THE SINGULAR POINTS OF CERTAIN CLASSES OF FUNCTIONS OF SEVERAL VARIABLES

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1. A great deal has been written about the singular points of particular classes of functions defined by Taylor's series in a single variable. Of the very numerous interesting results which have been obtained a large proportion have come from a development of an idea originally suggested by Hadamard. Hadamard first pointed out how, if we know the singular points of the function

$$V(x) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

we can discuss those of the function

$$f(x) = \int_0^1 \phi(t)\, V(tx)\, dt.$$

His original enunciation of the results thus obtained perhaps lacked something in precision, but all that was lacking has been amply supplied by later writers, and notably by Le Roy.* The result of the work in this and other directions of Hadamard and other writers has been that the theory of the singular points of Taylor's series in one variable may be said to be tolerably complete. Such general results as are likely to be proved by the methods at present at our disposal have been proved: and (what is almost equally important) large classes of particular functions have been discussed in detail; so that we are well supplied with interesting examples of all the theoretical possibilities, and can hope actually to determine, without serious difficulty, the nature of any special function which presents special points of interest.

The corresponding theory for functions of several variables is in a very different state. There are, of course, a number of general results which have been proved for functions of one variable and which can obviously be extended to those of several variables, and writers on the simpler theory have generally been content to point these out. Of tangible results which help us in the actual discussion of particular functions there are practically none.

* Some discussion of these results and some indications of further extensions will be found in two previous papers in these *Proceedings* (Ser. 2, Vol. 3, p. 381, and Vol. 5, p. 197).

Nothing is easier, on the other hand, than to write down any number of interesting special series which seem to invite discussion. Such series as

$$\Sigma \frac{x^\mu y^\nu}{(a+\mu\omega+\nu\omega')^\alpha}, \qquad \Sigma \frac{\Gamma(a+\mu\omega+\nu\omega')}{\Gamma(a+b+\mu\omega+\nu\omega')} x^\mu y^\nu,$$

$$\Sigma \frac{\mu+\nu!}{\mu!\,\nu!} \frac{x^\mu y^\nu}{(a+\mu\omega+\nu\omega')^\alpha}, \qquad \Sigma \frac{\mu+\nu!}{\mu!\,\nu!} \frac{\Gamma(a+\mu\omega+\nu\omega')}{\Gamma(a+b+\mu\omega+\nu\omega')} x^\mu y^\nu,$$

$$\Sigma \frac{\mu!\,\nu!}{\mu+\nu!} \frac{x^\mu y^\nu}{(a+\mu\omega+\nu\omega')^\alpha}, \qquad \Sigma \frac{\mu!\,\nu!}{\mu+\nu!} \frac{\Gamma(a+\mu\omega+\nu\omega')}{\Gamma(a+b+\mu\omega+\nu\omega')} x^\mu y^\nu$$

at once suggest themselves, with more general classes of functions of similar types.

2. The methods which I have employed in my former papers lend themselves naturally enough to the discussion of a large variety of classes of series in any number of variables of which those written above are, as particular examples, fairly typical. This I have indicated briefly in one of the papers referred to.*

I shall now consider the question in greater detail, beginning with some generalities.

3. Let us suppose that $V(x_1, x_2, \ldots, x_n)$

is any function of $x_1, \ldots, x_n$ that can be expanded in a Taylor's series

$$\Sigma c_{\mu_1, \ldots, \mu_n} x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$$

convergent for $|x_1| < r_1, \ldots, |x_n| < r_n$.

The *associated radii* $r_1, r_2, \ldots, r_n$ are in general connected by a single functional relation, which may be obtained (theoretically, at any rate) by a method devised by Lemaire.†

Now suppose that $\omega_1, \omega_2, \ldots, \omega_n$ are quantities whose real parts are positive, and that

$$a_{\mu_1, \mu_2, \ldots, \mu_n}$$

is an analytic function of certain parameters α, β, γ, capable, when the real parts of γ and $\alpha+\beta-1$ are positive, of being expressed in the form

$$a_{\mu_1, \ldots, \mu_n} = \int_0^1 \left(\log \frac{1}{u}\right)^{\alpha-1} (1-u)^{\beta-1} u^{\gamma-1+\Omega} \phi(u)\, du$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 3, pp. 387–9. I shall refer to this paper as I, and to the second paper as II.

† For a detailed discussion of the nature of the region of convergence of the power series, I may refer to two memoirs by F. Hartogs (*Inaugural Dissertation*, published by Teubner, 1904, and *Math. Annalen*, Bd. LXII.).

where
$$\Omega = \mu_1 \omega_1 + \ldots + \mu_n \omega_n,$$
$$u^{\gamma-1} = e^{(\gamma-1)\log u}, \qquad (1-u)^{\beta-1} = e^{(\beta-1)\log(1-u)}$$
$$\left(\log \frac{1}{u}\right)^{\alpha-1} = e^{(\alpha-1)\log\log 1/u},$$
and
$$u^{\mu_\nu \omega_\nu} = e^{\mu_\nu \omega_\nu \log u},$$
the logarithms being real. Let us, as in I (§ 1), consider the integral
$$\int_C (\log u)^{\alpha-1}(u-1)^{\beta-1} u^{\gamma-1+\Omega}\, \phi(u)\, du$$
where now $(u-1)^{\beta-1} = e^{(\beta-1)\log(u-1)}$, $(\log u)^{\alpha-1} = e^{(\alpha-1)\log\log u}$,

the logarithms being real when u is real and greater than 1 and rendered uniform by a cut from 1 to $-\infty$ along the real axis, and where C is a loop from 0 enclosing the line $(0, 1)$. Then, for sufficiently small values of $x_1, x_2, \ldots, x_\nu$,

(1) $F(x_1, x_2, \ldots, x_n)$
$$= \Sigma a_{\mu_1, \ldots, \mu_n} c_{\mu_1, \ldots, \mu_n} x_1^{\mu_1} \ldots x_n^{\mu_n}$$
$$= \frac{1}{2i \sin(\alpha+\beta)\pi} \int_C (\log u)^{\alpha-1}(u-1)^{\beta-1} u^{\gamma-1}\, \phi(u)\, V(x_1 u^{\omega_1}, x_2 u^{\omega_2}, \ldots, x_n u^{\omega_n})\, du,$$

provided $R(\gamma) > 0$, and $\alpha+\beta$ is not integral,* and C does not include any singular points of $\phi(u)$ or of
$$V(x_1 u^{\omega_1}, x_2 u^{\omega_2}, \ldots, x_n u^{\omega_n}).$$

The condition concerning $\phi(u)$ can certainly be satisfied if, as we shall assume, $\phi(u)$ is regular in a domain which includes the line $(0, 1)$ in its interior. The last condition is certainly satisfied for sufficiently small values of $x_1, \ldots, x_n$. In these circumstances the equation (1) provides a representation of $F(x, \ldots, x_n)$ certainly valid for sufficiently small values of the variables.

