



## XI. The asymmetrical probability-curve

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or a hurricane ; if heard from the third wire ( $d=0.240$  cm.), it would mean 160 miles per hour, a speed\* which fortunately transcends our storm nomenclature.

7. At the close of these experiments I put up a couple of these anemometers in my yard, but I have not yet obtained sufficient material for discussion. The means of registry is to bring the sounding wire on the whirling arm into unison with the exposed anemometer, and to let the former wire make its registry on the chronograph (§ 2).

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### XI. *The Asymmetrical Probability-Curve.*

By Professor F. Y. EDGEWORTH, M.A., D.C.L.†

THE Probability-Curve may be described as an approximation to the law of frequency which governs the set of values assumed by a function of numerous independently varying small quantities ; the function and the limits within which the variables range being such that the function may be regarded as approximately *linear* ; so that we have nearly

$$Q = Q_0 + Q_1'q_1 + Q_2'q_2 + \&c. + Q_n'q_n\ddagger;$$

where  $Q$  is the compound quantity under consideration ;  $q_1, q_2, \&c.$  are the elementary quantities ;  $Q_1'$  is what  $Q$  becomes when we differentiate with respect to  $q_1$ , and substitute zero for each of the variables  $q_1, q_2, \dots$  ;  $Q_2', Q_3', \dots$  are similarly defined ;  $Q_0$  is what  $Q$  becomes when zero is substituted in  $Q$  for each of the  $q$ 's—an absolute term which may usually be omitted.

The *symmetrical* probability-curve is a *first* approximation which is commonly written

$$y = \frac{1}{\sqrt{\pi c}} e^{-\frac{x^2}{c^2}};$$

where  $x$  is the abscissa along which the values of  $Q$  are

\* There is a possible optical analogy to which I may allude in passing. If in relation to speeds of the order of molecular velocities, the luminiferous æther may be considered as evidencing viscosity (following in the line of a well-known hypothesis of Lord Kelvin), we might then expect a molecule in its passage through æther to “sing” optically ; in other words, we might expect the æther to awake resonant vibrations in the molecule, in the way in which the transverse harmonic vibrations of a wire are evoked (§ 1 *et seq.*) when the air-tone is the fundamental.

† Read before the Royal Society, June 1, 1894. Communicated by the Author ; in a revised and abridged form.

‡ On this condition see the present writer's paper in the *Philosophical Magazine*, Nov. 1892.

measured from an origin such that the average value of each of the  $q$ 's, and therefore also the average value of  $Q$ , vanishes ( $Q_0$  being omitted);  $y \Delta x$  is the proportional number of values of  $Q$  occurring between  $x$  and  $x + \Delta x$ ;  $\frac{c^2}{2}$  is the sum of the  $n$  quantities each of which is the mean square of error (measured from the average value or centre of gravity) for one of the elements  $Q/q$ .

The *asymmetrical* probability-curve is a second approximation which may be written

$$y = \frac{1}{\sqrt{\pi}c} e^{-\frac{x^2}{c^2}} \left( 1 - \frac{2j}{c^3} \left[ \frac{x}{c} - \frac{2}{3} \frac{x^3}{c^3} \right] \right);$$

where  $x$ ,  $y$ , and  $c$  have each the same meaning as before;  $j$  is the sum of the  $n$  quantities each of which is the mean cube of error for one of the elements  $Q/q$ . I propose to give a new proof of this formula; after first adverting to one or two old proofs.

I. The formula may be derived from an analysis which Todhunter, following Poisson, has indicated. Todhunter inquires what is the probability,  $P$ , of the value of a certain quantity  $E$  occurring between  $c + \eta$  and  $c - \eta$ ;  $E$  being  $= \gamma_1 \epsilon_1 + \gamma_2 \epsilon_2 + \dots$ , where  $\gamma_1, \gamma_2, \dots$  are constants, and each of the quantities  $\epsilon_1, \epsilon_2, \&c.$  fluctuates between given limits,  $a$  and  $b$ , according to a law of frequency which is of the form

