

Point-Groups in relation to Curves. By F. S. MACAULAY, M.A.

Read June 18th, 1895. Received, in revised form, November 8th, 1895.

I.

1. INTRODUCTION.—The following paper deals with the properties of point-groups in relation to algebraic curves drawn through them; without considering any of their applications to the transformation or generation of curves. Its object is to treat the subject geometrically; and the work principally consists in developing and extending Sylvester's theory of residuation.* Geometrical proofs of some known theorems are given; but the greater part of the paper is taken up with proving new results; more especially, those on characterization, in Section III.

It is essential, at the outset, to adopt a proper convention with respect to the intersection of two curves at a common multiple point. Such a convention is attained, so far as it is needed in the paper, by proving that the intersection of two multiple points with p and q branches respectively, but no common tangents, may be regarded as the complete intersection of two infinitely small curves of orders p and q (Art. 8). It follows that the intersection of two such multiple points has practically the same properties as the complete intersection of two finite curves of orders p, q . We call this intersection a cluster of pq points, and divide it at will into smaller clusters.

The following are some of the more important consequences depending on the consideration of clusters. A point-group, regarded in general as any partial intersection of two algebraic curves with common multiple points, contains both ordinary points and clusters, the latter consisting of any parts of the several clusters common to the two curves. The condition that a curve should have an ordinary multiple point with p branches at a given point A , is shown (Art. 8) to be equivalent to the condition that the curve should pass through a general cluster of $\frac{1}{2}p(p+1)$ points, placed arbitrarily about A . Hence the condition that a curve should pass through given ordinary points, and have given multiple points, is equivalent to the condition that the curve should simply pass through a given point-group, con-

* SALMON, *Higher Plane Curves*, 2nd edition, Arts. 157-160.

taining ordinary points and clusters. Also the theorem that a curve through the complete intersection of two given curves C_l, C_m must be of the form $C_l S_{n-l} + C_m S_{n-m}$, is true, whether C_l, C_m have common clusters or not; from which it follows that the theorem of residuation (Art. 6) is applicable to all cases.

In considering the properties of a given point-group in relation to curves, we require to know how far the point-group affects the degrees of freedom of curves drawn through it; or, what is practically equivalent, the number of independent conditions supplied by the point-group for curves of any assigned order. For a group of N points this number of conditions cannot, under any circumstances, exceed N ; and for curves of sufficiently high order it is equal to N ; but for curves through the N points, of sufficiently low order, the number of conditions may be less than N . We call r_n the n -ic excess of a group of N points which supplies only $N - r_n$ independent conditions for n -ics. It will be found to be the rule, rather than the exception, that for point-groups such as we have to consider, viz. those obtained by the intersections of curves, the values of r_n are not zero for all values of n . Thus in the case of the complete intersection of two curves of orders l, m , the value of r_n is $\frac{1}{2}(l+m-n-1)(l+m-n-2)$, when $n \geq l \geq m < l+m$;^{*} and is zero when $n \geq l+m-2$ (Art. 4, iv). For four points on a straight line, we have $r_1=2$, $r_2=1$, and $r_n=0$ when $n \geq 3$. We are thus led to regard any given point-group as possessing a definite characterization, expressed by the number of its points N , and the values of r_n . The values of r_n are not connected by any law which holds in general; but for any given point-group, r_n diminishes as n increases, and cannot exceed certain limits.

The latter part of the paper consists of a general investigation of the characterization of point-groups; and it is shown from Theorem V (Art. 20), how to construct a non-composite point-group having any given characterization, by means of the intersections of curves.† Two other general problems are considered: viz., the determination of the absolute number of independent connexions of the points of a group whose construction is known; and the determination of the number of points that can be chosen arbitrarily on a curve of any given order which form part of such a point-group on the curve.

^{*} This is not to be taken to mean $l \geq m$; but only that $n \geq l$, $n \geq m$, and $n < l+m$. The same applies to all inequalities in the paper.

† In order, however, that the construction may be in all cases fully determinate, a knowledge is required of the relative position of multiple points on curves of given order, when their number is so great that they could not have an arbitrary position.