4. This representation of $F(x_1, \ldots, x_n)$ is, of course, often valid for a range of values of the variables far wider than that for which the power series converges. In order to discuss for what range of values the equation (1) holds, I shall begin by considering the simplest case, that in which
$$c_{\mu_1, \mu_2, \ldots, \mu_n} = 1,$$
$$V(x_1, x_2, \ldots, x_n) = 1/(1-x_1)(1-x_2)\ldots(1-x_n),$$

(2) $$F(x_1, x_2, \ldots, x_n) = \frac{1}{2i \sin(\alpha+\beta)\pi} \int_C \frac{(\log u)^{\alpha-1}(u-1)^{\beta-1} u^{\gamma-1}\, \phi(u)\, du}{(1-x_1 u^{\omega_1})(1-x_2 u^{\omega_2})\ldots(1-x_n u^{\omega_n})},$$

* If $\alpha+\beta$ is an integer k, the formula may be replaced by
$$F(x_1, \ldots, x_n) = \frac{(-1)^k}{2\pi i} \int (\log u)^{\alpha-1}(u-1)^{\beta-1} u^{\gamma-1} \log(u-1)\, \phi(u)\, V(x_1 u^{\omega_1}, \ldots, x_n u^{\omega_n})\, du.$$

The loop C may be taken so as to enclose the line (0, 1) as closely as we please. We must therefore consider first for what values of the variables (x) the part of the subject of integration which depends on (x) will have a singular point on the line (0, 1).

Let us suppose therefore

$$x_1 u_1^{\omega_1} = 1 \qquad (0 < u_1 \leqslant 1)$$

or

$$1/x_1 = e^{\omega_1 \log u_1}.$$

As u_1 varies from 0 to 1 the value of x_1 given by this equation varies from ∞ to 1. Its path is a certain equiangular spiral S_1 which, in the particular case in which ω_1 is real, reduces to the straight line $(1, +\infty)$. Similarly, we obtain spirals $S_2, \ldots, S_n$ in the planes of $x_2, \ldots, x_n$. By drawing barriers along these spirals we define a certain domain T for the variables (x).

Let T_1' be any finite domain in the plane of x_1, all of whose points lie at a distance from S_1 greater than some arbitrarily small fixed positive quantity δ_1. Similarly we define $T_2', \ldots, T_n'$. Let T' be the domain formed by the composition of $T_1', T_2', \ldots, T_n'$.

Then $F(x_1, x_2, \ldots, x_n)$ *is regular in* T'. For let $x_1^0, x_2^0, \ldots, x_n^0$ be a system of values inside the domain of convergence of the original power series; and $x_1', x_2', \ldots, x_n'$ any other system inside T'. We can suppose that (x) vary from (x^0) to (x') along a "path" which lies inside T'. If we suppose the loop C taken initially so closely surrounding (0, 1) that no root of

$$x_\nu u^{\omega_\nu} = 1 \quad (\nu = 1, 2, \ldots, n)$$

falls inside or on C for any set of values of (x) in T', it is plain that the integral (2) gives the analytical continuation of $F(x_1, x_2, \ldots, x_n)$ over a region which includes the path from (x^0) to (x'), and so over the whole region T'.

We may say shortly that $F(x_1, \ldots, x_n)$ is regular *within* T'.

It is equally easy to define different, though similar, domains within which $F(x_1, \ldots, x_n)$ is regular. We have only to take as fundamental a form of C enclosing, not the line (0, 1), but some other path (such as an arc of a circle) from 0 to 1 in the u-plane. We then obtain a modified domain T' within which $F(x_1, \ldots, x_n)$ is regular, bounded by a cut from 1 to ∞ in the plane of every x along a certain curve. To give a simple example, let us suppose every $\omega = 1$. Then it is easy to see that, if we take C to always include a fixed arc of a circle from 0 to 1, the cut in every x-plane must be along a straight line from 1 to ∞, *these lines making in each plane the same angle with the real axis.*

For example, the function

$$(x-y)^{-1}\log\{(1-x)/(1-y)\},$$

which, as we shall see, falls under the class of functions here considered, is regular in any domain T formed by cutting the planes of x and y along straight lines from 1 to ∞ *making the same angle with the real axis.*

The last condition is essential; otherwise we could find a pair of values x, y lying inside T, and for which

$$x = y, \qquad \log\{(1-x)/(1-y)\} = \pm 2\pi i,$$

and the function would no longer be regular inside T.

Let us return to the function $F(x_1, \ldots, x_n)$. It will in general be many-valued, and the system of branches of F will be far more complex than that of the branches of a many-valued function of a single variable. The branch which is regular in T, *i.e.*, the branch represented near $(0, 0, \ldots, 0)$ by the original power series, we call the *principal branch* and denote by $\bar{F}$.

We can now form some general conclusions with regard to the singularities of $\bar{F}$, or rather the *possible* singularities, since our method does not at present enable us actually to assert that any system of values does, in point of fact, correspond to a singularity.

(1) We can in no case form a domain within which $\bar{F}$ is regular and which includes any system of values for which any $x_\nu = 1$ or ∞. Hence

$$x_1,\ x_2,\ \ldots,\ x_\nu = 1,\ \ldots,\ x_n,$$

$$x_1,\ x_2,\ \ldots,\ x_\nu = \infty,\ \ldots,\ x_n$$

correspond to possible singularities of the principal branch of F. In other words, *possible singularities are given by* $x_\nu = 1, \infty$ *for any values of the other variables.*

(2) No value of x_ν other than $1, \infty$ can give a singularity of $\bar{F}$ for all possible values of the other variables. For, if we take any set of values of (x) of which none is 1 or ∞, we can define a domain which includes the domain of convergence of the original power series, and this point, and throughout which our contour integral gives the analytical continuation of $\bar{F}$.

(3) Let us consider the domain T bounded by a definite set of barriers S_ν. It is possible that when x_ν tends to a certain value ξ_ν on the barrier S_ν (other than $1, \infty$) $\bar{F}$ may tend to a singularity when the other variables have or tend to some particular values or system of values in some particular way. This is the case, for example, with the function

$$(x-y)^{-1}\log\{(1-x)/(1-y)\}$$

when x and y tend simultaneously and *from opposite sides* to equal values on the barriers in their respective planes.

Let us suppose, *e.g.*, that x_1 tends to a value ξ_1 upon S_1. There is one, and only one, value u_1 $(0 < u_1 < 1)$ such that

$$1-\xi_1 u_1^{\omega_1} = 0,$$

and it is obvious that as x_1 approaches ξ_1 our representation of $\bar{F}$ ceases to be valid.

When x_1 is nearly, but not quite, equal to ξ_1 the subject of integration has a singularity $\bar{u}_1$ (see the figure, Fig. *a*) lying a little off (0, 1). We can deform C slightly (Fig. *bc*) so as to leave $\bar{u}_1$, u_1 on one side of it. And the modified integral gives us the continuation of $\bar{F}$ over a region slightly passing the limits of T, in that its constituent part T_1 contains a small region *which includes* ξ_1.

