difficulty of forming these systems and then solving them; but this does affect one result: viz., that the auxiliary system may be reduced to systems involving $\mu-1, \mu-2, \ldots \mu'$ of the F's. These systems are not generally of the first order; though, as in case (A) of Art. 9, one of them may be so; and it seems not unlikely, bearing this case in mind, that a considerable simplification in the form of the subsidiary equations may be found when they can be written down.

The lack of symmetry in the above process is another objection to it; but it is not unprecedented.

Note on the Integral Solution of $x^2-2Py^2=-z^2$ *or* $\pm 2z^2$ *in certain Cases.* *By* SAMUEL ROBERTS, F.R.S.

[*Read March 11th*, 1880.]

1. I assume that P is a non-square number, the prime factors of which are of the form $8m+1$. The number is therefore uneven, and susceptible of the three forms $\alpha^2+16\beta^2$, $\alpha^2+8\beta^2$, and $\alpha^2-8\beta^2$.

The number of such representations is finite in the first two cases, and infinite in the last case.

The forms $\alpha^2+16\beta^2$ are comprised in

$$(8k\pm s)^2+16l^2,$$

where k, l remain unchanged in species throughout all the system for a given value of P, and s remains constant, having one of the values 1, 3.

In fact, if a^2+16b^2, $c^2+16d^2 \ldots m^2+16n^2$ are representations of the prime factors of P (and these may be identical in groups, so that one or more prime factors enter in powers higher than the first), then, giving positive values to the letters, we may write

$$(a\pm i4b)(c\pm i4d)\ldots(m\pm i4n) = \pm A \pm i4B \;\ldots\ldots\ldots\ldots(a)$$

with $P=A^2+16B^2$. If, therefore, $ac\ldots m \equiv s$ (mod. 8), we shall have, independently of the ambiguity of the signs,

$$A = 8k\pm s,\; B = l,$$

where k, l remain unchanged in species because the species of $\pm\mu\pm\nu\pm\pi\pm$ &c. is the same as that of $\mu+\nu+\pi+$&c. Moreover, all the representations of P as the sum of two squares can be obtained from (a). The sign affecting s in two representations of the same number may be different.

It appears, similarly, that the representations of P by $\alpha^2+8\beta^2$ are comprised in $$(8k\pm s)^2+8l^2;$$
where k, l remain unchanged in species in the same system, and s is constant.

With regard to the last form $\alpha^2-8\beta^2$, a new consideration enters, since the representations are infinite in number.

Suppose the prime factors of P are represented by a^2-8b^2, c^2-8d^2 ... m^2-8n^2, where we have $a^2>8b^2$, $c^2>8d^2$, &c., then all the similar representations of P immediately depending on these particular representations of the prime factors, are obtained by means of

$$(a\pm\sqrt{8}\,b)\,(c\pm\sqrt{8}\,d)\ldots(m\pm\sqrt{8}\,n)=\pm A\pm\sqrt{8}\,B\ldots\ldots(b).$$

Hence, if $ac\ldots m\equiv s \pmod{8}$,

$$A=8k+s,\ \ B=l,$$

where, as before, k, l remains unchanged in species, and s is constant, and remains of the same sign throughout. In this case, we suppose s to have one of the four values $+1$, -1, $+3$, -3.

In order to obtain all the other representations by means of the foregoing group derived from (b), it is necessary and sufficient to make use of $$(A+\sqrt{8}\,B)\,(p\pm\sqrt{8}\,q)=\pm A'\pm\sqrt{8}\,B',$$
where $p^2-8q^2=1$.

For all the solutions of $x^2-8y^2=p$, where p is a prime, are obtained by means of $$(x-\sqrt{8}\,y)\,(X\pm\sqrt{8}\,Y),$$
when $X^2-8Y^2=1$; consequently, when $x^2-8y^2=P'$, the prime factors of P', being of the form $\alpha^2-8\beta^2$, can be similarly obtained. As was noticed by Lagrange, and is now well known, this result does not hold always when P' is composite; because we have different conjugate pairs of incongruous roots satisfying $n^2-8\equiv 0 \pmod{P'}$.

The representations of $p^2-8q^2=1$ are of the forms

$$(8\mu+1)^2-8\,(2\nu+1)^2,$$

or

$$(8\mu+3)^2-8\,(2\nu)^2.$$

So that we may write $A'=8k+s,\ \ B'=l,$

or $A'=8k+3s,\ \ B'=l,$

the species k, l remaining unchanged, and s being constant and affected with the same sign in all the system.

