SOME NEWLY OBSERVED MANIFESTATIONS OF FORCES IN THE INTERIOR OF AN ELECTRIC CONDUCTOR.[1]

BY EDWIN F. NORTHRUP.

SOME months ago, my friend, Carl Hering, described to me a surprising and apparently new phenomenon which he had observed. He found, in passing a relatively large alternating current through a non-electrolytic, liquid conductor contained in a trough, that the liquid contracted in cross-section and flowed up hill lengthwise of the trough, climbing up upon the electrodes. With a further increase of current, he found that this contraction of cross-section became so great at one point that a deep depression was formed in the liquid with steeply-inclined sides like the letter V. This depression extended in the case of a liquid metal as deep as six inches. With a current of constant value, the condition was a stable one, but the liquid on the inclined surfaces showed great agitation. With a still greater increase of current, the depression reached the bottom of the trough, thereby rupturing the circuit; this, of course, resulted in the liquid flowing together again, and again breaking, a violent interrupter being thus formed. He thus concluded that there is a limit to the current which it is possible to pass through a liquid conductor and that this limit may be reached before the volitalization limit. He suggested that the blowing of fuse wires and the action of the Caldwell interrupter may, perhaps, be aided by this phenomenon. Mr. Hering suggested the idea that this contraction was probably due to the elastic action of the lines of magnetic force which encircle the conductor, which lines, he said, acted on the conductor like stretched rubber bands, tending to compress it, especially at its weakest point. As the action of the forces on the conductor is to squeeze or pinch it, he jocosely called it the "pinch phenomenon."

The phenomenon described appeared to the writer to have a considerable theoretical interest. The theory and the experiments de-

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scribed in the following are the results of the writer's analysis of the problem. In the development of the theory which follows, he has been much assisted by the discussions which he had with Mr. Hering who offered many useful suggestions.

Depression Obtained with a Small Current.

To show the phenomenon observed by Mr. Hering, the writer devised several laboratory experiments in which mercury was used as the liquid conductor. In a thick sheet of hard rubber, about five inches square, two square holes or depressions were cut. These depressions were about three quarters of an inch deep and about one and three quarter inches on a side. Their two adjacent sides were parallel and about one half inch apart. A channel one quarter inch wide and three quarter inch deep was cut so as to connect the depressions. The depressions and channel were nearly filled with mercury. Electrodes of brass placed in the depressions enabled a storage battery current to be passed in the mercury from one depression to the other through the narrow channel.

It was found that about 800 amperes caused the mercury in the channel to depress lengthwise of the channel in the form of a V, about one half inch below the general level of the mercury. A small increase in the current caused the angle of the depression to reach the bottom of the channel and rupture the circuit. The liquid would again flow together and again rupture thus forming a slow and irregular interrupter.

The following experiment was then devised by which the forces at work could be shown in a more striking manner and with much smaller currents.

The apparatus shown in Fig. 1 was constructed. This is a box made of wood. The box is divided as shown into two rectangular compartments which are connected by a deep channel. One side of the channel was formed of a sheet of transparent mica, so the height of any liquid in the box could be observed from the side. In each end of the box were fastened brass electrodes.

The interior of this box, thus arranged, was filled to a depth of two inches with a liquid alloy of potassium and sodium. The remaining space was filled with kerosene oil. The liquid alloy used has a good electrical conductivity and a specific gravity but little greater than kerosene. Hence, if, when current is passed from one electrode to the other, in either direction, forces act to cause the liquid metal conductor to depress in the narrow channel, *C*, a

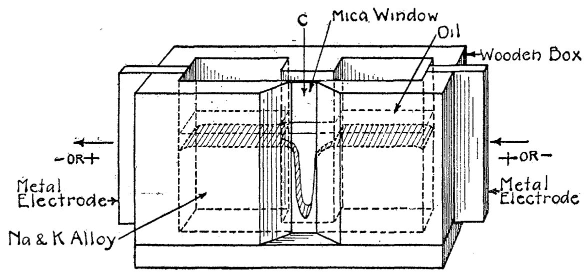


Fig. 1.

great depression should be obtained with a feeble force. It is evident that, while the conductor may depress, the level of the oil would remain unchanged. The experiment fully realized the results anticipated. Less than 180 amperes, when passed in either direction from one electrode to the other, caused the V-shaped depression to extend at least one and three quarter inches down. Regulating the current with a carbon rheostat enabled the depression to be held at any desired vertical depth. This proved that with a fixed current the depression formed in the liquid is stable. Fig. 1, drawn in perspective, shows the general appearance of what was observed.

Physical Views of the Forces in an Electric Conductor.

Several other experiments were made, but before describing them and explaining those just given, it will be useful to carefully consider the nature and magnitude of the forces which exist in the interior of a conductor carrying a steady electric current.

When a conductor carries an electric current, we may picture, with Faraday, a medium filled with lines of magnetic induction and consider all the forces connected therewith, or, we may consider, with Ampère, the conductor only, or the imaginary filaments of which it is composed, as giving rise to forces, which act through space according to laws ascertained by experiment. These two ways of viewing the matter are totally unlike, but, if properly treated, both should lead to identical conclusions regarding the action of conductors and elements of conductors upon each other. We shall consider the forces, in which we are interested, in each of these two ways. The results obtained should check each other.