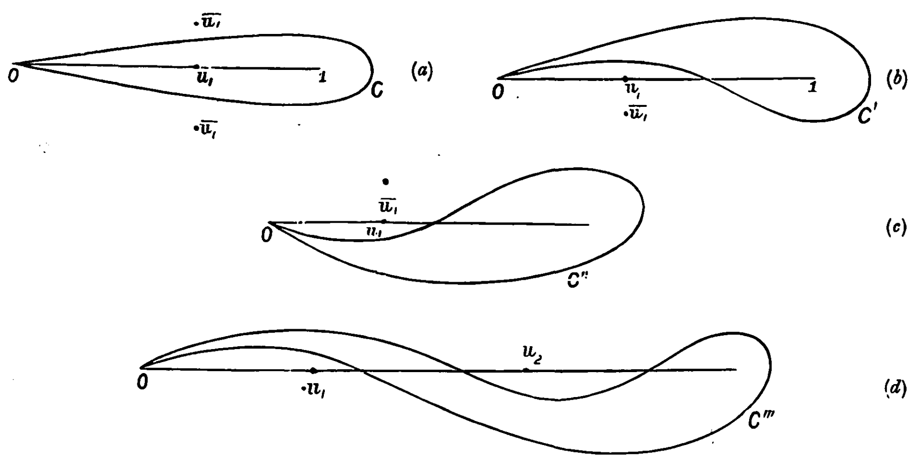

FIG. 1.

It follows that, if x_1 tends to ξ_1, the other variables remaining off the barriers in their planes, $\bar{F}$ does not tend to a singularity. The same conclusion *generally* holds even if some or all of the other variables tend at the same time to points on their barriers, as appears from a reference to Fig. 1, (*d*), where is shown the modification necessary in C to meet the case in which two variables x_1, x_2 tend to values ξ_1, ξ_2 such that

$$1-\xi_1 u_1^{\omega_1} = 0, \qquad 1-\xi_2 u_2^{\omega_2} = 0.$$

But there is an exception : *If* $u_1 = u_2$, *the suggested modification is impossible*; in other words, if x_1, x_2 vary in such a way that the contour C is *nipped* between two singularities of the subject of integration. In fact, the existence of singularities of $\bar{F}$ is indicated either (1) by the approach of a singularity of the subject of integration to an end of the contour or (2) by the nipping of the contour as described above.*

5. We are thus led to discriminate two kinds of singularities of F from among those which appear as singularities of $\bar{F}$, *viz.*, the singularities

$$x_\nu = 1, \ \infty$$

which we shall describe as *primary*, and the singularities given by

$$1-\xi_\mu u_\mu^{\omega_\mu} = 0, \qquad 1-\xi_\nu u_\nu^{\omega_\nu} = 0$$

where μ, ν are any two of $1, 2, \ldots, n$. These we shall describe as *secondary*. The distinction between these two classes is not precisely the same as that which I made in my earlier paper (I) between *principal* and *subsidiary* singularities in the case of functions of a single variable, since the secondary singularities here considered do, in fact, appear as singular when the variables (x) move in a prescribed manner towards special places on the boundary of T. Still, *qua* singularities of $\bar{F}$, they depend on the region adopted as fundamental in the definition of $\bar{F}$, and, from the point of view of the Taylor's series, there is a genuine distinction to be drawn between them and the primary singularities.

When we come to consider the function F as a whole these secondary singularities (as might be expected) also appear as a connected whole, viz., as the systems of values given by

$$x_\mu^{\omega_\nu} = x_\nu^{\omega_\mu} \quad (\mu, \nu = 1, 2, \ldots, n).$$

Besides these two classes of singularities there remains the possibility of a third, viz., a class of singularities of other branches of F which do not appear at all as singularities of $\bar{F}$. Thus we shall find that often $x_\nu = 0$ defines a singularity of all branches of F other than the principal branch, just as $x = 0$ is a singularity of all branches of

$$\frac{x}{1^a}+\frac{x^2}{2^a}+\frac{x^3}{3^a}+\ldots$$

other than the principal branch. Such singularities may more appropriately be called *subsidiary*.

6. The preceding arguments are merely an adaptation of those used by Hadamard and Le Roy for functions of a single variable. They use *line* instead of *loop* integrals; but, so far as these generalities are

* For the analogue of (1) for functions of one variable, see I and II′, *passim*. For that of (2), see II, § 6. The phenomenon of nipping occurs also in Hadamard's proof of the "multiplication theorem" for singularities or ordinary Taylor's series.

concerned, one form of integral has little advantage over the other.* It is in the further development of the theory, especially in the obtaining of more precise information as to the nature of the singularities, that the great advantage of the loop integral declares itself.

7. The arguments which we have used for a special form of $V(x_1, \ldots, x_n)$ may obviously be extended, *mutatis mutandis*, to the general case. But, in general, the statement is somewhat less simple. Let us therefore consider the case perhaps next in simplicity.

Let
$$c_{\mu_1, \mu_2, \ldots, \mu_n} = \frac{\mu_1+\mu_2+\ldots+\mu_n!}{\mu_1!\,\mu_2!\,\ldots\,\mu_n!};\dagger$$

so that
$$V(x_1, x_2, \ldots, x_n) = 1/(1-x_1-x_2-\ldots-x_n)$$

and
$$F(x_1, x_2, \ldots, x_n) = \frac{1}{2i\sin(\alpha+\beta)\pi}\int_C \frac{(\log u)^{\alpha-1}(u-1)^{\beta-1}u^{\gamma-1}\phi(u)\,du}{1-x_1u^{\omega_1}-\ldots-x_nu^{\omega_n}}.$$

Then the argument used above shows how we may define a branch of $F(x_1, \ldots, x_n)$ regular within any domain which excludes all sets of values of (x) for which
$$1-x_1u^{\omega_1}-\ldots-x_nu^{\omega_n} = 0$$

for a real $u<1$. *We cannot, however, draw fixed cuts in the planes of (x) in such a way as to secure this.* Hence there is a difficulty in giving a precise definition of a branch of F analogous to $\bar{F}$ above.

Let $x_1^0, x_2^0, \ldots, x_n^0$ be any system of values of the variables such that

$$|1-x_1^0u^{\omega_1}-\ldots-x_n^0u^{\omega_n}| \geqslant \delta > 0 \tag{1}$$

for $0 \leqslant u \leqslant 1$. And let
$$x_\nu = x_\nu^0\lambda^{\omega_\nu}$$

where $0\leqslant\lambda\leqslant 1$ and λ^{ω_ν}, like u^{ω_ν}, has its principal value. As λ varies from 0 to 1 each x_ν varies from 0 to x_ν^0 along a certain path C_ν. The aggregate of corresponding points of every C_ν we shall call the path C of the variables (x) from (0) to (x^0).