In writing the paper I have received invaluable help from Miss Scott, D.Sc., Professor of Mathematics at Bryn Mawr College, Pa., U.S.A. This has enabled me to make a number of alterations and corrections; but it is quite possible that the work is still not entirely free from numerical errors.

The theory of point-groups on curves is mostly contained in German and Italian mathematical publications.* Proofs of the principal known theorems are therefore given (Arts. 3-6, 9, 17-19), which serve also as examples of the methods followed in the paper.

2. EXPLANATION OF TERMS.—(a) We shall denote a given curve of order m by C_m , and one whose coefficients are partially or wholly at disposal by S_m .

The degree of a point-group is the number of points it contains; and is equal to the sum of the degrees of its clusters added to the number of its ordinary points.

A point-group is denoted by a single letter Q, R, N, \dots , either without a suffix, in which case the letter denoting the point-group denotes its degree also; or with a suffix, in which case the suffix denotes the degree of the point-group.

(b) Two point-groups Q, R on a given curve C_m are said to be residual to one another if they together make up the complete intersection of C_m with any other curve, proper or composite.

The point-group $Q + R$ is said to have a zero residual, since a curve can be drawn through it which does not cut C_m in any more points.

* The following memoirs contain fundamental portions of the subject, and illustrate the different methods that have been employed.

BRILL-NOETHER, "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie" (*Mathematische Annalen*, vii, p. 269). The greater part of this important memoir is reproduced with slight variation in the *Vorlesungen über Geometrie* of CLEBSCH-LINDEMANN; and in the translation of the same work by BENOIST, *Leçons sur la Géométrie*, Tome II, pp. 135-146, and Tome III, p. 31 ff. Several papers are contributed separately by the same authors to other volumes of the *Math. Ann.*, some of which are referred to below. The work is mostly analytical.

BACHARACH, "Ueber den Cayley'schen Schnittpunktsatz" (*Math. Ann.*, xxvi, p. 275); consisting chiefly of an examination of exceptions to Cayley's theorem, by the theory of residuation.

CAYLEY, "On the Intersection of Curves" (*Math. Ann.*, xxx, p. 85); a reply to the preceding paper.

CASTELNUOVO, "Ricerche generali sopra i sistemi lineari di curve piane" (*Mem. della R. Accademia delle Scienze di Torino*, March, 1891, XLII, p. 3).

BERTINI, "La geometria della serie lineari sopra una curva piana" (*Annali di Matematica*, April, 1894, xxii, p. 1); giving a valuable summary of known theorems on point-groups.

(c) The whole system of point-groups of the same degree, on a given curve C_m , which have a common residual, is called a complete *coresidual* or *equivalent* system. (Cf. Art. 6.)

The equation of residuation $R \equiv R'$, or $R - R' \equiv 0$, expresses the fact that R, R' are coresidual, i.e., have a common residual Q , whether R, R' are of the same degree or not; and $Q + R \equiv 0$, that Q, R are residual.

The curve C_m on which Q, R, R', \dots lie is called the *base-curve*.

(d) A *cluster* is any arrangement of points crowded infinitely near together; or a group of ordinary points on an infinitely small scale.

It has been already mentioned that we regard a point-group in general as containing both ordinary points and clusters; and it should be noticed that any ordinary point may be regarded as a cluster whose degree is unity. We shall only consider the simple case of those point-groups in which all the clusters and ordinary points are finitely separated; since the combination of any two gives rise to properties which require additional investigation.*

If on a given base-curve the ordinary points of a point-group containing clusters be denoted by Q , the whole point-group will be denoted by ΣQ .

(e) A curve *adjoined* to a given curve C_m is one which passes through every double point of C_m , and which has at each multiple point on C_m with three or more branches a multiple point with one branch less. Thus a curve C_n adjoined to C_m has an $(i-1)$ -ple point at each i -ple point on C_m . We shall suppose, however, for the reason mentioned above, that C_m, C_n have no common tangent at any multiple point.