First conceive a very long straight conductor of small cross-section and very far separated from its return conductor. The action of such a conductor carrying a unidirectional current upon a small magnet in its neighborhood shows that the conductor is surrounded by lines of magnetic force. These lines are concentric circles with the conductor at their common center, and they pass normally through any plane passing through the conductor. The law of the diminution of the intensity of the field of force with increasing distance from the axis of the conductor is given by the following experiment, taken from Maxwell, § 478.

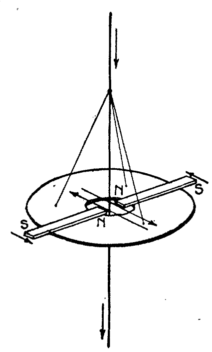


Fig. 2.

Two equal bar magnets are supported free to revolve around the conductor in the manner shown in Fig. 2. Under these conditions it is found that no movement of the magnets takes place whatever the strength of the current in the conductor. Hence, all the couples involved must balance. Thus, for the couples due to either magnet, we have

$$m_1 r_1 T_1 + m_2 r_2 T_2 = 0.$$

Here r_1 is the axial distance to the pole m_1 and r_2 to the pole m_2. T_1 and T_2 are the field intensities at these points. Other experiments show that in any magnet,

$$m_1 + m_2 = 0.$$

Hence

$$\frac{T_1}{T_2} = \frac{r_2}{r_1}.$$

Or the intensity of the field about an infinitely long straight conductor is inversely as the distance from the axis of the conductor.

Experiment also shows that the intensity of the field at any point in the neighborhood and outside of a conductor depends upon the strength of the current. This intensity may be taken as the measure of the current and in the electro-magnetic system of measurement,

$$T = \frac{2I}{r}, \tag{1}$$

where I is the strength of the current in the conductor, and r is the distance from the axis of the conductor to any point outside of the conductor where the intensity of the field is measured.

When the current in the conductor disappears, the lines of force which encircle it disappear, leading to the conclusion that to have lines of magnetic force a current must exist which they can encircle. The above experimental facts suffice for determining the forces acting both in and around a non-magnetic conductor carrying a steady unidirectional current.

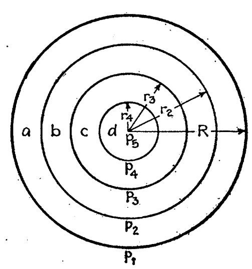


Fig. 3.

Besides the field of force exterior to the conductor, there is a field of force in the substance of the conductor. The intensity of this field, at any distance from the axis of a conductor of circular cross-section and of non-magnetic material, is easily found as follows: Let the outer circle of Fig. 3 represent the cross-section of such a conductor. Conceive this conductor to consist of concentric shells of internal radii, as r_2, r_3, r_4. Then at point p_1, located on the surface of the conductor, the intensity of the field will be,

$$T_1 = \frac{2I}{R},$$

I being the current in the entire conductor, and R its radius. At a point p_2 in the interior of the conductor, no line of force or field intensity can exist which is contributed by the outer shell, a. For we conceive p_2 to lie on the inside surface of the shell a, and on the interior of a hollow-conducting cylinder no lines of force can exist due to the current in the cylinder. This is proved by the experimental fact that a small magnet suspended in the interior of such a cylinder is uninfluenced by current in the cylinder. Moreover, lines of force cannot exist, because in the hollow space of the cylinder there are no currents which they can surround. Every line of magnetic force, due to a current, must surround current. It is concluded, therefore, that any field which exists at p_2 must be due to the current in that portion of the conductor having a radius r_2. Call i_2 the total current in that portion of the conductor which has a radius r_2 and we have the intensity of the field at p_2,

$$T_2 = \frac{2i_2}{r_2}.$$

Similarly, the intensities of the field at p_3 and p_4 are

$$T_3 = \frac{2i_3}{r_3}, \quad T_4 = \frac{2i_4}{r_4},$$

or, in general,

$$T_r = \frac{2i_r}{r}.$$

If it be assumed that the distribution of the current throughout the cross-section of the conductor is uniform and we make

$I = \pi R^2$, we will have

$i_2 = \pi r_2^2$, or, in general, $i_r = \pi r^2$,

Then it follows that

$$i_r = \frac{r^2}{R^2} I = \frac{rT_r}{2}$$

where T_r is the intensity at distance r from the axis due to the current i_r in the circle of this radius. Hence we obtain

$$T_r = \frac{2Ir}{R^2}. \tag{2}$$

By similar reasoning, it can be shown that the intensity of the field in the metal at any point p, distant r from the axis of a hollow cylinder of internal radius d and external radius R is

$$T = 2I \frac{r^2 - d^2}{(R^2 - d^2)r}. \tag{3}$$

If $d = 0$, this expression becomes the same as the one above for a solid conductor.

Experiment to Show the Action of the Field in the Interior of a Conductor.

Since in the interior of a solid cylindrical conductor the field intensity *increases* from zero at the axis to a maximum at the circumference, a condition exists the reverse of that on the outside of the conductor where the intensity of the field *decreases* with the distance from the axis. To show this difference the following experiment was arranged.

