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MODERN ADVANCED ANALYSIS.

Theory of Numbers. By G. B. Mathews, M.A. Part I. (Cambridge: Deighton, Bell and Co., 1892.)

THE book under review is a great contrast in many ways to the "Théorie des Nombres" of M. Edouard Lucas, the first volume of which has recently appeared under the ægis of Messrs. Gauthier-Villars. The latter, reminding the reader much of the same author's "Récréations Mathématiques," exhales human interest from well-nigh every page. The former is on severe philosophical lines, and may be greeted as the first work of the kind in the English language. That this should be a fact is somewhat remarkable. When the late Prof. H. J. S. Smith died prematurely many years ago he left his fellow-countrymen a very valuable legacy. Fortunately he had been commissioned by the British Association to frame a report on the then present state of the Theory of Numbers, a subject with which he was pre-eminently familiar, and in which his own original researches had won for him a great and world-wide renown. The pages of the reports for the years 1864–66 inclusive yield as a consequence a delightful account of modern research in this recondite subject. It is, however, much more than a recital of victories achieved by many able men in many special fields. Prof. Smith's fertile genius enabled him to marshal the leading facts of the theory, and to impress upon them his own personality in a manner that was scarcely within the reach of any other man. He contrived to impart a glamour to those abstract depths of the subject to which few mathematicians have sufficient faith and energy to penetrate. Since that day the scientific world has been yearly expecting his collected papers. There is no doubt that their appearance will greatly stimulate interest and research in Higher Arithmetic. The reports of the British Association are not sufficiently accessible. Doubtless the papers will soon emerge from the hands of those upon whom has devolved the responsibility of their production. In the meantime we welcome Part I. of the present work.

The theory of numbers is the oldest of the mathematical sciences, and may be regarded as their sire. Just as applied mathematics is based on pure, so pure mathematics rests on the theory of numbers. Every investigator finds that sooner or later his researches become a question of pure number. Continuous and discontinuous quantity are indissolubly allied. The theory of series, the theory of invariants, the theory of elliptic functions throw light upon and receive light from higher arithmetic. Algebra in its most general sense is everywhere pervaded by numbers. It may safely be affirmed that there is nothing more beautiful or fascinating in the wide range of mathematics than the interchange of theorem between arithmetic and algebra. A proposition in arithmetic is written out as a theorem in continuous quantity or conversely an algebraic identity is represented by a statement concerning discontinuous quantity. In this country the more recent advances in this attractive method are in large measure due to the labours of Sylvester and J. W. L. Glaisher. In a "Constructive Theory of Partitions," published some half-dozen years ago in the *American Journal of Mathematics*, Sylvester showed some beautiful progressions from arithmetic to algebra, and was followed in the same line by Franklin, Ely, and others, whilst in the pages of the *Quarterly Journal of Mathematics* and *Messenger of Mathematics* Glaisher has applied elliptic function formulas to arithmetical theory. The famous theorem which asserts that every number can be composed by four or fewer square numbers, was due to an application by H. J. S. Smith of elliptic functions to arithmetic. These interesting matters are not alluded to in this first volume.

Chapter I. discusses the divisibility of numbers and the elementary theory of congruencies. Euler's function $\phi(n)$, which denotes the number of positive integers, unity included, which are prime to and not greater than n, is not treated as fully as might be desired. Gauss's theorem

$$\phi(d) + \phi(d') + \phi(d'') + \dots = n$$

where $d, d', d'', \dots$ are all the divisions of n (unity and n included) is given, but not some interesting theorems connected with permutations, of which this is a particular case. Sylvester has written much about the same function, which he calls the "totient" of n. M. Ed. Lucas employs the term "indicateur" in the same sense, and believing that there is a great convenience in having a special name for the function, we regret that Mr. Mathews has not taken a course which would have familiarized students with Sylvester's nomenclature, and have enabled them to feel at home with much that has been written by him and others in this part of the theory of numbers.

The author states that this chapter is substantially a paraphrase of the first three sections of the "Disquisitiones Arithmeticæ," the classical work of Gauss; we are inclined to think that advantage would have been gained if the paraphrase had not been quite so close. The next succeeding chapters are occupied with "Quadratic Congruencies" and the theory of "Binary Quadratic Forms."

The account given is fairly complete. There are so many proofs of Legendre's celebrated "Law of Quadratic Reciprocity" that it must have been difficult to make a selection. A wise choice has, we think, been made of Gauss's third proof as modified by Dirichlet and Eisenstein; the latter's geometrical contribution to the proof taken from the twenty-seventh volume of Crelle is, in particular, of great elegance. Gauss's first proof is also given, as well as references to several others. In the difficult subject of Binary Quadratic Forms, the author keeps well in view the close analogy with the algebraic theory of forms; so many additional restrictions present themselves that a large number of definitions are requisite at the outset, and this circumstance is apt to repel a student who approaches the theory for the first time. The definitions, in fact, constitute the alphabet of the science which must be mastered before progress can be expected in the appreciation of the wonderful beauties which are inherent in it. In this subject, more almost than in any other, the initial drudgery must not be shirked, and it may be said in favour of the present work that clearness of definition and conciseness of statement help the learner much to get quickly over the wearisome preliminaries.