The definition of an adjoined curve is sometimes generalized as follows:—A curve adjoined to C_m is one which has at least $i-1$ branches at each i -ple point on C_m . (Art. 9, ii.)

(f) The term *general*, or *non-specialized*, is applied to curves, point-groups, &c., which only satisfy specified conditions. Thus we may

* Thus we exclude, for example, the consideration of the intersection of two curves which have contact at any common multiple point; which corresponds to the case of one or more of the ordinary points being infinitely near to a cluster. See BERTINI (*Math. Ann.*, xxxiv, p. 447); NOETHER (*Math. Ann.*, xl, p. 140); and BAKER (*Math. Ann.*, xlii, p. 601).

speak of N general points on a given curve; or of a general n -ic through a given point-group.

A *point-group of special form** is one whose degree N exceeds the number of independent conditions it supplies for any one or more curves which can be drawn through it. The simplest example is any group of $\frac{1}{2}(m+1)(m+2)$ or more general points on an m -ic; and the most typical, the complete intersection of two curves of higher order than the second. A point-group which is not of special form may still be specialized; but we shall nevertheless refer to all such as general point-groups.

The same distinction is made between a general cluster and a cluster of special form, the latter being a point-group of special form on an infinitely small scale.

(g)† Two point-groups N, N' in a plane, which make up the complete intersection of any two curves, are called *rest-groups*; and any rest-group of N' , e.g., any coresidual of N , is called a second derived rest-group of N .‡

(h) The n -ic excess, § r_n , of a point-group is the excess of its degree N over the number of independent conditions it supplies for n -ics.

An n -ic through $N - r_n$ of the N points passes necessarily through the remainder r_n , if the $N - r_n$ points supply $N - r_n$ independent conditions for n -ics (Art. 7).

The *characterization* of a point-group is expressed by its degree N and the several values of r_n .

(k) The n -ic defect, § q_n , of a point-group N is the number of

* The common term, *special point-group*, means a group of ordinary points on a given curve C_m , which has an excess for adjoined $(m-3)$ -ics, or, more generally, for any given linear system of curves. Point-groups of special form differ from these—(i) in including clusters, and (ii) in having excess for general curves, instead of for curves belonging to a given linear system. Thus (ii) involves a restriction, which necessitates the employment of a distinguishing term.

† Definitions g, k, l, m refer more especially to Section III.

‡ Roughly speaking, we take the letters N, N' to denote rest-groups in a plane, whether they contain clusters or not; and Q, R , or $\Sigma Q, \Sigma R$, to denote residuals, i.e. rest-groups, on a given base-curve.

§ The n -ic excess r_n corresponds to the term *sovrabbondanza* (Castelnuovo), the former referring to the point-group N , and the latter to the general system of n -ics drawn through N . Also $N - r_n$ is the *postulation*, and q_n the *postulandum* (Cayley), of an n -ic drawn through N . The values of q_{m-3}, r_{m-3} for a whole point-group ΣR on a curve C_m , containing a general cluster of degree $\frac{1}{2}i(i-1)$ at each i -ple point on C_m , are the same as the values of q, r for the ordinary point-group R , in the Brill-Noether notation.

general points through which n -ics can be drawn, in addition to passing through the N points. (Cf. Art. 15, ii.) Thus the n -ic defect of N is the same as the degree of freedom of an n -ic drawn through N , or the dimensions of the general system of n -ics through N .

(*l*) A *redundant* point-group is one of special form which contains one or more general points having no connexion with the rest.

A *complete** point-group is one which includes all points in the plane which in reality belong to it. Any n -ic drawn through an incomplete point-group, for which r_n is not zero, must pass through those other points in the plane which complete the point-group; and may possibly necessarily pass through a second complete point-group.

A *simple* point-group is one which is neither redundant, nor incomplete, nor composed of two or more complete point-groups.

(*m*) If N be a given point-group on a given curve C_m , then the number of general points that can be chosen arbitrarily on C_m which form part of any group of N points coresidual to the given one is called the *multiplicity* or *manifoldness* of the coresidual system. This multiplicity is a definite number, since a coresidual system of degree N is completely determined by any single point-group of the system (Art. 6); and we may therefore call it the multiplicity of the given point-group on C_m , or of any point-group of the system.