For all points of the path C
$$|1-x_1u^{\omega_1}-\ldots-x_nu^{\omega_n}| = |1-x_1^0(\lambda u)^{\omega_1}-\ldots-x_n^0(\lambda u)^{\omega_n}| \geqslant \delta.$$

Let $P_1, \ldots, P_n$ be a set of corresponding points on the paths C_ν. Round each P_ν describe a small circle of radius ϵ. The domain of the x's formed by all systems of values of which each corresponds to a point

* Using the loop enables us to avoid the annoying and irrelevant condition $R(\alpha+\beta)>1$. When $\alpha = 1$ the restriction on γ could be removed by the use of a double-circuit integral.

† The region of convergence of the original power series is in this case defined by
$$|x_1|+|x_2|+\ldots+|x_n|<1.$$

within a circle C_ν we denote by $C_\epsilon(\lambda)$. As λ varies from 0 to 1, $C_\epsilon(\lambda)$ varies from a small region enclosing the origin to a small region enclosing $(x_1^0, \ldots, x_n^0)$. At any point in $C_\epsilon(\lambda)$

$$x_\nu = x_\nu^0 \lambda^{\omega_\nu} + \xi_\nu$$

where $|\xi_\nu| < \epsilon$, and so

$$|\xi_1 u^{\omega_1} + \ldots + \xi_n u^{\omega_n}| < |\xi_1| + \ldots + |\xi_n| < n\epsilon.$$

Choose ϵ so that $\qquad n\epsilon < \tfrac{1}{2}\delta.$

Then, for every point in $C_\epsilon(\lambda)$,

$$|1 - \Sigma x_\nu u^{\omega_\nu}| = |1 - \Sigma x_\nu^0 (u\lambda)^{\omega_\nu} - \Sigma \xi_\nu u^{\omega_\nu}| > \delta - \tfrac{1}{2}\delta = \tfrac{1}{2}\delta,$$

for $0 \leqslant u \leqslant 1$.

It follows that, as λ varies from 0 to 1, $C_\epsilon(\lambda)$ defines a series of domains through each of which in succession we may analytically continue F until we have passed from the neighbourhood of the origin to the neighbourhood of $x_1^0, x_2^0, \ldots, x_n^0$. Hence *this system of values does not correspond to a primary singularity of F.*

We have thus excluded from the possible primary singularities of F all systems of values which satisfy the condition (1).

8. Now let us give $x_1, x_2, \ldots, x_{n-1}$ fixed values $x_1^0, x_2^0, \ldots, x_{n-1}^0$. The values of x_n for which

$$1 - x_1^0 u^{\omega_1} - \ldots - x_{n-1}^0 u^{\omega_{n-1}} - x_n u^{\omega_n} = 0$$

for some value of u between 0 and 1 form a continuous curve stretching from $1 - x_1^0 - \ldots - x_{n-1}^0$ to infinity. The values of x_n for which

$$|1 - x_1^0 u^{\omega_1} - \ldots - x_{n-1}^0 u^{\omega_{n-1}} - x_n u^{\omega_n}| \leqslant \delta$$

form a continuous domain, which includes this curve. By choosing δ sufficiently small, we can exclude from this domain any point in the finite part of the x_n-plane which does not actually lie upon the curve.

Hence the possible primary singularities of F can only be sought among the systems given by

$$x_1^0, x_2^0, \ldots, x_{n-1}^0, x_n = (1 - x_1^0 u^{\omega_1} - \ldots - x_{n-1}^0 u^{\omega_{n-1}})/u^{\omega_n},$$

for $0 \leqslant u \leqslant 1$.

But, as before, we are at liberty to vary our fundamental u-path from 0 to 1. And the only systems which arise, *however* we choose this path, are evidently

$$x_1^0, x_2^0, \ldots, x_{n-1}^0, 1 - x_1^0 - x_2^0 - \ldots - x_{n-1}^0; \quad x_1^0, x_2^0, \ldots, x_{n-1}^0, \infty.$$

Hence *the primary singularities of* $F(x_1, x_2, \ldots, x_n)$ *are given by*

$$x_1+x_2+\ldots+x_n = 1$$

or

$$x_\nu = \infty.$$

As in the previous case, these are the values for which a pole of

$$1/1-x_1u^{\omega_1}-\ldots-x_nu^{\omega_n}$$

approaches an *end* of C.

Secondary or subsidiary singularities will clearly be given by those values of (x) for which two roots of $1-x_1u^{\omega_1}-\ldots-x_nu^{\omega_n}=0$ become equal. In particular, when ω_1, ω_2, ... are real and rational (in which case we may, without loss of generality, suppose them integral), they will be given by the discriminant of this equation.

9. The arguments of the preceding sections may evidently be applied whatever be the function $V(x_1, x_2, \ldots, x_n)$. The cases of most interest to us at present are those in which V is a rational function P/Q regular at the origin. The primary singularities of the associated functions F are in this case given by

$$Q(x_1, x_2, \ldots, x_n) = 0.$$

If V is a many-valued function such as

$$1/(1-x_1)^{\delta_1}(1-x_2)^{\delta_2}\ldots(1-x_n)^{\delta_n}, \qquad 1/(1-x_1-x_2-\ldots-x_n)^{\delta},$$

some additional complications are introduced, but the general principle of the method is none the less applicable.

10. We are thus able to define, by means of a contour integral, a family of functions associated with each given function V. Each of these functions is generally many-valued, but possesses a branch whose properties bear a considerable resemblance to those of V.

The other branches of F will, however (as can be seen clearly enough from the case of one variable), have properties differing widely from those of V. In order to obtain a more accurate idea of the nature of these branches, as well as to obtain more precise information as to the nature of the primary singularities of F, further analysis is necessary. It is natural to attempt to use the method which I adopted in my first paper. That method (in its simplest form) was based on a change from the contour C to another contour C', between which and C lay a pole of the subject of integration. It is therefore best adapted to the case in which V is rational, and I shall, for the present, confine myself to that case.

10. Let, then, $V = P/Q$ be rational. And let

$$x_1^0,\ x_2^0,\ \ldots,\ x_n^0$$

be a set of values of the variables for which $Q(x_1, x_2, \ldots, x_n) = 0$, and let

$$x_\nu = x_\nu^0 + \xi_\nu$$

where ξ_ν is small.