We are glad to see the prominence given to the geometrical methods of Klein and Poincaré; that of the former is based on the theory of substitutions, reminding the reader much of the "Icosaeder"; that of the latter s the "Method of Nets," a most ingenious geometrical application throwing light on the theory of "Reduced Forms."

The "Composition of Forms" given in Chapter VI. is logically and judiciously developed, by means of the bilinear substitution, up to the point of showing the method of tabulating the primitive classes of regular and irregular determinants. The chapter on cyclotomy is one of the best written in the book. The discussion of the section of the periods of the roots of unity has engaged the attentions of mathematicians of the first rank since the time of Gauss, so that of necessity much has been written, and while the author states that he has given but an outline of an extensive theory which has not yet been completed, it may be said that the theory as given, with the references to authorities at the end of the chapter, will be quite sufficient to conduct the student bent upon research to the frontiers of the unknown country.

The determination of the number of properly primitive classes for a given determinant, applications of the theory of quadratic forms, and the distribution of primes complete the volume. Mr. Mathews may be congratulated on his resolve to include Sylvester's masterly contraction of Tchébicheff's limits with reference to the distribution of primes; the reader is taken from the "sieve" of Eratosthenes to the work of Legendre, Meissel, Rogel, Riemann, and to the latest researches of Sylvester and Poincaré, of which the ink is scarcely dry. English mathematicians will turn with delight to the account given on page 302 of Riemann's great memoir of 1859, which contains the only satisfactory attempt to obtain an analytical formula for the number of primes not exceeding a given numerical quantity.

In conclusion, though the sequence of the subject matter may be open to criticism, we regard the book as a most valuable contribution to the small library of higher mathematical treatises that, owing chiefly to the energy and enthusiasm of the rising generation of mathematicians, is being brought together. How woefully deficient that library was but a few years since those engaged in research know only too well, and greatly do they rejoice as they see the yawning gaps one by one efficiently filled up. Part II. of the task Mr. Mathews has set himself to accomplish will, we hope, soon appear, and we trust he will be as successful with it as with the present Part I.

P. A. M.

THE DARWINIAN THEORY.

Darwin and After Darwin; an Examination of the Darwinian Theory and a Discussion of Post-Darwinian Questions. By George John Romanes, M.A., LL.D., F.R.S. I. *The Darwinian Theory.* (London: Longmans, 1892.)

WE had hoped ere now to have received the second instalment of this work, and to have dealt with the two volumes in a single critical notice. Unforeseen causes, one of them deeply to be regretted, have presumably prevented the appearance of the discussion of Post-Darwinian questions so early as had been anticipated. We therefore propose to give a short expository notice of the present volume, reserving such criticism as we have to offer for a future occasion, when the second volume shall have come to hand.

The first section consists of an exposition of the scientific evidences of evolution as a fact independent of the Darwinian theory of the method by which this evolution has been brought about. It may be regarded as an expansion of the author's little volume in the "Nature Series," on "The Scientific Evidences of Organic Evolution," published ten years ago. Mr. Romanes has spared no pains in the collection and marshalling of his evidence. His object is to convince, by the abundance of facts and by logical inferences based thereon, those who still hold by the tenets of Special Creation. Whether those who still hold by these tenets are likely to be influenced by the facts or the inferences is a question we do not propose to discuss. The author evidently supposes that they are, and has written for them a good many pages in a strain of which we give a couple of examples:—"It would seem most capricious on the part of the Deity to have made the eyes of an innumerable number of fish on exactly the same ideal type, and then to have made the eye of the octopus so exactly like these other eyes, in superficial appearance, as to deceive so accomplished a naturalist as Mr. Mivart, and yet to have taken scrupulous care that in no one ideal particular should the one type resemble the other." Again, "Although in nearly all the numerous species of snakes there are no vestiges of limbs, in the Python we find very tiny rudiments of hind-limbs. Now, is it a worthy conception of Deity that, while neglecting to maintain his unity of ideal in the case of nearly all the numerous species of snakes, he should have added a tiny rudiment in the case of the Python—and even in that case should have maintained his ideal very inefficiently, inasmuch as only two limbs, instead of four, are represented?"

The second section of the volume is devoted to the setting forth of the theory of natural selection as it was held by the master. This, as was to be expected, is a well-ordered and lucid exposition. We could wish that Mr. Romanes had been more careful to avoid all appearance of personifying natural selection. He says, for example, "it is the business of natural selection to secure the highest available degree of adaptation for the time being." Such language is highly metaphorical, if not misleading. If we can talk of business at all we may say that it is the business of various eliminating agencies, in the struggle for existence, to weed out and exclude from any share in perpetuating their race all those individuals who are too weakly to stand the stress of the struggle. The survival of the fit is an incidental result of the stern business of elimination. It is here that the naturalistic hypothesis differs most markedly from the teleological interpretation of nature. In conversation a while since a friend observed to us: Since your school of thought admit that the eye of natural selection is ever on the watch for the slightest improvement in adaptation, why should they hesitate to say with us that it is the eye of Beneficence that is thus ever watchful? The misunderstanding of the naturalistic position here