On Multiply Infinite Series and on an Extension of Taylor's Series. By M. G. Mittag-Leffler. Communicated June 8th, 1899.*

I have introduced in a former publication† a new geometrical conception, the *star*-figure.

In the plane of the complex variable x let an area be generated in the following manner:—Round a fixed point a let a vector l (a straight line terminated at a) revolve once. On each position of the vector determine uniquely a point, say a_l, at a distance from a greater than a given positive quantity, this quantity being the same for all positions of the vector. The points thus determined may be at a finite or at an infinite distance from a. When the distance between a_l and a is finite, the part of the vector from a_l to infinity is excluded from the plane of the variable.

The *star* is the region which remains when all these sections (*coupures*) in the plane of x have been made. The fixed point a is called the *centre* of the star. It is convenient to name the points a_l the *summits* of the star, and to introduce the following definition:—One star is *inscribed* in another, if all the points of the first star belong to the second and if the two stars have common summits; the second star is then *circumscribed* about the first.

By the use of the well known notion of a *limiting value*, the results which were obtained in the paper referred to can be stated as follows:—

Let
$$F(a),\ F^{(1)}(a),\ \ldots,\ F^{(\nu)}(a),\ \ldots$$
be an array of quantities satisfying Cauchy's condition;‡ denote by
$$G_n(x \mid a)$$
the polynomial in x,
$$\sum_{h_1=0}^{n^2}\sum_{h_2=0}^{n^4}\cdots\sum_{h_n=0}^{n^{2n}} \frac{1}{h_1!\,h_2!\,\ldots\,h_n!}\,F^{(h_1+h_2+\ldots+h_n)}(a)\left(\frac{x-a}{n}\right)^{h_1+h_2+\ldots+h_n},$$

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* Received, in French, May 4th, 1900. The English translation, made by Mr. R. W. H. T. Hudson, received from Prof. Mittag-Leffler, June 19th, 1900.

† "Sur la représentation analytique d'une branche uniforme d'une fonction monogène (première note)," *Acta Mathematica*, tome XXIII., pp. 43–62.

‡ *Acta Mathematica*, *l.c.*, pp. 43, 44.

which is formed by means of these quantities. Consider the limiting value

$$\lim_{n=\infty} G_n(x \mid a).$$

There exists a star A, with centre a, which is uniquely determined when the quantities $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... are once given, and which possesses the following properties in relation to the limiting value $\lim_{n=\infty} G_n(x \mid a)$:—

This expression is uniformly convergent for every domain in the interior of A, but is not uniformly convergent for any two-dimensional continuum which includes a summit of A. It defines, within A, the branch $FA(x)$ of a monogenic function. This branch is regular within A, but has the summits of A for critical points.

Further, it possesses the property

$$\left(\frac{d^\mu FA(x)}{dx^\mu}\right)_{x=a} = F^{(\mu)}(a) \quad (\mu = 0, 1, 2, \ldots).$$

The functional branch $FA(x)$ can be defined, not only by $G_n(x \mid a)$, but also by an infinite number of other polynomials

$$g_n(x \mid a) = \sum_\nu c_\nu^{(n)} F^{(\nu)}(a)(x-a)^\nu,$$

in which each of the coefficients $c_\nu^{(n)}$ is a numerical quantity, given in terms of ν and n, and independent of a, of $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ..., and of x, and which possesses in relation to A the same properties as $G_n(x \mid a)$.

Among the stars which can be inscribed in the star A is to be specially noticed the circle C, centre a, the circumference of which passes through the summit of A nearest to the centre, in accordance with our definition. Corresponding to this circle we have the limiting value

$$\lim_{n=\infty} \sum_{\nu=0}^{n} \frac{1}{\nu!} F^{(\nu)}(a)(x-a)^\nu,$$

which is known as *Taylor's series*. Taylor's series possesses in relation to the circle C the same properties which I have just stated for

$$\lim_{n=\infty} \sum_{h_1=0}^{n^2} \sum_{h_2=0}^{n^4} \ldots \sum_{h_n=0}^{n^{2n}} \frac{1}{h_1!\, h_2! \ldots h_n!} F^{(h_1+h_2+\ldots+h_n)}(a) \left(\frac{x-a}{n}\right)^{h_1+h_2+\ldots+h_n},$$

in relation to A, with one very important exception. Every summit of A is a singular point of the functional branch $FA(x)$. This is not the case for C with respect to $FC(x)$. It can be asserted only that there exists at least one summit of C (*i.e.*, a point on the circumference of C) which is a singular point for the branch $FC(x)$.