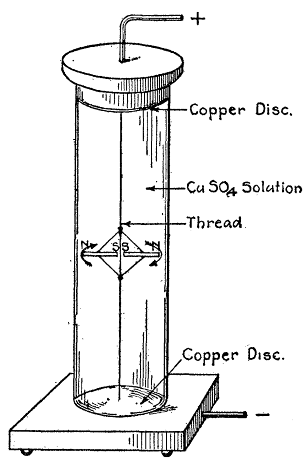


Fig. 4.

A glass tube 14 inches long and three inches inside diameter was fastened in a vertical position on a wooden base. In the top and bottom ends of the tube, copper disks were fitted. To each of these a lead wire was fastened, one passing out of the top and one out of the bottom of the tube. Between the top and bottom disks a silk thread was tightly stretched and located on the axis of the tube. At the center of the cylinder two small magnets were placed with their south poles pointing to the axis and their north poles to the circumference.

The magnets were fastened together by a light wire frame and supported by the thread in such a manner that they might rotate freely in a horizontal plane about the vertical thread as an axis. The arrangement described is made clear by referring to the diagram, Fig. 4. The glass cylinder was filled quite full with a strong solution of copper sulphate. A moderate current could be passed through the solution between the copper disks without generating gas.

As was expected, the magnets began to rotate as soon as current was sent through the liquid. When the current passed from the top to the bottom, the rotation was clockwise. Reversing the current reversed the rotation. With five amperes of current, the magnets made about two complete revolutions a minute. This result is demanded by the distribution of the field intensity in the interior of a conductor. For, call m and $-m$ the pole strengths of one of the magnets, T_1 the intensity of the field where m is located at the distance r_1 from the axis, and T_2 the intensity of the field where $-m$ is located at the distance r_2 from the axis. Then the couple acting to turn one of the magnets in a horizontal plane about the axis of the conductor is,

$$P = mr_1 T_1 - mr_2 T_2.$$

We have found

$$T_1 = \frac{2Ir_1}{R^2} \quad \text{and} \quad T_2 = \frac{2Ir_2}{R^2}$$

R being the radius of the conductor. Hence,

$$P = 2Im\frac{r_1^2 - r_2^2}{R^2},$$

is the couple acting upon one magnet, while twice this couple acts to turn the two magnets, if these are alike and similarly located in respect to the axis.

Physical View of the Pressure in the Interior of a Conductor.

Let Fig. 5 be a cross-sectional view of the liquid column of Fig. 4. Let f represent the section of a small wire, which extends throughout the length of the liquid column and parallel to its axis.

If the liquid column has a radius R and carries a current I, the wire, if located at a distance r from the axis of the liquid conductor, will be in a field of magnetic force of intensity

$$T_r = \frac{2Ir}{R^2}.$$

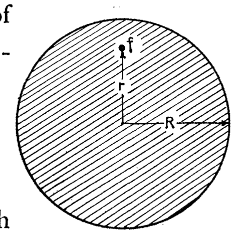


Fig. 5.

If current is passing *in the same direction* through the liquid conductor and the wire f, the wire will experience a force which will tend to move it across the circular lines of force in a direction to bring it toward and parallel to the axis of the liquid conductor. If the wire is free to move and the current density in it is greater than the current density in the liquid conductor, it will so move until it coincides with the axis of the liquid column.

This was proved experimentally as follows: In the apparatus shown in Fig. 4, a fine manganin wire of high resistance was placed in the liquid column. One end of the wire was fastened to the upper copper disk near its circumference and the other end to the corresponding point of the lower copper disk. The wire was slightly longer than the distance between the disks and hence hung loose between them. It was arranged to have it hang near the circumference of the liquid column. When current passed through the liquid column a current in the same direction and of greater current density also passed through the wire. On making the circuit, the loose wire was instantly urged from the circumference and formed a bow which bent toward the axis of the liquid column.

In place, now, of an actual wire in the inside of the liquid column, we may picture this liquid column as being made up of a great many filaments parallel to the column. Each filament will be a conductor which carries a current proportional to the total current and to its own cross-section. All the imaginary conducting filaments, except the one coinciding with the axis, will be in a field of magnetic force, the intensity of the field in which any filament lies being proportional to its distance from the axis. All the filaments, except the one on the axis, will experience a force which, if free to act, will urge them toward the axis of the conducting liquid column.

From this conception, it is evident that, if the conducting liquid column have elastic walls, it will contract so as to diminish in cross-section and in consequence increase in length. If the column cannot increase in length, it will tend to contract. In other words, there will be a pressure at the center greater than the pressure near the surface. This pressure is hydrostatic. Its existence and magnitude are shown by the following experiment.

Experiment to Show Pressure in the Interior of a Conductor.

An apparatus was constructed which is shown in section in Fig. 6. *T* is the tube of fiber, and *t* another tube of fiber having an internal diameter of 2.54 cm; *B* and *D* are brass plugs tightly fitted in the ends of the inside tube. Heavy copper terminals enable these plugs to be joined to a battery of large current capacity. At the center of the tube t was placed a fiber ring to reduce the cross-section of the tube at one place. The hole in the fiber ring had a diameter $2R$ or a radius of .635 cm.