$y = f_i(x)$ ; the value of  $\int_{b_i}^{a_i} x f_i(x) dx$  being  $k_i$ ; the corresponding value of mean *second* powers being  $k'_i$ , of mean *third* powers  $k''_i$ . For a first approximation to  $P$ , Todhunter finds

$$P = \frac{1}{2\kappa\sqrt{\pi}} \int_{-\eta}^{\eta} e^{-\frac{(l-c+v)^2}{4\kappa^2}} dv;$$

where  $l = \sum \gamma_i k_i$ , and  $2\kappa^2 = \sum \gamma_i^2 (k'_i - k_i^2)$ . This coincides with the first approximation given above\* when  $k_1, k_2$  each  $= 0$ , and when the interval  $2\eta$  is indefinitely small. For then  $P$  reduces to

$$\frac{1}{2\kappa\sqrt{\pi}} e^{-\frac{c^2}{4\kappa^2}} \times 2\eta;$$

where  $c^2$  may be replaced by our  $x^2$ ,  $2\eta$  by our  $\Delta x$ , and  $4\kappa^2$  = our  $c^2$ .

For a second approximation Todhunter indicates as the

\* P. 90.

correction of, or addendum to, the above expression an expression which is described as the third differential with respect to  $l$  of the value of  $P$  above written (as a first approximation) multiplied by a constant  $l_1$  which

$$= \frac{1}{6} \Sigma \gamma_i^3 (k_i'' - 3k_i k_i' + 2k_i^3).$$

Performing this operation\* and putting (as before) each of the  $k$ 's = 0, and therefore  $l = 0$ ,

$$l_1 = \frac{1}{6} \Sigma \gamma_i^3 k_i'' = \frac{1}{6} \text{ our } j;$$

and (as before) substituting  $x$  for  $c$ ,  $\Delta x$  for  $2\eta$ ,  $c^2$  for  $4\kappa^2$ , we find the addendum (to the formula for the symmetrical probability-curve), in our notation,

$$\frac{2j}{c} e^{-\frac{x^2}{c^2}} \left[ \frac{x}{c^2} - \frac{2}{3} \frac{x^3}{c^4} \right] \Delta x.$$

Adding the correction to the first approximation, we have our formula for the asymmetrical probability-curve†.

II. A second proof is derivable from the reasoning by which Mr. E. L. De Forest in the 'Analyst' (Iowa) (vol. ix. p. 163) obtains for the asymmetrical probability-curve a certain formula which has been independently discovered by Prof. Karl Pearson‡. The expansion of this formula in ascending powers of  $x$  will be found to coincide with the expansion of our formula in ascending powers of  $x$  ;

provided that the second and higher powers of  $\frac{j}{c^3}$  may be neglected—a condition which is employed by Mr. De Forest in his proof§, and may readily be established||.

\* This and subsequent operations are performed more fully in the MS. deposited in the Archives of the Royal Society.

† When writing this paragraph I had not adverted to the similar work in Galloway's treatise on *Probability* (forming the article on that subject in the seventh and eighth editions of the *Encyclopædia Britannica*), art. 136.

‡ Philosophical Transactions, 1894. Proceedings of the Royal Society, 1893, p. 331. Cf. 'Nature,' 1895, p. 317.

§ E. g. *loc. cit.* p. 138, regard being had to the definition of his symbols.

|| For example, let the elemental frequency-locus consist of two points, at a distance  $b$ , assumed with respective frequencies  $p$  and  $q$ . Then the mean square of error for a single element is  $pqb^2$ , and the sum of these means for all the elements, our  $k$ , is  $npqb^2$ . Also the mean cube of error for an element is  $\pm pq(p-q)b^3$  (cf. Phil. Mag. vol. xxi. (1886) p. 320); whence  $j \div k^{\frac{3}{2}} = (p-q) \div \sqrt{n} \sqrt{pq}$ . Which is small when  $\sqrt{n}$  is large relatively to  $(p-q) \div \sqrt{pq}$ , a quantity which vanishes when the element is symmetrical, and is finite for all but infinite degrees of asymmetry. By parity it will be found that  $j \div k^{\frac{3}{2}}$  in general = a finite quantity  $\div \sqrt{n}$ ; so that it becomes small when  $n$  is sufficiently large.