But, on the other hand, Taylor's series possesses in relation to C the property, which cannot in general be attributed to the limiting value

$$\operatorname*{Lim}_{n=\infty} \sum_{h_1=0}^{n^2} \sum_{h_2=0}^{n^4} \dots \sum_{h_n=0}^{n^{2n}} \frac{1}{h_1!\, h_2! \dots h_n!} F^{(h_1+h_2+\dots+h_n)}(a) \left(\frac{x-a}{n}\right)^{h_1+h_2+\dots+h_n}$$

in relation to A, that it does not converge for any point outside C.

Now between C and A there exist intermediary stars K, unlimited in number, each of which in succession circumscribes the preceding; with each star corresponds a limiting value possessing in relation to it the same properties which I have stated for Taylor's series in relation to C. These new limiting values have further the property that they include Taylor's series as a special case.

There are different classes of expressions which fulfil these conditions. My intention is to investigate in this note one of these classes, as follows:—

Let
$$f_{h_1 h_2 \dots h_n} \quad \left\{ \begin{array}{l} h_1 = 0, 1, 2, \dots, \infty \\ h_2 = 0, 1, 2, \dots, \infty \\ \dots \quad \dots \quad \dots \\ h_n = 0, 1, 2, \dots, \infty \end{array} \right\}$$

be an n-ply infinite array of functions of a certain number of variables.

The multiple series $\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \dots \sum_{h_n=0}^{\infty} f_{h_1 h_2 \dots h_n}$

is generally defined as being equal to a single series, the terms of which are the different functions $f_{h_1 h_2 \dots h_n}$ made to correspond with the integers 0, 1, 2, ..., according to a certain law.

It is, however, useful to study this series from another point of view, which, in spite of its great simplicity and the advantage to be obtained from it, does not appear so far to have attracted the attention of mathematicians.*

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* See "Om den analytiska framställningen, &c., Första meddelande, 11 Maj 1898," *Vet. Ak. Öfversicht.*

I make the following definition :—

I suppose that the series

$$f_{h_1 \dots h_{n-1}} = \sum_{h_n=0}^{\infty} f_{h_1 \dots h_n},$$

$$f_{h_1 \dots h_{n-2}} = \sum_{h_{n-1}=}^{\infty} f_{h_1 \dots h_{n-1}},$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$f_{h_1} = \sum_{h_2=0}^{\infty} f_{h_1 h_2},$$

$$f = \sum_{h_1=0}^{\infty} f_{h_1}$$

are all convergent for a certain value of the variables.

I say then that *the series*

$$f = \sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \dots \sum_{h_n=0}^{\infty} f_{h_1 h_2 \dots h_n}$$

is an n-ply infinite series which is convergent for this value.

I suppose further that the series

$$f_{h_1 h_2 \dots h_{n-1}},\ f_{h_1 h_2 \dots h_{n-2}},\ \dots,\ f_{h_1},\ f$$

are all uniformly convergent in a common domain B within the domain of existence of the functions $f_{h_1 \dots h_n}$.