The *absolute n -ic multiplicity*, x_n , of any point-group N which satisfies given conditions, is the number of general points that can be chosen arbitrarily on a given n -ic which form part of such a group of N points on the n -ic. It is implied that the given conditions are of such a kind that x_n has a definite (but not a given) value.

3. THEOREM I.—*The number of independent conditions supplied for l -ics by the complete intersection of two curves C_l, C_m is*

$$lm - \frac{1}{2}(m-1)(m-2);$$

provided C_l, C_m have no common factor, and l is not less than $m-2$.

In dealing with point-groups on a given base-curve C_m this property is of fundamental importance. Expressed more fully, the theorem states that through any $lm - \frac{1}{2}(m-1)(m-2) - 1$ of the lm

* The terms complete and incomplete, applied here to a single point-group, do not in any way correspond to the same terms when applied to a system of point-groups on a given curve. (See Def. c.)

points common to C_l , C_m an l -ic can be drawn which does not pass through all the rest; and that an l -ic through any number whatever of the lm points does or does not necessarily pass through the rest according as they supply $lm - \frac{1}{2}(m-1)(m-2)$ independent conditions for l -ics, or less. That the theorem is true when $l = m-1$ or $m-2$, follows at once from the fact that, in each of these cases, $lm - \frac{1}{2}(m-1)(m-2)$ is equal to $\frac{1}{2}l(l+3)$.

We shall suppose both here, and in Theorem II, that C_l , C_m have no common multiple points; leaving it to be shown in Art. 8 that the reasoning is also valid in the contrary case.

(i.) The number of independent conditions that all the points on C_m supply for l -ics is the number of general points on C_m which would require any l -ic drawn through them to be of the form $C_m S_{l-m}$; and this number is

$$\frac{1}{2}l(l+3) - \frac{1}{2}(l-m)(l-m+3) = lm - \frac{1}{2}(m-1)(m-2) + 1;$$

since the curve $C_m S_{l-m}$ is still capable of passing through $\frac{1}{2}(l-m)(l-m+3)$ general points in the plane.

(ii.) Again, if an l -ic S_l through the lm points common to C_l , C_m be made to pass through one more point on C_m , it must pass through all points on C_m . For S_l meets C_m in more than lm points, and must therefore have a common factor with C_m . Let

$$S_l \equiv C_p S_{l-p}, \quad \text{and} \quad C_m \equiv C_p C_{m-p}.$$

Then, since S_l passes through all points common to C_l and C_m , S_{l-p} passes through all points common to C_l and C_{m-p} ; hence

$$S_{l-p} \equiv C_{m-p} S_{l-m}, \quad \text{and therefore} \quad S_l \equiv C_m S_{l-m};$$

which had to be proved. Hence the number of independent conditions supplied by the lm points for l -ics is one less than that found in (i), viz. $lm - \frac{1}{2}(m-1)(m-2)$. In other words, the l -ic excess of the lm points is $\frac{1}{2}(m-1)(m-2)$, and the degree of freedom of an l -ic through the lm points is $\frac{1}{2}(l-m+1)(l-m+2)$.*

The theorem can be easily extended to those cases in which C_l , C_m have common factors, provided each has one factor at least which is not a factor of the other.

* Cf. ZEUTHEN, "Sur la détermination d'une courbe algébrique par des points donnés" (*Math. Ann.*, xxxi, p. 235), for a fuller development of the method followed here.

4. THEOREM II.*—Any n -ic which passes through the complete intersection of two given curves C_l, C_m must be of the form

$$S_n \equiv C_l S_{n-l} + C_m S_{n-m} = 0;$$

n being not less than l or m .

We shall suppose that C_l, C_m have no common factor. The truth of the theorem is, however, independent of any conditions imposed on the curves C_l, C_m ; or any restriction placed on the value of n , assuming that S_{n-l}, S_{n-m} are zero when $n-l, n-m$ are negative.