Hence the representations of P by $\alpha^2-8\beta^2$ are comprised in

$$(8k+s)^2-8l^2,$$

or

$$(8k+3s)^2-8l^2.$$

The species of l is at once determined by means of $P\equiv 1 \pmod{16}$, or $P\equiv 9 \pmod{16}$, as the case may be.

2. We have next to determine certain relations in which the three classes of representations stand to one another, and, for this purpose, proceed by way of exclusion.

We cannot have

$$(8a\pm 1)^2+16\,(2\beta+1)^2 = (8k\pm 1)^2+8\,(2l)^2 \;\ldots\ldots\ldots\ldots (c).$$

For suppose $8k\pm 1>8a\pm 1$, then

$$(8k\pm 1)^2-(8a\pm 1)^2 = 16\{(2\beta+1)^2-2l^2\}.$$

Any odd factor common to $(8k\pm 1)-(8a\pm 1)$ and $(8k\pm 1)+(8a\pm 1)$ is also common to $8k\pm 1$ and $8a\pm 1$. Also, $k\pm a$ must be odd, since

$$(k\pm a)\,\{4\,(k\mp a)\pm 1\} = (2\beta+1)^2-2l^2,$$

and since $4\,(k\mp a)\pm 1$ is of the form $8m\pm 3$, the first factor $k\pm a$ must have in common with the second a factor of the same form which will also be a factor common to $2\beta+1$ and l. It is also common to $8k\pm 1$ and $8a\pm 1$ by what precedes. This result is contrary to our primary supposition, and hence the equality (c) cannot hold.

A similar conclusion follows if we suppose $8a\pm 1>8k\pm 1$.

In the same way it will be found that the following equalities cannot hold

$$\begin{aligned}(8a\pm 1)^2+16\,(2\beta)^2 &= (8k\pm 3)^2+8\,(2l+1)^2,\\ (8a\pm 3)^2+16\,(2\beta+1)^2 &= (8k\pm 1)^2+8\,(2l+1)^2,\\ (8a\pm 3)^2+16\,(2\beta)^2 &= (8k\pm 3)^2+8\,(2l)^2.\end{aligned}$$

Again, we cannot have

$$(8a\pm 1)+16\,(2\beta+1)^2 = (8k+1)^2-8\,(2l)^2,$$

or

$$(8k+3)^2-8\,(2l+1)^2.$$

In fact, taking the first form of the equality, and supposing

$$8k+1>8a\pm 1,$$

we have

$$(k\mp a)\,\{4\,(k\pm a)+1\} = (2\beta+1)^2+2l^2,$$

and $k\mp a=2n+1$ gives $4\,(k\pm a)+1=8m-3$. It follows that a factor of this form must be common to $8a\pm 1$ and $2\beta+1$, contrary to our supposition. The result is similar if we suppose $8a\pm 1>8k+1$ and, for like reasons, also the second form of the equality is impossible. Following the same method, we find that the following equalities cannot hold:—

$$\begin{aligned}(8a\pm 1)^2+16\,(2\beta)^2 &= (8k-1)^2-8\,(2l)^2,\\ \text{or}\quad &= (8k-3)^2-8\,(2l+1)^2;\\ (8a\pm 3)^2+16\,(2\beta+1)^2 &= (8k+3)^2-8\,(2l)^2,\\ \text{or}\quad &= (8k+1)^2-8\,(2l+1)^2;\\ (8a\pm 3)^2+16\,(2\beta)^2 &= (8k-3)^2-8\,(2l)^2,\\ \text{or}\quad &= (8k-1)^2-8\,(2l+1)^2.\end{aligned}$$

On the other hand, we have the following compatible coexistent forms:

I.

$$(8a\pm 1)^2+16\,(2\beta+1)^2,\quad (8k\pm 3)^2+8\,(2l+1)^2,\quad \begin{matrix}(8k-1)^2-8\,(2l)^2\\ (8k-3)^2-8\,(2l+1)^2.\end{matrix}$$

II.

$$(8a\pm 1)^2+16\,(2\beta)^2,\quad (8k\pm 1)^2+8\,(2l)^2,\quad \begin{matrix}(8k+1)^2-8\,(2l)^2\\ (8k+3)^2-8\,(2l+1)^2.\end{matrix}$$

III.

$$(8a\pm 3)^2+16\,(2\beta+1)^2,\quad (8k\pm 3)^2+8\,(2l)^2,\quad \begin{matrix}(8k-1)^2-8\,(2l+1)^2\\ (8k-3)^2-8\,(2l)^2\quad .\end{matrix}$$

IV.

$$(8a\pm 3)^2+16\,(2\beta)^2,\quad (8k\pm 1)^2+8\,(2l+1)^2,\quad \begin{matrix}(8k+1)^2-8\,(2l+1)^2\\ (8k+3)^2-8\,(2l)^2\quad .\end{matrix}$$

3. We have next to apply the foregoing results to the equations $x^2-2Py^2=-z^2$ or $\pm 2z^2$, in which we shall suppose x to be prime to y.