III. The new proof of the formula for the asymmetric probability-curve, which is offered here, is analogous to that which has been given by Mr. Morgan Crofton for the symmetrical probability-curve\*. The proof consists in determining  $y$ , the required error-function, as the solution of a system of partial differential equations which must be satisfied by such a function. Put  $y = F(x, k, j)$  where  $x$  and  $j$  have the same signification as before and  $k$  now = our  $c^2 \div 2$ , (= Todhunter's  $\kappa^2 \times 2$ ). A first equation is obtained from the condition that, if each of the constituent elements  $q_1, q_2 \dots$  be multiplied by a constant  $\gamma$ †,

$$\frac{y}{\gamma} = F(\gamma x, \gamma^2 k, \gamma^3 j).$$

Putting  $\gamma = (1 + \omega)$ , where  $\omega$  is indefinitely small, expanding and neglecting powers of  $\omega$  above the first, we have

$$y + x \frac{dy}{dx} + 2k \frac{dy}{dk} + 3j \frac{dy}{dj} = 0. \quad . \quad . \quad (1)$$

Two more equations are given by the conditions that, if  $y = F(x, k, j)$  is the law of frequency for the sum of the  $n$  elements  $Q_1' q_1 + Q_2' q_2 + \dots$ , then the superposition of a new element of the form  $Q_{n+1}' q_{n+1}$  for which the mean-square-of-error (measured from its centre of gravity) is  $\Delta k$ , and the mean-cube-of-error (similarly measured) is  $\Delta j$ , must obey the law of frequency

$$y + \Delta y = F(x, k + \Delta k, j + \Delta j).$$

Let  $\eta = f(\xi)$  be the law of frequency for the new element. Then the law of frequency for the compound (of  $n + 1$  elements) ‡ is

$$\int_b^a f(\xi) F(x - \xi) d\xi,$$

where  $a$  and  $b$  denote the extreme limits of the range of  $f(\xi)$ —limits which are by hypothesis finite§. Expanding  $F(x - \xi)$  in terms of  $\xi$ , and neglecting powers of  $\xi$  above the third (upon the hypothesis that the range of  $\xi$  is comparatively

\* In the article on *Probability*, 'Encyclopædia Britannica,' 9th edit., vol. xix. p. 781.

† Cf. Mr. Morgan Crofton, *loc. cit.*

‡ According to the rule for compounding laws of error indicated by Mr. Morgan Crofton in the article referred to. Compare the present writer, *Camb. Phil. Trans.* vol. xiv. p. 141.

§ Above, p. 90, and *Phil. Mag.* vol. xxxiv. (Nov. 1892).

small); and observing that

$$\int_b^a f(\xi) \xi d\xi = 0$$

(since  $\xi$  is measured from the centre of gravity of the corresponding curve),

$$\int_b^a f(\xi) d\xi \xi^2 = \Delta k, \quad \int_b^a f(\xi) d\xi \xi^3 = \Delta j;$$

we have

$$y + \Delta y = F + \frac{1}{2} \Delta k \frac{d_2 F}{dx^2} - \frac{1}{6} \Delta j \frac{d_3 F}{dx^3}.$$

This expression ought to be identical with

$$F + \Delta k \frac{dF}{dk} + \Delta j \frac{dF}{dj}.$$

Here  $\Delta k$  and  $\Delta j$  may be regarded as independent observations; it is therefore proper to equate the coefficients of  $\Delta k$  in the two expressions for  $y + \Delta y$ ; and similarly the coefficients of  $\Delta j$ . Thus we obtain two additional partial differential

\* The reasoning requires that the expansion of  $y + \Delta y$  should form a descending series. This condition is fulfilled by our solution. For let the mean square of error for each element be of the order  $\frac{1}{n}$ , then  $k$ , the sum of these mean squares, will be of the order unity. Accordingly if  $y$ , as proposed,  $= \frac{1}{\sqrt{\pi} \sqrt{2k}} e^{-\frac{x^2}{2k}}$  approximately, then

$$\begin{aligned} \frac{d_2 y}{dx^2} \div y &= -\frac{1}{k} + \frac{x^2}{k^2}, \\ \frac{d_3 y}{dx^3} \div y &= \frac{3x}{k^2} - \frac{x^3}{k^3}. \end{aligned}$$