I say then that the series is *an n-ply infinite series which is uniformly convergent in the domain B.* Having laid down this definition, I can enunciate the following theorem :—*

The functions $$f_{h_1 h_2 \dots h_n}(x_1 \dots x_m) \quad \left\{\begin{matrix} h_1 = 0, 1, 2, \dots, \infty \\ h_2 = 0, 1, 2, \dots, \infty \\ \dots \quad \dots \quad \dots \quad \dots \\ h_n = 0, 1, 2, \dots, \infty \end{matrix}\right\}$$

being analytic functions of the variables $x_1, \dots, x_m$, uniform and regular in the domain B, and the series

$$f = \sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \dots \sum_{h_n=0}^{\infty} f_{h_1 h_2 \dots h_n}$$

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* This theorem is an immediate consequence of the theorem demonstrated by Weierstrass in his memoir "Zur Functionenlehre," § 2, *Werke*, Bd. II., pp. 205–208.

being an n-ply infinite series, which is uniformly convergent in the domain B, then the series represents, in the same domain B, a uniform regular function of the variables $x_1, \ldots, x_m$.

It is easy to see that the conception of an n-ply infinite series is very different from the generally received notion of a multiple series, and that the region of convergence of the first series is generally of much greater extent than that of the second.

Take as an example

$$f(x) = \frac{1}{1-x}.$$

Denoting by ζ a point on the real axis between 0 and -1, we have

$$f(x) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} f^{(\nu)}(\zeta)(x-\zeta)^{\nu}.$$

This equality holds for $|x-\zeta| \leqslant |\zeta|$.

Then $$f(2\zeta) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} f^{(\nu)}(\zeta)\,\zeta^{\nu} = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu!\,\mu!} f^{(\nu+\mu)}(0)\,\zeta^{\nu+\mu}.$$

If we regard the series

$$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu!\,\mu!} f^{(\nu+\mu)}(0)\,\zeta^{\nu+\mu}$$

as a double series in the ordinary sense, we shall have

$$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu!\,\mu!} f^{(\nu+\mu)}(0)\,\zeta^{\nu+\mu} = \sum_{h=0}^{\infty} \frac{1}{h!} f^{(h)}(0)\,(2\zeta)^{h},$$

and this series will be then convergent only for $0 \geqslant \zeta > -\frac{1}{2}$. If, on the contrary, the series

$$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu!\,\mu!} f^{(\nu+\mu)}(0)\,\zeta^{\nu+\mu}$$

is regarded as a doubly infinite series, the convergence holds for $0 > \zeta > -1$.

It is by an application of my conception of an n-ply infinite series in the case where the functions $f_{h_1 h_2 \ldots h_n}$ involve only one variable x that I obtain a class of limiting values possessing the properties which I have stated above.

We have, in fact, the following theorem :—

THEOREM.—*Let $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... be any quantities satisfying Cauchy's condition, and let*

$$\sum_{h_1=0}^{\infty}\sum_{h_2=0}^{\infty}\dots\sum_{h_n=0}^{\infty} c_{h_1 h_2 \dots h_n} F^{(h_1+h_2+\dots+h_n)}(a)\,(x-a)^{h_1+h_2+\dots+h_n}$$

be an n-ply infinite series in which $c_{h_1 h_2 \dots h_n}$ denote certain numerical constants, independent of the quantities $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... as well as of a and of x.

It is always possible to choose these constants in such a way that the series may possess the following properties :—

It will have a star of convergence $A^{(1/n)}$, such that the series is uniformly convergent for every domain within $A^{(1/n)}$, but converges nowhere outside $A^{(1/n)}$.

This star $A^{(1/n)}$ is inscribed in the star A of the elements $F(a)$, $F^{(1)}(a)$, ..., $F^{(n)}(a)$, ..., and when $n \geqslant \bar{n}$, where the positive integer $\bar{n}$ is taken sufficiently great, it contains within itself any finite domain which is contained in A.*

Further, the star $A^{(1/n)}$ is inscribed in the star $A^{(1/n')}$, when $n < n'$.

The equation

$$FA^{(1/n)}(x) = \sum_{h_1=0}^{\infty}\sum_{h_2=0}^{\infty}\dots\sum_{h_n=0}^{\infty} c_{h_1 h_2 \dots h_n} F^{(h_1+h_2+\dots+h_n)}(a)\,(x-a)^{h_1+h_2+\dots+h_n}$$

holds throughout the interior of $A^{(1/n)}$. For $n = 1$ the series becomes Taylor's series.