(i.) To show that, when $n \geq l+m-2$, the lm points common to C_l, C_m supply lm independent conditions for n -ics.

If they do not, then either the lm points are such that an $(l+m-2)$ -ic through all but *any* one must necessarily pass through the last, or some part N of the lm points possesses this property. We have then only to prove that this cannot be true of any N of the lm points.

Suppose $l \geq m$. Then among the lm points it is clear that $\frac{1}{2}m(m+1)$ can be chosen through all but any one or more of which a curve S_{m-1} can be drawn without passing through any of the rest. The other $lm - \frac{1}{2}m(m+1) \leq (l-1)m - \frac{1}{2}(m-1)(m-2)$ points lie on a curve S_{l-1} which does not contain C_m as a factor (Theorem I), and therefore does not pass through all the lm points. Now, if S_{l-1} passes necessarily through some of the $\frac{1}{2}m(m+1)$ points, we can still choose S_{m-1} so as to pass through all but one of the rest. The composite curve $S_{l-1}S_{m-1}$ then passes through $lm-1$ of the lm points without passing through the last. The same proof holds for any N of the lm points, except when the N points all lie on an $(m-1)$ -ic or an $(l-1)$ -ic; and each of these last cases is a simpler one than that already considered, to which similar reasoning applies.

Hence the degree of freedom of the general n -ic through the complete intersection of C_l, C_m is $\frac{1}{2}n(n+3) - lm$, when $n \geq l+m-2$.

(ii.) To find the degree of freedom D of the curve

$$S_n \equiv C_l S_{n-l} + C_m S_{n-m} = 0.$$

When $n \geq l+m$, choose the origin at a point not lying on C_m ;

* The proof of this theorem was originally given in a memoir by Noether, "Ueber einen Satz aus der Theorie der algebraischen Functionen" (*Math. Ann.*, vi, p. 351). For other references, see Note to Art. 2 (d).

and consider the n -ic*

$$S'_n \equiv C_l S_{n-l}^{n-l-m+1} - C_m S'_{n-m} = 0,$$

where $S_{n-l}^{n-l-m+1}$ denotes any algebraic expression of order $n-l$, whose lowest terms are of order $n-l-m+1$. Let D' be the degree of freedom of S'_n ; then S'_n can be made to pass through D' general points chosen arbitrarily in the plane. Let the coordinates of these and one more arbitrary point be substituted in $S'_n = 0$; then the resulting $D'+1$ independent equations for the coefficients of S'_n will necessitate that S'_n vanishes identically; i.e., they will require that

$$C_l S_{n-l}^{n-l-m+1} \equiv C_m S'_{n-m}$$

Now C_l has no factor in common with C_m , and $S_{n-l}^{n-l-m+1}$ cannot be divisible by C_m , since the origin is not on C_m . Hence the above identity requires that $S_{n-l}^{n-l-m+1}$ and S'_{n-m} should both vanish identically, i.e., that all their coefficients, whose number is

$$\begin{aligned} \frac{1}{2}(n-l+1)(n-l+2) - \frac{1}{2}(n-l-m+1)(n-l-m+2) \\ + \frac{1}{2}(n-m+1)(n-m+2) = \frac{1}{2}n(n+3) - lm + 1, \end{aligned}$$

should be zero. Hence the $D'+1$ equations are equivalent to $\frac{1}{2}n(n+3) - lm + 1$ independent equations; and therefore

$$D' = \frac{1}{2}n(n+3) - lm.$$

But D is not less than D' , since S_n is not less general than S'_n ; and D is not greater than $\frac{1}{2}n(n+3) - lm$ by (i), since S_n is not more general than the general n -ic through the lm points. Hence also

$$D = \frac{1}{2}n(n+3) - lm.$$

When $n < l+m$, we can prove in the same way that S_n cannot vanish identically unless all the coefficients of S_{n-l} , S_{n-m} are zero; so that

$$D+1 = \frac{1}{2}(n-l+1)(n-l+2) + \frac{1}{2}(n-m+1)(n-m+2).$$

Hence finally we have

$$D = \frac{1}{2}n(n+3) - lm + \frac{1}{2}(l+m-n-1)(l+m-n-2), \text{ when } n < l+m;$$

$$\text{and } D = \frac{1}{2}n(n+3) - lm, \text{ when } n \geq l+m-2.$$

* It can be easily shown that S_n can be changed to the form S'_n ; but it is not required for the proof of the theorem.

