.* four. At the same time, every triple point takes the place of three apparent intersections. Thus we have four triple points substituted for twelve apparent intersections, leaving three unaccounted for. It does not follow, however, that every kind of sextuple point on a nodal curve is of this value. In like manner, we might take m planes intersecting in one point, which would be a point of the order m on the surface, but of the order $\frac{m(m-1)}{2}$ on the nodal curve. In the particular case, the points appear to be quadruple points on the surface, equivalent in each instance to $\frac{4.3.2}{1.2.3}$ triple points.

We may regard the point as one where the tangents form a cone of the order 4, with six double edges touching the nodal curve. For such a point Dr. Zeuthen gives the addition $y(\mu-2)$ to the expression for $b(n-2)$. The number of double edges touching the nodal curve is

y, and the order of the conic μ. Putting $y=6$, $\mu=4$, we get a correct result.

It would perhaps be better, in these cases of multiple points, to follow Dr. Zeuthen, in denoting by k the Plückerian number of the double edges of a cone through the double curve, and substituting $k-f-3t+\Sigma\ldots$ for k in the expression for $b(n-2)(n-3)$.

II. The cuspidal curve is the intersection of a quartic and a sextic surface, less the intersection of a quadric and a cubic surface; and $h=96$. The class $q=6$ of the nodal curve is due to three nodal conics, and the class of the residue is zero, indicating that it is made up of eight right lines. In the case of the parallel, the cuspidal curve is reduced to the intersection of a quartic and a quintic surface, less a conic; and $h=96$. The class q is 4, because a nodal conic at infinity consists of right lines, *i.e.* has an actual double point, or $f=1$.

The singular point on the axis at infinity counts as a union of eight points β, two in each of the principal planes and four in the plane at infinity. The number 28 is thus made up of four points finite and in the two principal planes, the eight united points, and the 16 points in the four special planes. There are four triple points t in each of the principal planes and the plane at infinity.

The 'cnictrope' is not a proper one, and corresponds to a point in the nature of a cnicnode on the reciprocal quartic surface.

The order $q'=0$; for, referring to the reciprocal surface, we see that the nodal conic has an actual double point, or $f'=1$.

For the manner in which the complementary nodal lines are reduced in number, see the memoir before alluded to, of Herr Rudolf Sturm, on the quartic surface with one cnicnode and a nodal conic, &c.

When there is a cnicnode, four lines on the quartic surface become 'binary,' or equivalent to two ordinary lines. These 'binary' lines are such that planes through them touch at one point. The eight ordinary lines correspond to the eight double lines of the reciprocal surface.

III. The cuspidal curve is, in general, the intersection of two quartic surfaces, less the intersection of two quadric surfaces; and $h=38$. In the case of a parallel, we have two points at infinity, each of which may be regarded as the intersection of three branches of the nodal curve with six branches of the cuspidal curve; that is to say, we have three nodal conics and six cuspidal circles intersecting in two common points. One nodal conic at infinity is the circle, and one is a conic having a double contact with the circle.

The case of a parallel of revolution is simple geometrically; but, as far as I am aware, does not come within existing general formulæ for characteristics.

IV. The cuspidal curve consists of four conics which intersect in two common points. Hence r is reduced 24 by these, in addition to the reduction by 12 points h. These points also present the singularity $\theta = 16$ which enters the formula for $c\ (n-2)$ in the same way that $3t$ enters the formula for $b\,(n-2)$. If a is the number of such singular points, the addition is $8a$. In the case of the parallel, r is reduced to 6 because the cuspidal conic at infinity contains an actual double point. In like manner $r' = 0$. In place of $j' = 4$, we now have $\chi' = 2$, on account of the multiple lines of the reciprocal being cuspidal.

V. The torsal conics of the parallel surface pass through the cnicnodes, and the same is the case in the reciprocal surface. Also $q' = 0$, on account of $f' = 1$.

V′. For the parallel, we have a particular case of the general tore, and the characteristics are the same as for a surface generated by a conic revolving about an axis in its plane. The speciality is that, considering a section by any plane through the axis, the generating conic and its reflection intersect in two points at infinity.