Among the different modes of fixing $c_{h_1 h_2 \dots h_n}$, I will bring forward one, with which corresponds a star of convergence $A^{(1/n)}$, obtained by the following simple construction:—Select one vector l through the centre a of the star A. Construct a system of circles having their centres a, η_1, η_2, ..., η_{n-1} on l, so that each passes through the centre of the preceding. I shall denote the radii by r, r_1, r_2, ..., r_{n-1}. The centres η_1, η_2, ..., η_{n-1} are chosen in such a manner that each circle cuts the preceding in the points of contact of tangents from a to the former, and that $|\eta_1 - a| = r_1 = r$. It is evident that, if the radius r be taken sufficiently small, this system of circles will be included in A. The star $A^{(1/n)}$ is obtained by measuring along l the length

* *I.e.*, the star belonging to the elements $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... See "Sur la représentation analytique d'une branche uniforme d'une fonction monogène (première note)," *Acta Mathematica*, tome XXIII., p. 48.

$|\eta_{n-1}-a|+r_{n-1}$, then substituting for r its superior limit ρ and making l revolve once round a. With this form of star of convergence, the coefficients $c_{h_1h_2\ldots h_n}$ are

$$c_{h_1h_2\ldots h_n}=\left(\frac{\sin^2 a_{n-1}}{1+\sin a_{n-1}}\right)^{h_1+h_2+\ldots+h_{n-1}}\frac{\left(\frac{1}{\sin a_{n-1}}\right)^{h_1+h_2}}{h_1!\,h_2!}\,\frac{\left(\frac{2}{\sin a_{n-2}}\right)^{h_3}}{h_3!}$$

$$\times\frac{\left(\frac{2^2}{\sin a_{n-3}}\right)^{h_4}}{h_4!}\cdots\frac{\left(\frac{2^{\nu-2}}{\sin a_{n+1-\nu}}\right)^{h_\nu}}{h_\nu!}\cdots\frac{(2^{n-2})^{h_{n-1}+h_n}}{h_{n-1}!\,h_n!}$$

$$(n=2,\,3,\,\ldots),$$

$$\sin a_{\mu+1}=-\tfrac{1}{4}\sin^2 a_\mu+\tfrac{1}{4}\sin a_\mu\sqrt{8+\sin^2 a_\mu}\quad(\mu=1,\,2,\,3,\,\ldots),$$

$$\sin a_1=1,$$

$$c_{h_1}=\frac{1}{h_1!}.$$

The root which occurs in the expression for $\sin a_{\mu+1}$ is the positive root.

We have then

$$c_{h_1}=\frac{1}{h_1!},$$

$$c_{h_1h_2}=\frac{1}{h_1!\,h_2!},$$

$$c_{h_1h_2h_3}=\frac{1}{h_1!\,h_2!\,h_3!},$$

$$c_{h_1h_2h_3h_4}=\frac{1}{h_1!\,h_2!\,h_3!\,h_4!}\left(\frac{\sin a_3}{1+\sin a_3}\right)^{h_1+h_2}\left(\frac{4\sin^2 a_3}{1+\sin a_3}\right)^{h_3+h_4}$$

$$\left(\sin a_3=\frac{\sqrt{33}-1}{16}\right),$$

... ...

Since the n-ply infinite series

$$\sum_{h_1=0}^{\infty}\sum_{h_2=0}^{\infty}\cdots\sum_{h_n=0}^{\infty}\frac{1}{h_1!\,h_2!\ldots h_n!}F^{(h_1+h_2+\ldots+h_n)}(a)\,(x-a)^{h_1+h_2+\ldots+h_n}$$

has a star of convergence in the same sense as Taylor's series has, when $n=1,\,2,\,3$, the question arises whether the same is true for $n>3$. It is not so, if the elements $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... are chosen arbitrarily. It may happen then that the series converge at a point $x=x'$, without converging at $x=ax'$, a being a real positive quantity less than unity.

This is a new example of the danger in analysis of drawing general conclusions from special cases.