(iii.) If $n \geq l+m-2$, it follows from (i) and (ii) that the general n -ic through the lm points common to C_l, C_m has the same degree of freedom as S_n , and can therefore be written in the form S_n .

If $n \geq l \geq m < l+m-2$, and O_n is an n -ic through the lm points, and $O_{l+m-n-1}$ another curve which does not pass through any of the lm points, then $C_n O_{l+m-n-1}$ is an $(l+m-1)$ -ic through the lm points; and we therefore have

$$C_n O_{l+m-n-1} \equiv C_l S_{m-1} + C_m S_{l-1}.$$

Hence all the points common to $O_{l+m-n-1}$ and C_l lie on S_{l-1} ; none of them lying on C_m , by hypothesis. Hence $O_{l+m-n-1}$ is a factor of S_{l-1} , and similarly of S_{m-1} , and, by dividing it out, we have

$$C_n \equiv C_l S_{n-l} + C_m S_{n-m},$$

which proves the theorem.

(iv.) The n -ic excess of the lm points common to C_l, C_m is

$$\frac{1}{2} (l+m-n-1)(l+m-n-2),$$

when $n \geq l \geq m < l+m$; and is zero, when $n \geq l+m-2$.*

For any n -ic through the lm points common to C_l, C_m has the same degree of freedom D as S_n ; and the number of independent conditions supplied by the lm points for n -ics is $\frac{1}{2}n(n+3)-D$; i.e., is $lm - \frac{1}{2}(l+m-n-1)(l+m-n-2)$ when $n \geq l \geq m < l+m$, and lm when $n \geq l+m-2$; whence the theorem follows.

Cayley's theorem; viz., that an n -ic ($n \geq l \geq m < l+m$) through $N = lm - \frac{1}{2}(l+m-n-1)(l+m-n-2)$ points common to C_l, C_m must pass through the remainder; is not accurate without the addition of the proviso that the N points supply N independent conditions for n -ics. It is perhaps still more important to notice that, Cayley's theorem does not prove that an n -ic can be drawn through any $N-1$ points common to C_l, C_m , without passing through all the rest.

The simplest criterion as to whether the general n -ic, through any given group of points common to C_l, C_m , does or does not pass through the rest, is given at the end of Art. 20.

* The n -ic excess of the lm points can be written in the form

$\left[\frac{1}{2} (l+m-n-1)(l+m-n-2) \right] - \left[\frac{1}{2} (l-n-1)(l-n-2) \right] - \left[\frac{1}{2} (m-n-1)(m-n-2) \right]$; the square brackets indicating that the terms enclosed by them are only to be retained when their factors are positive. This formula is correct for all values of n , reducing to $lm - \frac{1}{2}(n+1)(n+2)$ when $n < l < m$. (See Note, p. 507.)

5. THEOREM III.—If a group of $m(n+n')$ points on a base-curve C_m have a zero residual (Art. 2, b); and mn of the $m(n+n')$ points have a zero residual; the remaining mn' points have a zero residual.

Let $C_{n+n'}$, C_n be curves through the point-groups $m(n+n')$, mn respectively, which do not cut C_m in any more points. Then $C_{n+n'}$ passes through all the points common to C_m , C_n . Hence, if $n+n' \geq m$, we have

$$C_{n+n'} \equiv C_m S_{n+n'-m} + C_n S_{n'}.$$

Hence the curves $C_{n+n'}$, $C_n S_{n'}$ cut C_m in one and the same point-group, viz. $m(n+n')$; but C_n cuts C_m in the mn points, and in no others; hence $S_{n'}$ cuts C_m in the mn' points, and in no others; i.e., the point-group mn' has a zero residual. Again, if $n+n' < m$, we must have $C_{n+n'} \equiv C_n S_{n'}$; and the same result follows.