Through the sides of the tube and fiber ring several small holes, h, were drilled radially to the axis. The bottom of the annular space between the tubes T and t was closed by a fiber piece P. The two tubes were then filled with mercury to the level shown in the cut. According to the conception given above, when current is passing through the mercury column in the tube t, a hydrostatic pressure should exist in its interior, which is greatest along its axis. Hence, as the mobile conductor has no other escape than the hole H provided in the plug D, it should rise in this hole from the pressure at the axis, more being at the same time drawn in at the holes, h. Sufficient pressure should raise the mercury in the hole H until it overflows the upper surface of the plug D, and falls back into the well from whence it came. Thus, a continuous stream of mercury should flow by the path, through the holes h, up the tube t and the hole H, into the well and thence again to the holes h.

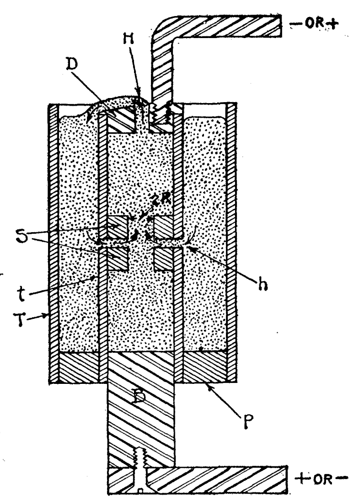


Fig. 6.

In trying this experiment, these results were amply realized. With about 1,800 amperes, the mercury flowed in a continuous and rapid stream, as described.

To test the magnitude of the pressure, a glass tube was tightly fitted in the hole H, and the mercury column was found to stand, with 1,800 amperes, six tenths of an inch in the tube above the general level of the mercury in the well.

So large a current soon heated the mercury and it was not found practical to keep it on more than a minute or two at a time. It is scarcely necessary to remark that the flow and the rise of the mercury is independent of the direction of the current.

It was thought that iron might be used to increase the induction and so get the mercury to flow continuously with less current. The previous experiment was modified as follows.

A disk of soft iron was inserted in the narrow portion of the tube *t* in the manner shown in Fig. 7, *a* being a vertical section and *b* a horizontal section of the middle portion of the tube *t* and its fittings. A slot was cut in the iron disk parallel to its axis. This slot was 1.6 mm. wide and extended in radically to the center of the disk. The iron disk was located in the narrow portion of the tube *t* so that the end of the slot registered with one of the holes, *h*, Fig. 6. It is evident that the intensity of the magnetic field in the space of this narrow slot should be greatly increased by the presence of the iron. Also that the conducting mercury filaments in the slot should be urged with increased force toward the axis with a resulting greater hydrostatic pressure at the axis.

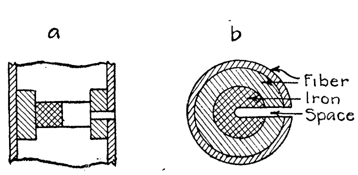


Fig. 7.

In trying the experiment, it was found that the force previously obtained was about doubled and that with 1,800 amperes the mercury would rise in the glass tube about one and a quarter inches. With 900 amperes and with the tube removed, the mercury would flow continuously, the heating being now reduced so that the current could be kept on for many minutes.

Law of the Pressure in the Interior of a Conductor.

Let Fig. 8 represent the cross-section of a cylindrical conductor. Conceive this cross-section to be made up of a large number of annular spaces as 1, 2, 3, etc. Let the radial depth of each of these annular spaces be called *dr*. Let *R* be the radius of the cylindrical conductor, and *r* the radial distance from the axis to any point within the circular cross-section. It should further be supposed that the conductor is of very great length, having its axis perpendicular to the plane of the paper, and far removed from any return circuit. By the theory and experiments cited above, all portions of the conductor, as the annular sections, 1, 2, 3, etc., are under the influence of a force, when the conductor is carrying a current, which tends to urge them radially toward the axis. This force, as measured in dynes, will have at any point distant r from the axis, a definite value per unit of area. The total force or pressure which acts upon the surface of any imaginary cylinder of radius r and length l, will be, evidently, the force per unit of area multiplied by the total area of a length l, of this imaginary cylinder.

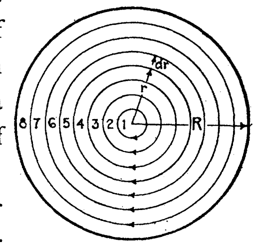


Fig. 8.

It is first required, to find the force, g, per unit of area, at any point within the conductor distant r from the axis, and second, the force or pressure P on the surface of an imaginary cylinder of radius r, and axial length l.

This problem may be solved by conceiving the action of the magnetic field on all portions of the conductor carrying current, or by considering only the mutual attractions of all the elements of the conductor.

By the first method, we may proceed as follows:

At a point within the conductor distant r from the axis the intensity of the magnetic field is by equation (2).

$$T_r = \frac{2Ir}{R^2},$$

where I is the total current flowing in the conductor. The lines of force of this field are circles, having as their common axis, the axis of the conductor.