These and higher differentials may be regarded as being of the order unity for values of  $x$  between limits  $\pm 1$ . Whence it follows that the terms of the expansion in the text form a descending series; since  $\frac{d_2 F}{dx^2}$ ,  $\frac{d_3 F}{dx^3}$ , &c. are of the order unity, and  $\Delta k$ ,  $\Delta j$ , &c., being integrals of  $\xi^2 f(\xi)$ ,  $\xi^3 f(\xi)$ , &c. between limits separated by a very small interval, will in general form a descending series. The reasoning is not affected if we change the unit: *e. g.* suppose the range of the elements to be of the order unity; in which case  $\frac{d_2 F}{dx^2}$ ,  $\frac{d_3 F}{dx^3}$ , &c. will form a descending series, while  $\Delta k$ ,  $\Delta j$ , &c. will be of the same order. Also the order of  $\frac{d_2 y}{dx^2} \div y$ ,  $\frac{d_3 y}{dx^3} \div y$ , &c. is not affected by taking into account the second term of approximation to the value of  $y$  given in the text; it being observed that  $j \div c^3$  is small, and that the formula only professes to be applicable for values of  $x$  which are of the same order as  $c$ .

equations ; the whole system being

$$y + x \frac{dy}{dx} + 2k \frac{dy}{dk} + 3j \frac{dy}{dj} = 0, \quad . \quad . \quad . \quad (1)$$

$$\frac{dy}{dk} = \frac{1}{2} \frac{d_2 y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{dy}{dj} = -\frac{1}{6} \frac{d_3 y}{dx^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To these data are to be added the conditions (a) that  $j \div k^{\frac{1}{3}}$  is small\*, and (b) that

$$\int_{-\infty}^{+\infty} dyx = 1 \dagger.$$

Of the problem thus stated the following two solutions are offered :—

*First Method.*—From equation (1) we have by a familiar method

$$y = \frac{1}{\sqrt{k}} \phi \left( \frac{x}{\sqrt{k}}, \frac{j}{k^{\frac{1}{3}}} \right), \quad . \quad . \quad . \quad . \quad (4)$$

where  $\phi$  is an arbitrary function.

From equation (2) by a known method ‡ we have

$$y = \left( 1 + \frac{1}{2} x^2 \left( 2 \frac{d}{dk} \right) + \frac{1}{4} x^4 \left( 2 \frac{d}{dk} \right)^2 + \dots \right) \psi_1 \\ + \left( x + \frac{1}{3} x^3 \left( 2 \frac{d}{dk} \right) + \dots \right) \psi_2 ; \quad . \quad . \quad (5)$$

where  $\psi_1$  and  $\psi_2$  are arbitrary functions of  $k$  (and  $j$ ). These functions are restricted by equation (4) to the form

$$\frac{1}{\sqrt{k}} \phi \left( \frac{j}{k^{\frac{1}{3}}} \right).$$

To further determine the forms of  $\psi_1$  and  $\psi_2$  we must utilize equation (3); from which, by combination with equation (2), we have

$$\frac{dy}{dj} = -\frac{1}{3} \frac{d_2 y}{dx dk} \quad . \quad . \quad . \quad . \quad (6)$$

This condition may be fulfilled by assuming  $\psi_1$  and likewise  $\psi_2$  to consist of a series of ascending powers of  $\frac{j}{k^{\frac{1}{3}}}$ ; which is permissible by condition (a).

\* As shown above, p. 92 note. † A condition of a probability-curve.

‡ Forsyth's 'Differential Equations,' Art. 256.

To determine the absolute term in the expansion of  $y$ , it may be observed that in the case of symmetry  $\psi_2$  vanishes (since  $y$  cannot involve odd powers of  $x$ ); whence it appears, since  $j$  also vanishes in this case, that  $\psi_2$  has no absolute term,  $\psi_1$  reduces to  $\frac{A_0}{\sqrt{k}}$ .  $A_0$  is found by condition (b) to be  $\frac{1}{\sqrt{2\pi}}$ , while the value of  $y$  is (the expansion in powers of  $x$  of) the well-known (symmetrical) probability-curve.