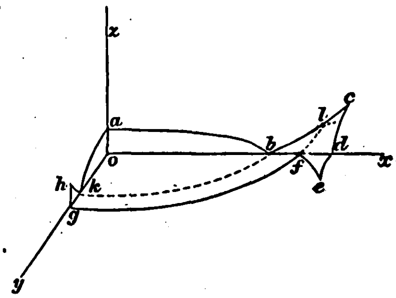
VI. The cuspidal curve is the intersection of a quadric and a cubic surface, and $h = 6$. In the case of the parallel $q' = 0$, because the nodal line of the reciprocal surface is made up of two coincident lines or $f' = 1$. The three 'cnictropes' are represented by three points in the nature of cnicnodes on or adjacent to the nodal lines.

37. It will be remarked that in several instances the characteristics of a tangent cone are the same as those of a general plane parallel of a conic. When the primitive is a central conicoid, having therefore no special relation to the line at infinity, we may determine these characteristics by considering a tangent cone from a point at infinity. This of course is a cylinder, and will be the parallel of the corresponding tangent cone (cylinder) to the primitive. Hence a section of the former cylinder perpendicular to its axis will be a parallel of the corresponding section of the latter. This appears to be generally true for a surface of order m, not specially related to the plane at infinity, and enables us to infer the characteristics of the general tangent cone of the parallel surface.

38. The general form of a parallel surface of a conicoid can be perceived without much difficulty, when we know that the real portion of the nodal curve consists of the real nodal conics.

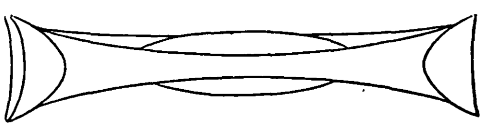
For the purpose of illustration, I refer to the case of an ellipsoid as primitive.

I will suppose that the modulus is less than the least of the principal semiaxes, but greater than the radii of curvature at the vertices on the major axis of any one of the three principal sections. Let Oxy, Oxz, Oyz (Fig. 1) represent the principal planes enclosing a quadrant of the primitive. The sections of the parallel by these planes will consist of an outer oval which is not represented and an interior curve. The three outer ovals belong to the exterior sheet of the surface, which presents no visible singularity, its contour being similar to that of an ellipsoid. The interior curves are represented, so far as they lie within the quadrantal space $Oxyz$, by the lines $Oabcd$, $Odefg$, and $Oghka$. Through the points b, k, which are double on the corresponding plane parallels, we have a real nodal ellipse kb; and through f, also double on the corresponding plane parallel, a nodal hyperbola passes, lying in the plane Oxz, and touching the plane parallel, suppose, in l. Beyond this point of contact, which is a stationary point on the cuspidal curve, the hyperbola is the intersection of imaginary sheets.



(Fig. 1.)

If now, with a free hand, we join ch and ce, the figure gives us a general notion of a quadrant of the interior sheet of the parallel surface. The complete sheet will, in its general features, resemble the Figure 2.



(Fig. 2.)

Outside this, we shall have, as I have said, a surface presenting no visible singularity, but resembling an ellipsoid.

The form of the interior sheet will vary by the disappearance of (1) the nodal ellipse, (2) the nodal hyperbola, (3) both. And again, the form will vary by the disappearance of (1) two twisted ovals of the cuspidal curve, (2) four. It would seem that we cannot have more than four of such ovals, or more than two actual nodal conics. The general form of the surface, when the primitive is an hyperboloid of two sheets, can be perceived in a similar way. The outer superficies will be hyperbolic, and the two hyperbolic sheets of this portion of the surface may intersect in a real nodal ellipse. The remaining portion of the surface due to negative value of the modulus may exhibit a pair of cuspidal ovals and two hyperbolic sheets.

When the primitive is an hyperboloid of one sheet, the parallel surface may exhibit cuspidal edges on both the principal sheets.

We may have an exterior surface with an elliptic ring bounded by the real nodal ellipse and a pair of cuspidal ovals. This ring and the exterior surface generally will enclose a surface of the general form of an hyperboloid of one sheet, but with annexed surfaces bounded by the actually nodal part of the nodal hyperbola and two cuspidal ovals. There would be no great difficulty in determining all the possible variations of forms of the surfaces, but the subject requires either drawings or models for its satisfactory discussion.