Consider any single annular space as 5. It will carry current di, which is in the same ratio to the total current as the area of the annular space is to the total area of the cross-section of the conductor. If we call da the area of the annular space which has an inside boundary of radius r, we have

$$da = \pi(r + dr)^2 - \pi r^2 = 2\pi r dr,$$

since the square of dr can be neglected. Then as πR^2 is the total area, we obtain

$$di = \frac{2Irdr}{R^2}.$$

This current, di, is disposed in a field of intensity

$$T_r = \frac{2Ir}{R^2}.$$

If the current I flows downward, the lines of force would act on a unit north pole to move it in a clockwise direction around the axis; and an element of the conductor carrying a downward current would tend to move radially toward the axis. If the direction of the current were reversed, the direction of the magnetic lines would be reversed but the current elements would still tend to move toward the axis. The force in dynes with which a length l of any conductor is acted upon to move at right angles to the lines of magnetic force is, in electro-magnetic measure, numerically equal to the product of the length of the conductor, to the strength of the current in the conductor and to the field intensity where the conductor is located.[1] Thus calling dF the force with which the element carrying the current di tends to move radially toward the axis, we have

$$dF = ldiT_r = \frac{4I^2lr^2dr}{R^4}.$$

It should be noted that the force, dF, is distributed so as to act normally on a surface the area of which is $2\pi rl$. Hence, calling dg the force per unit of area, or the force intensity acting radially inward, we have

$$\frac{dF}{2\pi rl} = dg = \frac{2I^2rdr}{\pi R^4}.$$

This is the intensity of the pressure at distance r from the axis, due to the current in a single annular space of radial depth, dr. It is necessary in order to obtain the total intensity, g, at the distance r from the axis, to take the sum of the force intensities due to the currents in all the annular spaces, each of radial depth dr, which lie in the space included between the radius R and the radius r. This sum is given by the integral

$$g = \frac{2I^2}{\pi R^4}\int_r^R rdr.$$

[1] Consult Maxwell, Vol. II., § 490.

Performing the integration, we have

$$g = \frac{I^2}{\pi R^4}(R^2 - r^2). \tag{5}$$

Equation (5) gives the inward radial pressure per unit of area on the material of the conductor itself at any distance r from the axis of the conductor.

If I is the current per unit of cross-section, or the current density in the conductor, $I^2 = I_1^2\pi^2R^4$, and we obtain as another expression for the pressure

$$g = \pi I_1^2(R^2 - r^2). \tag{6}$$

The pressure at the center, where $r = 0$, is therefore equal to the area of the cross-section of the conductor multiplied by the square of the current density.

The total pressure on the surface of an imaginary cylinder of length, l, and radius, r is, evidently, the force per unit of area multiplied by the total area of the surface of the cylinder considered. Calling P this pressure, we have:

$$P = \frac{2lI^2r}{R^4}(R^2 - r^2) \tag{7}$$

or

$$P = 2\pi lI_1^2r(R^2 - r^2). \tag{8}$$

These same expressions are easily obtained by conceiving the imaginary conducting shells to exert an attraction according to the law of mutual attraction between currents flowing parallel and in the same direction. The law of attraction between any two infinitely long parallel conductors distant r from each other and carrying currents i and i_1 is

$$F = 2ii_1\frac{l}{r}. \tag{9}$$

Here l is the portion considered of the length ot *one* of the conductors.[1]

In the case under consideration, we can take one of the conductors to be the cylinder of radius r, and the other conductor to be the annular area of inside radius r and outside radius $r + dr$.

[1] See Maxwell, Vol. II., § 495.

If I is the total current in the large conductor of radius R, the current carried by the conductor of radius r, is

$$i = \frac{r^2}{R^2} I,$$

uniform current density being supposed.

$2\pi r dr$ is, as previously found, the cross-section of the annular area and the current which it carries is therefore

$$di = \frac{2rdr}{R^2} I.$$

The currents i and di attract and produce a pressure over the surface of the cylinder of radius r. Calling dF this pressure, we have

$$dF = \frac{2idil}{r} = \frac{4lI^2r^2dr}{R^4}.$$

The pressure per unit of area is:

$$\frac{dF}{2\pi rl} = dg = \frac{2I^2rdr}{\pi R^4},$$

whence, in integrating, we obtain

$$g = \frac{2I^2}{\pi R^4}\int_r^R rdr = \frac{I^2}{\pi R^4}(R^2 - r^2)$$

and

$$P = \frac{2lI^2r}{R^4}(R^2 - r^2).$$

These last two expressions are seen to be identical with (5) and (7).

ATTRACTION IN THE INTERIOR OF A CONDUCTOR.

The attraction in the interior of a cylindrical conductor carrying I_1 units of current per unit of cross-section which would be exerted upon a unit length of a unit of current distant r from the axis, may be found as follows:

In equation (9) let $l = 1$, and we have

$$F = \frac{2ii_1}{r}.$$

The current carried by the conductor of radius r is

$$i = \frac{r^2}{R^2} I,$$

which combined with the above gives

$$F = \frac{2ri_1 I}{R^2}.$$

Calling I_1 the current per unit of cross-section of the conductor and taking i_1 equal to unity, we find

$$F_t = 2\pi I_1 r. \qquad (10)$$

Equation (10) expresses the attraction in the interior of a conductor carrying current. It is analogous to the gravitational attraction in the interior of a body of the same form as the conductor, due to its mass only.