By carrying out the process of continuation described in § 7, we arrive at an equation

$$F(x_1, x_2, \ldots, x_n)$$
$$= \frac{1}{2i \sin(\alpha+\beta)\pi} \int_C (\log u)^{\alpha-1}(u-1)^{\beta-1} u^{\gamma-1} \phi(u) \frac{P(x_1 u^{\omega_1}, \ldots)}{Q(x_1 u^{\omega_1}, \ldots)}\, du.$$

In general, there will be just one pole of P/Q (considered as a *function of* u) *which tends to* $u = 1$ *as* $\xi_1, \ldots, \xi_\nu$ *tend to zero.*

For, let $u = 1+t$, and consider the equation

$$Q\left\{(x_\nu^0+\xi_\nu)(1+t)^{\omega_\nu}\right\} = 0.$$

In general, this can be expanded in the form

$$0 = a_1\xi_1 + a_2\xi_2 + \ldots + a_n\xi_n + \lambda t + \text{higher powers of } \xi_\nu,\ t \quad (\lambda \neq 0);$$

and it follows from the general implicit function theorem that this equation has just one solution of the type

$$t = b_1\xi_1 + \ldots + b_n\xi_n + \text{higher powers of } \xi_\nu.$$

But, of course, in special cases there may be several solutions of different forms. If, *e.g.*,

$$Q = (1-x_1)(1-x_2)\ldots(1-x_n),$$

each of the equations

$$1 - x_n u^{\omega_\nu} = 0$$

or

$$1-(1+\xi_\nu)(1+\omega_\nu t+\ldots) = 0$$

has one root which vanishes with the ξ's.

Considering the general case, let

$$u = u^0(x_1, \ldots, x_n)$$

be the pole in question. Then

$$\int_C = \int_{C'} + 2\pi i R$$

where R is the residue for $u = u^0$. The second integral (cf. I, § 1) is regular near $x_1^0, \ldots, x_n^0$. We can thus isolate the irregular part of F. In order to throw light on the results thus obtained (as well as for the sake of the interest of the actual results), I shall consider some particular cases in detail.

11. The simplest case is that in which

$$V(x_1, \ldots, x_n) = 1/(1-x_1)(1-x_2) \ldots (1-x_n).$$

In this case there are n poles between C and C', viz.,

$$u = x_\nu^{-1/\omega_\nu} \quad (\nu = 1, 2, \ldots, n).$$

The residue at this pole is

$$-(\omega_\nu)^{-a} \frac{\left\{\log\left(\frac{1}{x_\nu}\right)\right\}^{a-1} (1-x_\nu^{1/\omega_\nu})^{\beta-1} x_\nu^{(1-\beta-\gamma)/\omega_\nu} \phi(x^{-1/\omega_\nu})}{\prod\limits_{(\mu)}' (1-x_\mu x_\nu^{-\omega_\mu/\omega_\nu})}$$

where the dash denotes that the product applies to values of μ different from ν, and the values of the many-valued functions have been already specified. Let this quantity be denoted by Ω_ν. Then

$$F(x_1, x_2, \ldots, x_n) = \frac{\pi}{\sin(\alpha+\beta)\pi} \Sigma\Omega_\nu + \frac{1}{2i\sin(\alpha+\beta)\pi} \int_{C'},$$

We thus isolate what we may call the part of $F(x_1, \ldots, x_n)$ irregular near $(1, 1, \ldots, 1)$. The form of Ω_ν exhibits explicitly the singularities of F other than the primary singularities. It will be noticed that besides the *secondary* singularities already noted, which correspond to the factors

$$1-x_\mu x_\nu^{-\omega_\mu/\omega_\nu},$$

there are *fixed subsidiary* singularities given, *e.g.*, by $x_\nu = 0$.

12. Let us specialise further by supposing $n = 2$ and write x, y, ω, ω' for x_1, x_2, ω_1, ω_2 to avoid suffixes. There are two particularly interesting special cases:

(i.) Let $\phi(u) \equiv 1$, $a = 1$, so that

$$F(x, y) = \Gamma(\beta) \Sigma \frac{\Gamma(\gamma+\mu\omega+\nu\omega')}{\Gamma(\beta+\gamma+\mu\omega+\nu\omega')} x^\mu y^\nu.$$

The primary singularities are given by $x = 1$, $y = 1$; the function

$$F(x, y) - \frac{\pi}{\sin\beta\pi} \left\{ \frac{(1-x^{1/\omega})^{\beta-1} x^{(1-\beta-\gamma)/\omega}}{\omega(1-yx^{-\omega'/\omega})} + \frac{(1-y^{1/\omega'})^{\beta-1} y^{(1-\beta-\gamma)/\omega'}}{\omega'(1-xy^{-\omega/\omega'})} \right\}$$

is regular near $(1, 1)$; secondary or subsidiary singularities are given by

$$x = 0, \quad y = 0, \quad 1-x^{1/\omega} = 0, \quad 1-y^{1/\omega'} = 0, \quad 1-yx^{-\omega'/\omega} = 0, \quad 1-xy^{-\omega/\omega'} = 0.$$

Thus, *e.g.*, $x = e^{2\pi i\lambda}$, where $\lambda = k+k'\omega$, and k and k' are any integers, gives a subsidiary singularity.

These results are easily verified by supposing $\omega = \omega' = 1$, when the irregular part reduces to

$$\frac{\pi}{\sin\beta\pi} \frac{1}{x-y} \left\{(1-x)^{\beta-1} x^{2-\beta-\gamma} - (1-y)^{\gamma-1} y^{2-\beta-\gamma}\right\};$$

for a function of the form

$$F(x, y) = \Sigma \frac{x^\mu y_\nu}{\phi(\mu+\nu)} = \sum_{k=0}^{\infty} \frac{1}{\phi(k)} \sum_{(\mu+\nu=k)} x^\mu y^\nu$$

may be expressed in the form $\{xf(x)-yf(y)\}/(x-y)$,

where $f(x) = \Sigma x^k/\phi(k)$;

and so we can use the known results for functions of a single variable.

In calculating the form of Ω_ν, it should be observed, we have assumed that the poles between C and C' are all distinct. If some of them coincide, the form of the residues corresponding to them becomes illusory, and must be calculated afresh, *ab initio* or by a passage to the limit.

A particularly interesting case is that in which $\beta = 1$. In this case the preceding formulæ fail. The proper formulæ can be easily constructed by (a) a passage to the limit, or (b) by an independent investigation, starting from the integral

$$\frac{1}{2\pi i}\int_C \frac{\log(u-1)\, u^{\gamma-1}du}{(1-xu^{\omega})(1-yu^{\omega'})}.$$

The result is that the irregular part of $\Sigma\Sigma \dfrac{x^\mu y^\nu}{\gamma+\mu\omega+\nu\omega'}$

is

$$-\frac{1}{\omega}\,\frac{x^{-\gamma/\omega}\log(x^{-1/\omega}-1)}{1-yx^{-\omega'/\omega}}-\frac{1}{\omega'}\,\frac{y^{-\gamma/\omega'}\log(y^{-1/\omega'}-1)}{1-xy^{-\omega/\omega'}}.$$

It is instructive to consider the special case in which $\omega = \omega' = 1$, $\gamma = 1$. We then obtain the function

$$\Sigma\Sigma\frac{x^\mu y^\nu}{1+\mu+\nu} = \frac{1}{x-y}\log\left(\frac{1-y}{1-x}\right).$$

The *irregular part* turns out to be $\dfrac{1}{x-y}\log\left\{\dfrac{x(1-y)}{y(1-x)}\right\}$,

while the term due to the contour integral round C', which is regular near 1, 1, must therefore be

$$-\frac{\log x-\log y}{x-y}.$$

We thus obtain the formula

$$\frac{i}{2\pi}\int_C\frac{\log(u-1)\,du}{(1-xu)(1-yu)} = \frac{\log x-\log y}{x-y},$$

the path of integration being a loop from 0 round 1, $1/x$, and $1/y$. This may be verified independently. The decomposition

$$\frac{1}{x-y}\log\left(\frac{1-y}{1-x}\right) = \frac{1}{x-y}\log\left\{\frac{x(1-y)}{y(1-x)}\right\}-\frac{1}{x-y}\log\left(\frac{x}{y}\right)$$

gives a very simple and tangible illustration of the relations between the functions defined by the two contour integrals and the terms arising from the residues.