To proceed another step, let the first term of  $\psi_2$  be  $B_1 \frac{j}{k^{\frac{3}{2}}}$ , a form which is prescribed by equation (1); and the second term of  $\psi_1$ ,

$$\frac{A_1}{\sqrt{k}} \frac{j}{k^{\frac{3}{2}}}.$$

By equation (6) combined with (5) we have

$$\begin{aligned} \left(x + \frac{1}{3} x^3 \left(2 \frac{d}{dk}\right) + \dots\right) \frac{B_1}{k^{\frac{5}{2}}} + \left(1 + \frac{1}{2} x^2 \left(2 \frac{d}{dk}\right) + \dots\right) \frac{A_1}{k^{\frac{3}{2}}} \\ = -\frac{1}{3} \left(x \left(2 \frac{d}{dk}\right) + \frac{1}{3} x^3 \left(2 \frac{d}{dk}\right)^2 + \dots\right) \frac{d}{dk} \frac{1}{\sqrt{2\pi k}}. \end{aligned}$$

This identity evidently requires that  $A_1=0$ . The identity is then satisfied by  $B_1 = -\frac{1}{2\sqrt{2\pi}}$ .

It is unnecessary to proceed further with the determination of the coefficients, since the higher powers of the expansion may be neglected. We have therefore for the solution

$$\begin{aligned} y = \left(1 + \frac{1}{2} x^2 \left(2 \frac{d}{dk}\right) + \dots\right) \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{k}} \\ + \left(x + \frac{1}{3} x^3 \left(2 \frac{d}{dk}\right) + \dots\right) \frac{-j}{2\sqrt{2\pi} k^{\frac{3}{2}}}; \end{aligned}$$

an expression of which the expansion in powers of  $x$  proves to be identical with the expansion of the formula given above, when  $k$  is replaced by  $\frac{c^2}{2}$ .

*Second Method.*—The following is perhaps a simpler solution. Put as the correction of the first approximation

$$(\text{viz. } y = \frac{1}{\sqrt{\pi} \sqrt{2k}} e^{\frac{-x^2}{2k}})$$



the expression  $\theta(x, k, j)$ . Then if we put  $F_1$  for the first approximation,

$$y = F_1 + \theta; \quad \frac{dy}{dx} = F_1' + \theta'; \quad \&c.;$$

where  $F_1', \theta' \dots$  denote partial differentials with respect to  $x$ . Now, if  $\theta$  is small with respect to  $F_1(x)$ , then the functions being continuous,  $\theta'$  will be small with respect to  $F_1'(x)$ . And we may likewise assume that  $\theta''$  and  $\theta'''$  are small in comparison with  $F_1''$  and  $F_1'''$  respectively. Therefore in the expression for  $\frac{d_3 y}{dx^3}$  it is allowable to neglect  $\theta'''$ . But it is not equally allowable to neglect  $\theta''$ . For considering the expansion of  $y + \Delta y$ , viz.

$$F + \frac{1}{2} \Delta k \frac{d_2 F}{dx^2} - \frac{1}{6} \Delta j \frac{d_3 F}{dx^3}$$

(above, p. 94), we could not be sure that the neglected quantity  $\Delta k \theta''$  is not of the same order as the retained quantity  $\Delta j \frac{d_3 F}{dx^3}$ , the terms of the expansion forming a descending series. Rejecting therefore only  $\theta'''$ , we have approximately

$$\frac{d_3 y}{dx^3} = \frac{d_3 F_1}{dx^3} = \frac{1}{\sqrt{2\pi k}} e^{\frac{-x^2}{2k}} \left( \frac{3x}{k^2} - \frac{x^3}{k^3} \right);$$

whence by equation (3),

$$\frac{dy}{dj} = -\frac{1}{6} \frac{1}{\sqrt{2\pi k}} e^{\frac{-x^2}{2k}} \left( \frac{3x}{k^2} - \frac{x^3}{k^3} \right).$$

Integrating, we have

$$y = -j \frac{1}{\sqrt{2\pi k}} e^{\frac{-x^2}{2k}} \left( \frac{x}{2k^2} - \frac{x^3}{6k^3} \right) + \chi,$$

$\chi$  being a "constant" with regard to  $j$ ; which by equation (1) must be of the form

$$\frac{1}{\sqrt{k}} \phi \left( \frac{x}{\sqrt{k}} \right).$$

Put  $j=0$ ; then the first term of the value for  $y$  vanishing while the curve becomes symmetrical, the second term, the "constant"  $\chi$ , must be the expression for the ordinary probability-curve, viz.

$$\frac{1}{\sqrt{\pi} \sqrt{2k}} e^{\frac{-x^2}{2k}}.$$

Thus the required expression for  $y$  is

$$\frac{1}{\sqrt{2\pi k}} e^{-\frac{x^2}{2k}} \left( 1 - j \left( \frac{x}{2k^2} - \frac{x^3}{6k^3} \right) \right);$$

which, when  $c^2$  is substituted for  $2k$ , coincides with the expression given above.