Gravitational Analogy.

It is interesting to compare the forces produced by an electric current in a conductor of circular cross-section with the forces produced by gravity in the interior of an infinitely long cylinder of the same radius and section as the electric conductor.

For the sake of comparing methods, we may use Poisson's equation to find the expressions giving the forces in the cylinder produced by gravity.

It is shown in Thomson and Tait's Natural Philosophy, Vol. II., page 37, that for the case considered, this equation takes the form

$$rF = C - 4\pi M \int_r^R r dr = C - 2\pi M(R^2 - r^2).$$

Here F is the force of gravitational attraction which acts radially toward the axis of the cylinder; M is the mass per unit of volume, and C a constant. Since we know, by symmetry, that F is zero at the axis, where r is 0, we obtain

$$C = 2\pi MR^2.$$

Hence

$$F = 2\pi Mr. \qquad (11)$$

Equation (11) is an expression of the same form as (10).

If we express F in dynes and r in centimeters, we must take for the unit of mass, that mass which at one centimeter from a like mass will cause an attraction between the two of one dyne. The value of this unit in grams is given in Everett's "C.G.S. System of Units," page 73, 1891 edition, as equal to 3,928. Hence, if m be a mass expressed in grams, we have

$$M = \frac{m}{3928},$$

or

$$F = \frac{2\pi m r}{3928} \text{ dynes.} \tag{12}$$

We shall now have for the force acting on the matter in an annular space of radius r, radial depth dr and length l,

$$dF_1 = \frac{2\pi m r}{3928} \cdot \frac{2\pi l m r dr}{3928} = \frac{4\pi^2 l m^2 r^2 dr}{3928^2}.$$

The intensity of this force, or the dynes pressure per unit of area at distance r from the axis is

$$\frac{dF_1}{2\pi r l} = dg_1 = \frac{2\pi m^2 r dr}{3928^2}.$$

And by integrating, we obtain

$$g_1 = \frac{2\pi m^2}{3928^2} \int_r^R r dr = \frac{\pi m^2}{3928^2} (R^2 - r^2). \tag{13}$$

Or, if we call G the number of grams of matter in a unit of length of the cylinder of radius R,

$$g_1 = \frac{G^2}{\pi R^4 \, 3928^2} (R^2 - r^2). \tag{14}$$

Comparing equations (5) and (14), we note that

$$g = 1.513 \times 10^7 g_1 \frac{I^2}{G^2}, \tag{15}$$

which means that the hydrostatic pressure in the interior of a conductor at any distance r from the axis when carrying I electromagnetic units of current is 1.513×10^7 times the pressure in this conductor, having G grams of matter per unit length, due to gravitational attraction alone.

If we compare equations (10) and (12), we note that the attraction due to the electric current, which would act on a unit length of a unit current, is of the same nature and in the same direction as the gravitational attraction which would act on a gram of matter, but 3,928 times as great, the number of units of current per unit of cross-section of the conductor being equal to the number of grams of matter per unit of cross-section in a unit of length of the cylinder.

Graphs.

The leading features respecting the forces discussed above are summarized in the curves shown in Fig. 9. The curves in full line apply to the cylindrical conductor which is shown in cross-section in full line. The curves in broken line, and the curve 2 apply to the conductor shown in broken line, having one half the diameter of the first conductor. In both cases, the larger and smaller conductor are supposed to carry the *same* current. Curve 1 gives the intensity of the field in the interior of the conductor. Curve 2, the intensity of the field exterior to the conductor. Curve 5, expresses the pressure in dynes per square centimeter. This pressure is a maximum at the center and zero at the circumference. At one quarter of the distance from the center to the circumference, the pressure has diminished but 6¼ per cent. — a fact having importance in certain practical constructions.

When the conductor is diminished in diameter to one half, its area is one fourth and the current density is increased four times, hence it follows from equation (6) that the pressure at the center is increased four times. This is shown by curve 6. The intensity of the field in the inside of the conductor is doubled, curve 4. On the outside of the conductor, the intensity of the field is not altered beyond the radius of the larger conductor, curve 2; but in the space outside the smaller and inside the larger, there is a field the intensity of which is given by curve 3.

From what has been given, the following deductions are easily made.