39. Instead of considering a single parallel surface, we may consider the system of parallels for a given primitive.

It appears that the locus of the cuspidal curves of the parallels is the common surface of centres. Moreover the locus of the stationary points of the parallels will be the cuspidal curves of the surface of centres in the principal planes and in the 8 special planes. The locus of the nodal right lines will be the 8 special planes. I am supposing the primitive to be a central conicoid, but similar conclusions follow in the other cases.

We can see that the discriminant with regard to k^2 of the discriminant with regard to θ of (1) will be of the gross order 60.

This discriminant is made up of the principal planes and the plane at infinity twice over, the 8 special planes twice over, and the surface of centres three times over. It is evident that the curves of curvature and geodesics are intimately connected with the theory of parallels. A system of normals along a line of curvature of the primitive will trace a line of curvature on the parallel; and when in the unwinding of a series of threads from the surface of centres their extremities generate the primitive, we may in precisely the same way generate a parallel surface. The subject, however, is full of interesting matter, which awaits investigation.

40. It is also obvious that, as in the theory of plane parallels, the parallel of the parallel of a surface will break up into two parallels. The discussion of the characteristics in this case belongs however to the general theory with regard to surfaces of a higher order than the second. The order can be determined by the help of the previous results. It appears that the degree of a parallel is diminished by $8p$ when the primitive has the circle at infinity for a multiple curve of the order p. Also the degree is diminished by 4 for a pair of simple contacts with the circle at infinity. The order in the general case is in terms of characteristics of the primitive, twice

$$\text{order} + \text{class} + \text{order of tangent cone}.$$

In the present instance, therefore, we have for the order of the parallel of the parallel of a central conicoid

$$2\,(12+4+8)-8\,.\,2-4\,.\,2 = 24.$$

For the circle at infinity is a nodal line, and the surface otherwise has 4 contacts with the circle. And generally for a surface of the order m,

$$\begin{aligned}&\text{Order of parallel of parallel}\\ &= 2\{2[m+m(m-1)^2+m(m-1)]+2m(m-1)^2+2m^2(m-1)\}\\ &\qquad -8m(m-1)^2-2.2m(m-1)\\ &= 4\{m+m(m-1)^2+m(m-1)\}\\ &= 2\times\text{order of parallel.}\end{aligned}$$

February 8th, 1872.

Prof. CAYLEY, V.P., F.R.S., in the Chair.

The Chairman stated that the President had made enquiries at the Home Office as to the mode of procedure requisite for obtaining a Charter for the Society, and that the matter would come on for consideration at the next subsequent meeting of the Society, when members would have an opportunity of stating their views upon the desirability of incorporation.

Mr. J. W. L. Glaisher, B.A., F.R.A.S., was elected a Member of the Society.

Mr. T. Cotterill, M.A., gave an account of his paper "On an Algebraical Form, and the Geometry of its dual connection with a Polygon, plane or spherical."

The Chairman, Dr. Hirst, and Prof. Clifford took part in a discussion on the paper.

The following presents were received:—

"Monatsbericht," Sept., Oct., and Dec., 1871.

"Journal of London Institution," Nos. 10, 11, 12.

The first 22 numbers of the "Bulletin des Sciences Mathématiques et Astronomiques," from the commencement to October, 1871: from M. G. Darboux, the Editor.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," tome viii., I^{er} et 2^{me} cahier, 1870.

March 14th, 1872.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. W. Paice, M.A. Lond., was proposed for election.

The President made a statement to the effect that it had been deemed desirable to apply for a Charter, and that he had taken the requisite steps for ascertaining the right mode of procedure; and, in reply to questions by Mr. Sprague and Mr. A. J. Ellis, said that the expense would most likely be small, as there was little prospect at the present time of the application being favourably entertained; and that even if the application were acceded to, it would not be at the Society's expense. The President's proposal—that application should be made to the Council Office for the grant of a Charter, the draft of which had been drawn up by Prof. Cayley—was then agreed to unanimously, and the subject dropped.

The Treasurer then proposed and Prof. Clifford seconded a vote of thanks to Mr. Drach for his present to the Society of two rather rare and interesting early works on Mathematics, by Vieta and Ubaldi respectively.

Prof. Clifford gave a full account of his paper "On a new expression of Invariants and Covariants, by means of alternate numbers." Reference was specially made by the author to the "Vorlesungen über die complexen Zahlen und ihre Functionen," of Dr. Hermann Hankel (1867, part I.).