(ii.) An even more interesting special case is that in which $\alpha = 1$, $\phi(u) \equiv 1$. We find then the corresponding results for the function

$$\Sigma\frac{x^\mu y^\nu}{(\gamma+\mu\omega+\nu\omega')^\alpha}.$$

The irregular part of the principal branch is

$$\Gamma(1-\alpha)\left\{\omega^{-\alpha}\frac{(\log 1/x)^{\alpha-1}\,x^{(1-\gamma)/\omega}}{1-yx^{-\omega'/\omega}}+\omega'^{-\alpha}\frac{(\log 1/y)^{\alpha-1}\,y^{(1-\gamma)/\omega'}}{1-xy^{-\omega/\omega'}}\right\}.$$

13. The formulæ which we have obtained can of course be applied to find *asymptotic* formulæ of a simpler kind valid near $x = 1$, $y = 1$. Let us suppose (to take the simplest and most interesting case) that ω, ω', and α are real and positive and $\alpha < 1$, and that x and y tend simultaneously to 1 by real quantities in such a way that the ratios

$$1-x : 1-y : \omega(1-y)-\omega'(1-x)$$

remain between certain positive constants H, K. Let $1-x = \xi$ and $1-y = \eta$. Then

$$\left(\log\frac{1}{x}\right)^{\alpha-1} = \left(\log\frac{1}{1-\xi}\right)^{\alpha-1} \sim \xi^{\alpha-1}, \qquad \left(\log\frac{1}{y}\right)^{\alpha} \sim \eta^{\alpha-1},$$

$$1-yx^{-\omega'/\omega} \sim (\omega\eta-\omega'\xi)/\omega, \qquad 1-xy^{-\omega/\omega'} \sim (\omega'\xi-\omega\eta)/\omega',$$

and $$\Sigma \frac{x^\mu y^\nu}{(\gamma+\mu\omega+\nu\omega')^a} \sim \frac{\Gamma(1-a)}{\omega(1-y)-\omega'(1-x)} \left\{ \left(\frac{\omega}{1-x}\right)^{1-a} - \left(\frac{\omega'}{1-y}\right)^{1-a} \right\},$$

which is the analogue for two variables of Appell's well known formula

$$\Sigma \frac{x^\mu}{(\gamma+\mu)^a} \sim \frac{\Gamma(1-a)}{(1-x)^{1-a}},$$

and may be proved independently by methods which do not involve complex variables.

14. Let us pass now to the function

$$(1) \quad F(x_1, x_2, \ldots, x_n) = \frac{1}{2i\sin(a+\beta)\pi} \int_C \frac{(\log u)^{a-1}(u-1)^{\beta-1} u^{\gamma-1} \phi(u)\, du}{1-x_1u^{\omega_1}-x_2u^{\omega_2}-\ldots-x_nu^{\omega_n}}.$$

We have already seen that, if

$$x_1+x_2+\ldots+x_n \neq 1,$$

we can continue F from (0) to (x), and that in the neighbourhood of (x) the branch $\bar{F}$ which we are considering will be given by an equation of the above form.

If now we suppose that Σx_ν is nearly, though not quite, equal to 1, we can apply the argument of §§ 9, 10. This is a case in which there is only *one* pole between C and C'.

In order to obtain tangible results let us again take $n=2$ and use the notation of § 12, and consider some cases in which the part of $\bar{F}$ irregular near x_0, $y_0 = 1-x_0$ can be actually calculated in finite terms, which requires that we should be able to solve the equation

$$1-xu^\omega-yu^{\omega'}=0$$

explicitly. The simplest cases are given by (i.) $\omega=\omega'$, (ii.) ω (or ω') $=0$, (iii.) $\omega=1$, $\omega'=2$.

If (i.) $\omega=\omega'$ or (ii.) ω or $\omega'=0$, the results are of interest only for purposes of verification.

(i.) If $\omega=\omega'$, F is a function of $x+y$ only.

(ii.) If $\omega'=0$ (in which case the expression of F as a contour integral requires a little reconsideration, as in obtaining it we supposed the real parts of ω and ω' positive), F is a function of the form

$$\Sigma \frac{\mu+\nu!}{\mu!\,\nu!} \frac{x^\mu y^\nu}{\phi(\mu)} = \frac{1}{1-y} \sum_{(\mu)} \frac{1}{\phi(\mu)} \left(\frac{x}{1-y}\right)^\mu.$$

(iii.) Let $\omega=1$, $\omega'=2$. Then $1-xu-yu^2=0$ gives

$$u=u_0=-\frac{1}{2y}\{x+\sqrt{(x^2+4y)}\},$$

the sign of the radical being such that u reduces to 1 when $y=1-x$. And the irregular part is

$$-\frac{1}{\sqrt{(x^2+4y)}}(\log u_0)^{a-1}(u_0-1)^{\beta-1}u_0^{\gamma-1}\phi(u_0).$$

An example is given by $$\Sigma \frac{\mu+\nu!}{\mu!\,\nu!} \frac{x^\mu y^\nu}{(\gamma+\mu+2\nu)^a},$$

of which the simplest case is

$$\Sigma \frac{\mu+\nu!}{\mu!\,\nu!} \frac{x^\mu y^\nu}{1+\mu+2\nu} = -\frac{1}{\sqrt{(x^2+4y)}} \log \left\{ \frac{-\sqrt{(x^2+4y)}+x+2y}{-\sqrt{(x^2+4y)}-x-2y} \cdot \frac{-\sqrt{(x^2+4y)}+x}{-\sqrt{(x^2+4y)}-x} \right\},$$

where $\sqrt{(x^2+4y)}$ is negative for small real values of x and y, and tends to $x-2$ as y approaches $1-x$.