Suppose a conductor to have originally a radius R. Imagine the conductor to contract in cross-section until its radius becomes aR, a being a number less than unity. Further, assume the conductor

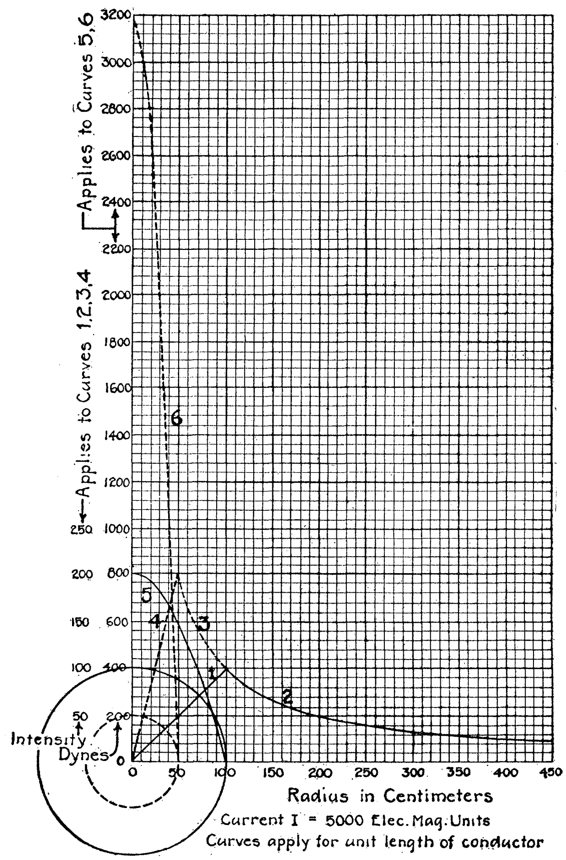


Fig. 9.

to always carry the *same* total current I and to preserve a circular cross-section. Let a portion of the length of the conductor be called l. Then the total number of lines of force in the interior of a length, l, of the conductor is given by the expression

$$N = \frac{2Il}{(aR)^2}\int_0^{aR} r dr = lI. \qquad (16)$$

Hence, we conclude that, however, the conductor may change in diameter, the total number of lines of force in its interior remains the same, provided the current is always the same, and that per unit length of the conductor the total number of lines of force in the conductor is numerically equal to the current in electromagnetic measure.

An inspection of curves 2 and 3 shows that when the conductor contracts from a radius R to a radius aR, the current remaining the same, lines of force are developed. These all lie in the medium around the conductor and their number per length l of the conductor is given by the expression

$$N_0 = 2Il \int_{aR}^{R} \frac{dr}{r} = 2Il \log_e \frac{1}{a}. \tag{17}$$

From this, it follows that the total number of lines exterior to the conductor is increased when the conductor diminishes in diameter, and as the number inside the conductor remains the same, we conclude that the coefficient of self-induction of a small wire is greater than that of a large wire.

It may be further concluded that, if the current be maintained constant, when the conductor diminishes in diameter, electromagnetic energy is stored up in the medium around the conductor.

It may be shown that the electromagnetic energy in the interior of a length l of the conductor is given by the expression

$$E = \frac{lI^2}{R^4} \int_0^R r^3 dr = \frac{lI^2}{4}. \tag{18}$$

Thus the energy stored in the conductor, as well as the number of lines of force, is independent of the diameter of the conductor. When the conductor contracts from radius R to radius aR, the energy added to the medium is stored in the space occupied by the conductor previous to its contraction less the space occupied by the conductor after its contraction. The energy so added per length l of the conductor is given by the expression

$$E_0 = I^2 l \int_{aR}^{R} \frac{dr}{r} = lI^2 \log_e \frac{1}{a}. \tag{19}$$

The above expressions (16), (17), (18) and (19) are given without proof, for they may be readily obtained from the facts stated above, and the energy relations given by Maxwell.

Adding Pressures in Series.

The writer soon perceived that to obtain forces of sufficient magnitude to be useful in practical applications means should be devised whereby the internal pressure at one point in a conductor might be added to that in another point and so on. This result may be

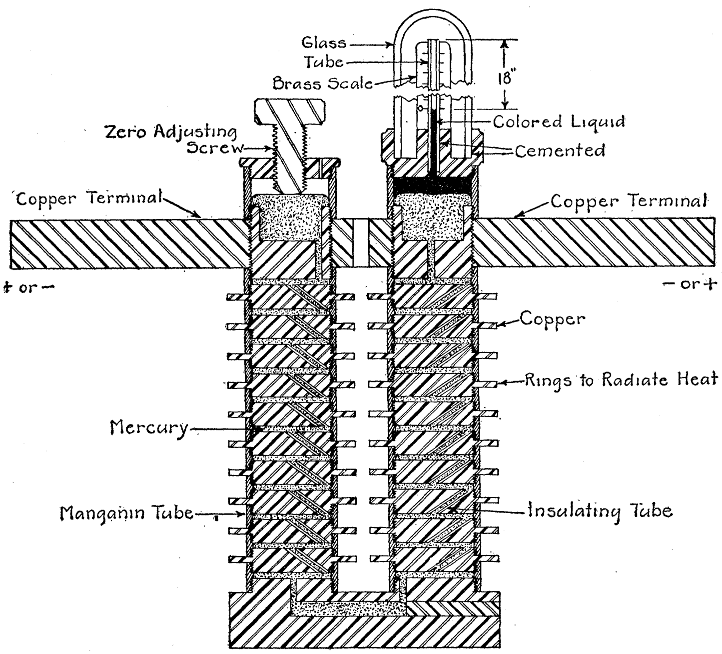
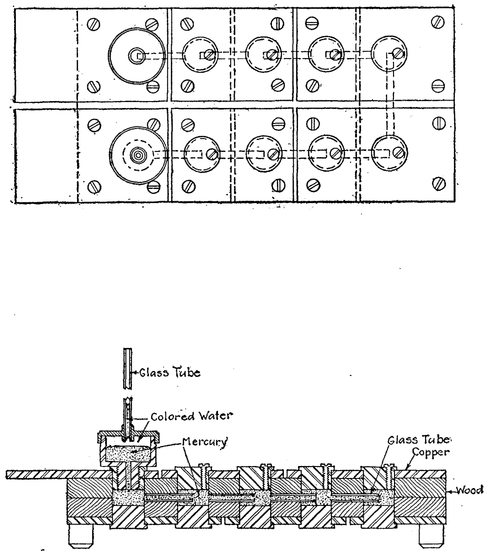