This theorem may be stated as follows:—Of the three equations $N \equiv 0$, $N' \equiv 0$, $N+N' \equiv 0$, any one is a consequence of the other two.*

For the theorem proves that, if $N \equiv 0$ and $N+N' \equiv 0$, then $N' \equiv 0$; and it is evident that, if $N \equiv 0$ and $N' \equiv 0$, then $N+N' \equiv 0$.

Hence the algebraic laws of addition and subtraction hold for equations of residuation.

For, if $M \equiv N$, $M' \equiv N'$; and Q be residual to M , N ; and R to M' , N' ; then $Q+R$ is residual to $M+M'$ and to $N+N'$; hence we have $M+M' \equiv N+N'$. It follows also that $M+N' \equiv M'+N$, which is equivalent to $M-M' \equiv N-N'$ (Art. 2, c).

Points which occur both in the positive and negative terms on the same side of an equation cancel. If, for example, $L+M-L \equiv N$, then $L+M \equiv L+N$; from which it evidently follows that $M \equiv N$.

As examples in addition we may notice the following:—

(i.) Of the three equations $M \equiv N$, $M' \equiv N'$, $M+M' \equiv N+N'$, any one is a consequence of the other two; i.e., two point-groups are coresidual if any parts of them are coresidual, and the remainders coresidual; and, if parts of two coresidual point-groups are coresidual, the remainders are coresidual.

(ii.) If $L \equiv M+N$ and $N+N' \equiv 0$, then $L+N' \equiv M$; i.e., if two point-groups are coresidual, the point-groups obtained by taking away any part of one, and adding any residual of it to the other, are also coresidual.

* Cf. BERTINI, *loc. cit.*, p. 497, for this way of presenting the subject.

6. THE THEOREM OF RESIDUATION.—*If two point-groups on a given base-curve have a common residual, then any residual of one is a residual of both.*

Let R, R' be the two point-groups, and Q their common residual; and let Q' be any residual of R . Then we are given the three equations $Q + R \equiv 0$, $Q + R' \equiv 0$, $Q' + R \equiv 0$; hence, by adding the last two and subtracting the first, we have $Q' + R' \equiv 0$; which proves the theorem. This theorem is therefore an immediate consequence, and may be regarded as another form, of Theorem III; and it is convenient to regard both theorems as included in the theorem of residuation.

Curves through a point-group Q on C_m determine so many co-residual point-groups R, R', \dots on C_m , whose properties are to a certain extent independent of the particular point-group Q ; since to any curve through R , which cuts C_m again in Q' , corresponds a curve through R' , which cuts C_m again in the same point-group Q' . In other words, two coresidual point-groups R, R' on a base-curve C_m are equivalent in respect to the point-groups Q, Q', \dots determined on C_m by curves drawn through R, R' ; and any two point-groups Q, Q' thus determined are also coresidual or equivalent. Also the condition that a point-group R' should have a given coresidual R is the same as that it should have a given residual, viz. any fixed residual of R ; and a coresidual system is fully determined by any single point-group of the system.

From the theorem of residuation we have the following:—

If any number of point-groups be taken in succession on a base-curve C_m , each of which is residual to the preceding, then the p^{th} and q^{th} point-groups are residual or coresidual according as the difference between p and q is odd or even.

7. POINT-GROUPS OF SPECIAL FORM.—An essentially fundamental property of curves, usually assumed without proof, is that an n -ic can be drawn through any $\frac{1}{2}n(n+3)$ given points in a plane, no matter how they may be placed. In other words, if the coordinates of any $n(n+3)$ points be substituted in the general equation of an n -ic whose constants (as we shall call the coefficients for the time being) are all at disposal, the resulting $\frac{1}{2}n(n+3)$ equations for the ratios of the $\frac{1}{2}(n+1)(n+2)$ constants have always one finite solution at least. It is clear that we may suppose that the equations for the constants have all their coefficients finite, whether any of the given points are at infinity or not; and that it cannot follow from the equations