Mr. Tucker (Hon. Sec.) then read portions of a communication from the Hon. J. W. Strutt, "On the Vibrations of a Gas contained within a Rigid Spherical Case."

Mr. A. J. Ellis, F.R.S., read out the following problem, which had been proposed to him by Prof. Haldeman, of Pennsylvania, United States (who is writing a treatise on English Versification): "The number of lines in a rhymed stanza being given, how many variations of rhyme-distribution does it admit of, supposing no line to be left without a rhyme?" It may be interesting, further, to state, that the Professor remarks, "Of seven-line stanzas, *I have observed in actual use twenty-eight.* The sonnet stanza, of fourteen lines, is very rich; the examples within my reach having given me two hundred and twenty varieties."

The following presents were received:—

"Francisci Vietæ Opera Mathematica": edited by F. A. Schooten, 1646.

"Guidi Ubaldi Mecanicorum liber," Venice, 1615: from Mr. S. M. Drach, F.R.A.S., F.R.G.S.

"Geometria rigorosa di Pietro Dott. Cassani": from A. Stein, Venezia.

"Crelle's Journal," 74 Band, zweites Heft.

"Annali di Matematica," Serie 2ª. tom. v. fasc. 1, Nov. 1871.

"Proceedings of the Royal Society," Vol. xx. No. 132.

"Bulletin des Sciences," Nov. Dec. and Index, 1871, and Jan. 1872.

"Journal of the London Institution," No. 13.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich, redigirt von Dr. Rudolf Wolf.," 15 Jahrgänge (mostly 4 Hefte in each) from 1863, to the 16th Jahrgang (1871), three parts. Duplicate of 8th Jahrgang viertes Heft.

*On the Vibrations of a Gas contained within a Rigid Spherical Envelope.** By the Hon. J. W. Strutt, M.A.

(*Read March 14th*, 1872.)

Whatever may be the motion of air within a sphere, it may always be resolved into a series of simple vibrations represented analytically by terms involving circular, or imaginary exponential, functions of the time. These exist in perfect independence of one another, so that it is sufficient to consider only one at a time. Moreover, the function containing the time (e^{ikat}) will run through the expressions for the velocity-potential,† and its differentials with respect to space as a simple factor, and may therefore be omitted from the beginning.

If λ be the wave-length, $k = 2\pi \div \lambda$, the conditions to be satisfied are

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\psi + k^2\psi = 0 \quad \text{.................. (1)},$$

throughout the interior, and at the surface of the sphere

$$\frac{d\psi}{dr} = 0 \quad \text{.................................. (2)}.$$

The main problem before us is the determination of the possible values of k or λ in terms of the radius of the sphere.

Let ψ be expanded in La Place's Series

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots + \psi_n + \dots \quad \text{.................. (3)},$$

then‡

$$\frac{d^2\psi_n}{dr^2} + \frac{2}{r}\frac{d\psi_n}{dr} - \frac{n(n+1)}{r^2}\psi_n + k^2\psi_n = 0 \quad \text{.............. (4)},$$

of which the solution is

$$r\psi_n = S_n e^{-ikr}\left\{1 + \frac{n(n+1)}{2ikr} + \frac{n(n-1)(n+1)(n+2)}{2.4\,(ikr)^2} + \dots\right\}$$

$$+ S'_n e^{+ikr}\left\{1 - \frac{n(n+1)}{2ikr} + \frac{n(n-1)(n+1)(n+2)}{2.4\,(ikr)^2} - \dots\right\} \quad \text{.....(5)},$$

* The problem here discussed was referred to in a paper on "the Theory of Resonance," Phil. Trans., 1871. Its publication seems of interest, as it is the only case of the vibration of air within a closed vessel which has hitherto been solved with complete generality, except perhaps that of a rectangular parallelepiped. Considerable alterations and additions have been made since the paper was sent in to the Mathematical Society, partly in accordance with the advice of Sir W. Thomson and Professor Clerk-Maxwell, and partly in consequence of my own further reflection on the subject.

† It is assumed that the motion is irrotational, so that a velocity-potential exists; and, further, that it is so small that the square may be neglected throughout.

‡ Stoke's Phil. Mag. Dec. 1868, or Phil. Trans., 1868.