15. In the general case we cannot solve the equation

$$1-xu^{\omega}-yu^{\omega'}=0$$

explicitly, but we can investigate the character of the root

$$u_0 = 1+a(x-x_0)+b(y-y_0)+c(x-x_0)^2+d(x-x_0)(y-y_0)+\dots$$

(where $x_0+y_0=1$) as a function of x_0 and y_0, and so determine the general character of the irregular part of $\bar{F}$. Thus, if ω and ω' are *positive and rational*, u_0 is an *algebraic* function of x and y, and the research of the further singular points of $F(x, y)$, beyond its primary singularities, is reduced to that of finding those of a finite expression involving this algebraic function.

This last remark obviously applies to the general case in which $V(x, y)$ (here $1/1-x-y$) is any rational function $P(x, y)/Q(x, y)$: the root u_0 of

$$Q(xu^{\omega}, yu^{\omega'})=0$$

being an algebraical function of x and y.

To give a definite example, consider the function

$$\Sigma\frac{\mu+\nu!}{\mu!\,\nu!}\,\frac{x^{\mu}y^{\nu}}{(\gamma+m\mu+n\nu)^{a}}.$$

We are led to recognise as singular, besides the primary singularities $x+y=1$, $x=\infty$, $y=\infty$:

(*a*) Values of x and y which make a root of $1-xu^{m}-yu^{n}=0$ equal to 0, 1, or ∞; the only additional value thus given is $y=0$ (if $n>m$), $x=0$ (if $m>n$), or $x+y=0$ (if $m=n$).

(*b*) Values of x and y which make two roots equal; these are given by

$$D(1-xu^{m}-yu^{n})=0$$

where D denotes the discriminant.

16. (i.) Whatever values ω and ω' may have, we can obtain a simple *asymptotic* formula for $F(x, y)$ valid near x_0, $1-x_0$. For let

$$x=x_0+\xi, \qquad y=y_0+\eta,$$

$$\delta=1-x-y=-\xi-\eta,$$

and $u_0=1+t$. We find

$$t=\frac{\delta}{\omega x_0+\omega' y_0}+P_2(\xi, \eta),$$

from which we can at once deduce the formula

$$F(x, y)\sim-\frac{\pi}{\sin(\alpha+\beta)\pi}\,\frac{\phi(1)}{(\omega x_0+\omega' y_0)^{\alpha+\beta-1}}\,\frac{1}{(1-x-y)^{2-\alpha-\beta}}.$$

If, in particular, $\phi\equiv 1$, $\beta=1$, we find

$$\Sigma\frac{\mu+\nu!}{\mu!\,\nu!}\,\frac{x^{\mu}y^{\nu}}{(\gamma+\omega\mu+\omega'\nu)^{a}}\sim\frac{\Gamma(1-a)}{(\omega x_0+\omega' y_0)^{a}}\,\frac{1}{(1-x-y)^{1-a}}.$$

Similarly we have

$$\Sigma \frac{\mu_1+\mu_2+\ldots+\mu_n!}{\mu_1!\ldots\mu_n!} \frac{x_1^{\mu_1}\ldots x_n^{\mu_n}}{(\gamma+\omega_1\mu_1+\ldots+\omega_n\mu_n)^a} \sim \frac{\Gamma\ 1-a)}{(\omega_1 x_1^0+\ldots+\omega_n x_n^0)^a} \frac{1}{(1-x_1-\ldots-x_n)^{1-a}},$$

as $x_1, \ldots, x_n$ approach values $x_1^0, \ldots, x_n^0$ such that $\Sigma x_\nu^0 = 1$.

(ii.) It will be found instructive to verify all these conclusions on such functions as

$$\int_0^1 \frac{du}{1-xu^\alpha-yu^\beta-zu^\gamma}$$

where α, β, γ are any of 0, 1, 2.

17. The functions which we have considered in §§ 11–16 are all derived from the base series

$$1/(1-x)(1-y), \qquad 1/(1-x-y).$$

Any other base series will give rise to a corresponding set of functions. The following are some of the simplest and most interesting forms :—

(i.) $V(x, y) = \Sigma F(-\mu, -\nu, 1, \lambda)\, x^\mu y^\nu = 1/\{(1-x)(1-y)-\lambda xy\}$;

(ii.) $V(x, y) = \Sigma \dfrac{\mu!\ \nu!}{\mu+\nu+1!} x^\mu y^\nu = \dfrac{1}{x+y-xy} \log \left\{\dfrac{1}{(1-x)(1-y)}\right\}$;

(iii.) $$V(x, y) = \Sigma \frac{\mu!\ \nu!}{\mu+\nu!} x^\mu y^\nu$$
$$= \frac{1}{(1-x)(1-y)} - \frac{xy}{(x+y-xy)(1-x)(1-y)}$$
$$+ \frac{xy}{(x+y-xy)^2} \log \left\{\frac{1}{(1-x)(1-y)}\right\} ;$$

(iv.) $$V(x, y) = \Sigma \frac{\Gamma(a+\mu)\,\Gamma(b+\nu)}{\Gamma(a+b+\mu+\nu)} x^\mu y^\nu$$
$$= \frac{1}{x+y-xy} \{xF_{a,b}(x)+yF_{b,a}(y)\},$$

where $$F_{a,b}(x) = \Gamma(b)\, \Sigma \frac{\Gamma(a+\mu)}{\Gamma(a+b+\mu)} x^\mu.$$

In case (1) the finite primary singularities of all the derived functions are given by

$$(1-x)(1-y)-\lambda xy = 0 ;$$

in the other cases by $\quad x = 1, \quad y = 1,$

the factor $x+y-xy = 0$ not yielding primary singularities, as is independently obvious.

Any number of other base series may be constructed without difficulty. Of course, when the base series itself does not behave like a rational function in the neighbourhood of its primary singularities, we cannot investigate the precise nature of the corresponding singularities of the derived functions with the simplicity of §§ 10–16.

It is natural to attempt to construct series whose primary singularities are given by

$$1-xy=0.$$

If, however, we take $1/(1-xy)$ as our base series, our method fails, all the derived series turning out to be mere functions of $1-xy$.

It is, however, easy to prove that

$$\Sigma\,\theta(\mu-\nu)\,x^\mu y^\nu = \frac{1}{1-xy}\left\{\theta(0)+\sum_1^\infty \theta(k)\,x^k+\sum_1^\infty \theta(-k)\,y^k\right\},$$

and any series of this kind may be taken as a base series and will give results of the kind desired.