In the first equation, let $y=p^2+4q^2$ or $y^2=(p^2-4q^2)^2+16p^2q^2$, $2y^2=(p^2-4q^2+4pq)^2+(p^2-4q^2-4pq)^2$, which last is of the form $(8\mu\pm 1)^2+(8\nu\pm 1)^2$, p being odd and q odd or even.

If, now, P belongs to set I., we get

$$2Py^2=(8u\pm 3)^2+(8v\pm 3)^2$$

by means of the product

$$\{8a\pm 1+i4\,(2\beta+1)\}\{8\mu\pm 1+i\,(8\nu\pm 1)\}.$$

Again, if y is of the form p^2+2q^2,

$$y^2=(p^2-2q^2)^2+8p^2q^2=(8\mu\pm 1)^2+8\nu^2,$$

we have Py^2 of the form $(8u\pm 3)^2+8u^2$ or $2Py^2=16u^2+2\,(8v\pm 3)^2$.

Lastly, if $y=p^2-2q^2$, $y^2=(p^2+2q^2)^2-8p^2q^2=(8\mu+1)^2-8\,(2\nu)^2$, or $(8\mu+3)^2-8\,(2\nu+1)^2$, Py^2 is of the form $(8k-1)^2-8\,(2l)^2$ or $(8k-3)^2-8\,(2l+1)^2$, that is to say, of the form

$$2\,(8k-1\pm 4l)^2-(8k-1\pm 8l)^2$$

or

$$2\,\{8k-3\pm(4l+2)\}^2-\{8k-3\pm 2\,(4l+2)\}^2,$$

and we get

$$2Py^2=4u^2-2\,(8v\pm 1)^2.$$

Similar results are arrived at when P belongs to one of the three remaining sets; so that we get finally the following systems corresponding to the four sets:

I.

$$2Py^2=(8u\pm 3)^2+(8v\pm 3)^2,$$
$$2Py^2=16u^2+2\,(8v\pm 3)^2,$$
$$2Py^2=4u^2-2\,(8v\pm 1)^2.$$

II.

$$2Py^2 = (8u\pm 1)^2+(8v\pm 1)^2,$$
$$2Py^2 = 16u^2+2\,(8v\pm 1)^2,$$
$$2Py^2 = 4u^2-2\,(8v\pm 1)^2.$$

III.

$$2Py^2 = (8u\pm 1)^2+(8v\pm 1)^2,$$
$$2Py^2 = 16u^2+2\,(8v\pm 3)^2,$$
$$2Py^2 = 4u^2-2\,(8v\pm 3)^2.$$

IV.

$$2Py^2 = (8u\pm 3)^2+(8v\pm 3)^2,$$
$$2Py^2 = 16u^2+2\,(8v\pm 1)^2,$$
$$2Py^2 = 4u^2-2\,(8v\pm 3)^2.$$

4. If P is a prime or the odd power of a prime, one of the three following equations must hold in integers: $x^2-2Py^2=-1$ or $+2$ or -2. And the forms of the last article shew that—

(i.) If P is of the set III., the equation $x^2-2Py^2=-1$ is always possible.

(ii.) If P is of the set I., the equation $x^2-2Py^2=+2$ is resoluble.

(iii.) If P is of the set IV., the equation $x^2-2Py^2=-2$ is resoluble.

Lastly, if P is of the set II., it remains doubtful which of the three forms of the equation is resoluble.

In stating negative results, we may regard P as having its wider meaning, and then we have the following :—

(i.) If P belongs to the set I., $x^2-2Py^2=-1$ or -2 is impossible in integers.

(ii.) If P belongs to the set III., $x^2-2Py^2=+2$ or -2 is impossible in integers.

(iii.) If P belongs to the set IV., $x^2-2Py^2=-1$ or $+2$ is impossible.

We get also impossible forms involving fourth powers from the consideration that $(8k-1)^2$, $(8k\pm 3)^2$ cannot be integer fourth powers.

Notes (1) on a Geometrical Form of Landen's Theorem with regard to a Hyperbolic Arc; (2) On a Class of Closed Curves whose arcs possess the same property as two Fagnanian Arcs of an Ellipse. By John Griffiths, M.A., Jesus College, Oxford.

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In the following notes the quantities r, p, t refer to a point on a curve, viz., r is the radius vector of the point, measured from the