Fig. 10.

attained as follows: A cylindrical conductor is constructed which consists of alternate strata of solid and liquid, as copper and mercury. This heterogeneous conductor is tightly encased in a tube of insulating or high resistance material. In each of the solid portions of the conductor small tubes of insulating material are placed in a manner to connect the axis of one liquid layer with the circumference of the next liquid layer. This forms a continuous passage which goes in steps from the axis to the circumference of each liquid portion. A tube from one end of this conductor connects with a reservoir of the liquid and another tube of glass is raised from the other end of the conductor which connects with the end liquid layer and in which the liquid may rise. Heavy copper terminals lead the current, which may be direct or alternating, in, and out of, the compound conductor. When the electric current flows, the pressures produced at the axes of the liquid portions of the conductor add in series and the liquid may be thus raised to useful heights in the glass tube. Fig. 10 shows in section such a contrivance which it is evident might serve as a direct or alternating current ammeter of large capacity. A construction similar to that described was made and operated, and is a useful alternating current ammeter.



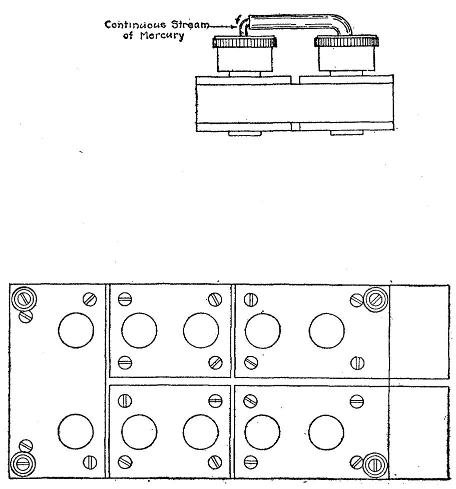


Fig. 11.

Fig. 11 shows another embodiment of the same principle. This apparatus was constructed and the calculated height of about 22 cm. to which the colored water would rise in the tube with 600 amperes was practically attained.

It should be noted that the hydrostatic pressure obtained is independent of the axial length of the liquid strata, while the heat developed by the current is directly as this length. With this consideration in mind, the axial lengths of the liquid portions of the apparatus shown in Fig. 11 were afterwards reduced to 1.6 millimeters each. The pressure was not lessened and the efficiency or cool running was greatly improved. The theoretical pressure or height of mercury that may be obtained in any particular case may be deduced easily from the curves (Fig. 9), or calculated from the expression

$$g_s = nAI_1^2 \tag{20}$$

which is derived from equation (6), when $r = 0$, $\pi R^2 = A$, the cross-section, $I_1 =$ the current density and $n =$ the number of times the pressures at the axis are added in series.

If C is the current density per square cm. expressed in amperes and g_p is the pressure in pounds per square cm. and A is the cross-section in square cm. the expression (20) takes the form

$$g_p = \frac{nAC^2}{44479100}. \tag{21}$$

In the apparatus shown in Fig. 11, $n = 10$, $A = 1.2$, $C^2 = 250{,}000$, which corresponds to nearly 600 amperes in the conductor. Whence we get $g_p = .067$ lb. per square cm. or .432 lb. per square inch. This pressure should give a rise of .88 inch of mercury or about 11.9 inches of water. The rise obtained with 600 amperes in the apparatus of Fig. 11 was about 9 inches, but this lessened height is due to the fact that the mercury itself had to raise some to produce the rise in the water. The point to note, however, is that with 600 amperes, the apparatus runs cool. Hence we can conclude that with a heterogeneous conductor of one half inch diameter and made up of 1,000 cells, or mercury sections of say, 1 mm. length each, we could obtain with 600 amperes the large pressure of 43.2 lbs. per square inch, and this without iron in the circuit. This pressure would sustain a column of mercury over seven feet high.

Some Final Remarks.

One is naturally led to inquire if the recognition of this force that exists in the interior of electric conductors and the establishment of its laws of action is likely to result in new and useful applications. It is quite certain that this force which depends only on the square of the current and linear dimensions may be used as an accurate measure of the current, either direct or alternating. As the forces can be enhanced by a proper disposition of iron in the heterogeneous circuit, motors without brushes or slip rings may be made operative. And remembering that power is the product of current and electromotive force, it is not impossible, even if large operative currents are required, that fair efficiencies may be obtained.

Further thought along an engineering line may bring about developments of a unique and useful character.