For example, let θ be an even function, such that

$$\Theta(x) = \Sigma\,\theta(k)\,x^k$$

is an integral function of x, and let

$$\Phi(x, y) = \theta(0)+\Theta(x)+\Theta(y);$$

also let $$F(x, y) = \frac{1}{2i\sin(\alpha+\beta)\pi}\int_C (\log u)^{\alpha-1}(u-1)^{\beta-1}u^{\gamma-1}\phi(u)\,\frac{\Phi(xu^\omega, yu^{\omega'})}{1-xyu^{\omega+\omega'}}\,du.$$

Then (i.) we can infer that the only primary singularities of F are given by

$$xy = 1,$$

and (ii.) we can write down the part of F irregular near (x_0, y_0), where $x_0 y_0 = 1$. Thus, *e.g.*, if

$$\omega = \omega' = 1, \quad \beta = 1, \quad \phi \equiv 1,$$

we find that $$\Sigma\,\frac{\theta(\mu-\nu)}{(\gamma+\mu+\nu)^\alpha}\,x^\mu y^\nu$$

is a function of this type, and that the irregular part is

$$2^{-\alpha}\,\Gamma(1-\alpha)\left\{\log\left(\frac{1}{xy}\right)\right\}^{\alpha-1}(xy)^{-\frac{1}{2}\gamma}\left\{\theta(0)+\Theta\left(\sqrt{\frac{x}{y}}\right)+\Theta\left(\sqrt{\frac{y}{x}}\right)\right\}.$$

If Θ is not integral, we get other primary singularities: thus those of

$$\Sigma\,\frac{x^\mu y^\nu}{\{a^2+(\mu-\nu)^2\}^s(\gamma+\mu\omega+\nu\omega')^\alpha}$$

are $x = 1$, $y = 1$, and $xy = 1$.

Similarly with $$\Sigma\,\frac{\Gamma(a+\mu-\nu)}{\Gamma(a+b+\mu-\nu)}\,x^\mu y^\nu$$

(in this case θ is not an *even* function).

18. A ready method of constructing interesting base series is the following:—Consider the integral

$$\frac{1}{2\pi i}\int_C \frac{R(u)\,du}{\{1-x\,S(u)\}\{1-y\,T(u)\}},$$

where R, S, T are rational. Suppose that the contour C includes the poles $u = \lambda_1, \ldots, \lambda_p$, and excludes the poles $u = \mu_1, \ldots, \mu_q$. Then (generally under certain restrictions as to *which* poles lie inside C) the integral can be expanded as a power series $\Sigma\,a_{\mu,\nu}\,x^\mu y^\nu$, where

$$a_{\mu,\nu} = \frac{1}{2\pi i}\int RS^\mu T^\nu\,du,$$

which can often be calculated in a simple form. The singularities are easily assigned: the *primary* ones come from making

$$\lambda_i = \mu_j, \qquad \lambda_i = \infty,$$

the rest from making two λ's or two μ's equal or a μ infinite.

We are thus able to construct base series whose singularities have a prescribed form. Thus the integral

$$\frac{1}{2\pi i}\int\frac{du}{(1-xu)(u-y)}$$

gives rise to $1/(1-xy)$.

Sometimes we can use integrals containing irrational functions. Thus the integrals

$$\frac{i}{2\sin a\pi\,\Gamma(b)}\int\frac{(-u)^a(1-u)^{b-1}}{(1-xu)(u-y)}\,du, \qquad \frac{1}{4\sin a\pi\sin b\pi}\int\frac{(u-1)^{a-1}(-u)^{b-1}}{(1-xu)\{1-y(1-u)\}}\,du,$$

taken round appropriate contours, give rise to the functions

$$\Sigma\frac{\Gamma(a+\mu-\nu)}{\Gamma(a+b+\mu-\nu)}x^\mu y^\nu, \qquad \Sigma\frac{\Gamma(a+\mu)\,\Gamma(b+\nu)}{\Gamma(a+b+\mu+\nu)}x^\mu y^\nu$$

considered above.

19. I conclude with three remarks of a general character:—

(1) Various generalisations of the procedure here adopted at once suggest themselves. It is sufficient to refer to §§ 4–7 of the paper II. My object has been to consider only a few of the very simplest cases with a view to making some sort of a beginning. The general problem of attempting to classify such simple types of power series in two or more variables as naturally suggest themselves, according to the nature of their singular points, seems to me an extremely attractive, but an extremely difficult, one.

(2) There is one particularly interesting type of series which does not seem to be amenable to analysis of so simple a character. I refer to the series obtained by replacing such a factor as $(\gamma+\omega\mu+\omega\nu)^{-a}$—which has frequently occurred in this paper—by a factor such as

$$(a\mu^2+2h\mu\nu+b\nu^2+2g\mu+2f\nu+c)^{-a}.$$

Even the simplest case in which the factor is $(\mu^2+\nu^2)^{-a}$, defies us, owing to the non-existence of any simple expression of this quantity as a definite integral of the type required. Even if $a=1$, it only appears to lend itself to analysis of an awkward and unsymmetrical type.

In order to deal with such a series as

$$\Sigma\frac{x^\mu y^\nu}{(\mu^2+\nu^2)^a}$$

it seems necessary to have recourse to entirely different methods. There is no difficulty in establishing *asymptotic* formulæ analogous to those of §§ 13 and 16 (i.). Thus, if

$$\lim x = 1, \quad \lim y = 1, \quad \lim(1-y)/(1-x) = \tan\zeta \quad (x,\, y \text{ real and } < 1),$$

we find

$$\lim \{(1-x)^2+(1-y)^2\}^s \Sigma\Sigma \frac{x^\mu y^\nu}{(\theta+a\mu^2+2\beta\mu\nu+\gamma\nu^2)^s}$$

$$= \Gamma\{2(1-s)\} \int_0^{\frac{1}{2}\pi} \frac{d\lambda}{(a\cos^2\lambda+2\beta\cos\lambda\,\sin\lambda+\gamma\sin^2\lambda)^s\{\cos^2(\lambda-\zeta)\}^{1-s}},$$

and in particular

$$\lim[-\log\{(1-x)^2+(1-y)^2\}]\,\Sigma\Sigma \frac{x^\mu y^\nu}{a^2+\mu^2+\nu^2} = \tfrac{1}{4}\pi.$$

These results I have proved by methods depending on real variables only; and I have not yet been completely successful in finding a method which furnishes, with regard to these functions, results analogous to those obtained in this paper. I have, however, made sufficient progress to shew, for example, that singularities of

$$\Sigma \frac{x^\mu y^\nu}{(\theta+a\mu^2+2\beta\mu\nu+\gamma\nu^2)^s}$$

are given by $x = 1$, $y = 1$, and $\gamma(\log x)^2-2\beta\log x\log y+a(\log y)^2 = 0$.

(3) It might be thought that a great deal might be learnt by means of Hadamard's "multiplication of singularities" theorem, which holds for functions of any number of variables. But (as Hadamard has himself pointed out) this is not the case. Thus

$$x = 1 \ (any\ y), \qquad y = 1 \ (any\ x)$$

are singular points for

$$\Sigma \frac{x^\mu y^\nu}{\gamma+\mu+\nu}, \qquad \Sigma \frac{x^\mu y^\nu}{\delta+\mu+\nu}.$$

Multiplying, we obtain *any* x, *any* y

as possible singularities of

$$\Sigma \frac{x^\mu y^\nu}{(\gamma+\mu+\nu)(\delta+\mu+\nu)},$$

and our knowledge is in no way advanced.