This solution may be completed by observing that it satisfies the fundamental equations (1) and (2) unconditionally, as well as (3) when account is taken of condition (a).

A further verification of the theory is afforded by showing that if the sum of  $m$  independent elements obeys the law of frequency  $y = F(x, k_1, j_1)$ ,  $F$  having the form which has been found, and  $k_1$  and  $j_1$  being the sum of the respective mean squares of error and mean cubes of error for the  $m$  elements; and likewise the sum of another set of  $m$  independent elements,  $n$  in number, obeys the similarly defined law of frequency,  $y = F(x, k_2, j_2)$ ; then the sum of  $(m+n)$  elements of which  $m$  are of the first class and  $n$  of the second obeys, as it should, the law

$$y = F(x, k_1 + k_2, j_1 + j_2)^*.$$

A particularly interesting case of the asymmetrical probability-curve is that in which an element has only two possible values, say zero and unity, occurring with the respective probabilities  $p$  and  $q$ —the case considered in a former number of the Philosophical Magazine (vol. xxi. p. 318, 1886). Observing that the mean square of error for this elementary locus is  $pq^2 + qp^2 = pq(p+q) = pq$ , and the mean cube of error  $= pq(p-q)$ , we have by the general formula for the curve representing the law of frequency for the sum of  $n$  such elements, an expression in terms of those constants which, *mutatis mutandis*, proves to be identical with the expression which Todhunter, after Laplace, has obtained by a method peculiar to the Binomial †.

The general or multinomial probability-curve, involving (in addition to the centre of gravity) only two constants  $k$  and  $j$ , may always be replaced by a binomial; through the equations

$$npqi^2 = k, \quad npq(p-q)i^3 = j,$$

where  $i$  is the length of each element‡. There are thus only two equations for three quantities,  $n$ ,  $i$ , and  $p+q$  ( $p+q=1$ ).

When it is proposed to construct a binomial from a given

\* The work is given in the original paper.

† History of Probabilities, Art. 993.

‡ Cf. above, note to p. 92.

set of observations, there is given a third condition, namely, that  $ni$  must be greater than the distance between the greatest and least observations. But this inequation (coupled with the other equations) is not sufficient to determine  $n$ ,  $i$ , and  $p \div q$  with any precision.

Of course, whether a binomial or multinomial probability-curve is to be adapted to a given set of observations, the set must fulfil the condition that  $j \div k^{\frac{1}{2}}$  should be a small fraction\*. In fact the condition is frequently unfulfilled: for instance, in the statistics of the duration of American marriages†, where the observed  $j \div k^{\frac{1}{2}}$  forms a large integer‡. In such cases it may be inferred that the number of independent elements is too small (or their asymmetry too great) to generate a probability-curve.

XII. *On the Existence of Vertical Earth-Air Electric Currents in the United Kingdom.* By A. W. RÜCKER, M.A., F.R.S.§

IN a paper by Dr. Adolph Schmidt, read before Section A of the British Association at Oxford (Report Brit. Assoc. 1894, p. 570), the author stated that he had expanded the components of the earth's magnetic force in series, and had deduced expressions, two of which give the magnetic potential on the surface of the earth in so far as it depends on (1) internal, and (2) external forces. "The third series represents that part of the magnetic forces which cannot be expressed in terms of a potential, but must be due to electric currents traversing the earth's surface." The author concludes that such currents amount on the average to about 0.1 ampere per square kilometre.

It appeared therefore desirable that this conclusion, drawn from the magnetic state of the earth as a whole, should be tested by means of those portions which have been most fully studied.

\* Above, p. 92.

† Given by Dr. W. F. Wilcox in "The Divorce Problem" (Studies in History &c., Columbia College, vol. i.).

‡ Many other instances in which the condition fails are given by Prof. Karl Pearson in his masterly "Contributions to the Mathematical Theory of Evolution," No. II. (Philosophical Transactions, 1895). For some criticism of Prof. Pearson's theory of asymmetric frequency-curves see the present writer's paper on "Recent Contributions to the Theory of Statistics," in the Journal of the Royal Statistical Society, Sept. 1895.

§ Communicated by the Physical Society: read December 13, 1895.