that all the $\frac{1}{2}(n+1)(n+2)$ constants are necessarily zero, since $\frac{1}{2}(n+1)(n+2)$ independent equations cannot be deduced from $\frac{1}{2}n(n+3)$ linear equations only. Again, it cannot follow from the equations that a single one of the constants is necessarily infinite; for, if the ratio of two constants is infinite, the antecedent of the ratio may still be assumed finite by taking the consequent zero. Hence the only hypothesis on which the equations cannot have a single finite solution is that they are inconsistent among themselves. But this is impossible, since any elimination of the constants from the equations would not lead to any inconsistency, but to an identity, viz. $0 = 0$.

It is, however, quite possible that the given points might be so placed that the elimination spoken of could be effected. When this is the case, the equations are not independent, and the given points do not all supply independent conditions for n -ics, while the point-group determined by them is of special form (Art. 2, f). The number of independent conditions supplied by the point-group for n -ics is exactly equal to the number of independent equations to which the system we have been considering is equivalent. It is important also to notice that the greatest number of points that can supply independent conditions for n -ics is $\frac{1}{2}(n+1)(n+2)^*$; the coordinates of such points, when substituted in the equation $S_n = 0$, requiring all the coefficients of S_n to vanish.

If the coordinates of $\frac{1}{2}p(p+1)$ given points, which do not lie on a $(p-1)$ -ic, be substituted in the general equation of the n^{th} order, viz.

$$S_n \equiv S_{p-1} + u_p + u_{p+1} + \dots + u_n = 0,$$

u_q denoting a homogeneous function in x, y of the q^{th} order, the resulting $\frac{1}{2}p(p+1)$ equations will determine the coefficients of S_{p-1} in terms of the remaining coefficients of S_n ; for, in eliminating all but one of the coefficients of S_{p-1} , the last will not disappear; since the given points do not lie on a $(p-1)$ -ic. Hence we have the following theorem.

If an n -ic be drawn through $\frac{1}{2}p(p+1)$ given points, which do not lie on a $(p-1)$ -ic, the coefficients of the terms in S_n of the $(p-1)^{\text{th}}$ and lower orders will be thereby completely determined in terms of the rest, which last can be chosen arbitrarily.

* Hence we may say that the n -ic excess of any point-group N which does not lie on an n -ic is $N - \frac{1}{2}(n+1)(n+2)$, or more simply that the n -ic defect is -1 . (Art. 16, ii.)

But if the $\frac{1}{2}p(p+1)$ given points lie on a $(p-1)$ -ic, or more generally, if they supply only $\frac{1}{2}p(p+1) - \rho$ ($\rho \geq 1$) independent conditions for $(p-1)$ -ics, then the coefficients of the terms in S_n of the $(p-1)$ and lower orders can be eliminated, giving rise to ρ equations among the coefficients of the terms in S_n of the p^{th} and higher orders; so that these last cannot be chosen arbitrarily.

The ρ equations will be equivalent to $\rho - \rho'$ independent equations only, if the $\frac{1}{2}p(p+1)$ points supply only $\frac{1}{2}p(p+1) - \rho'$ independent conditions for n -ics.

This theorem shows that a point-group of special form has special properties in relation to any curve drawn through it; since any such point-group, in contrast to a general one of the same degree, is connected with the more complex shape of the curve dependent on the terms of higher order in its equation.

8. MULTIPLE POINTS.—Ordinary multiple points, as well as multiple points of higher singularity, may be supposed to have complex shapes, which are not apparent because they are confined within infinitely small limits. Our object is to assign a shape to an ordinary multiple point with two or more branches, which is not inconsistent with theory, and which will provide a basis for reasoning about the intersection of two curves at a common multiple point with p and q branches respectively.

Suppose that we have a cluster of $\frac{1}{2}p(p+1)$ points at the origin, which do not lie on an infinitely small $(p-1)$ -ic. Let the coordinates of any point in the cluster be denoted by $\kappa x, \kappa y$, where κ is an infinitely small constant, and x, y are finite. Then, just as in the last article, an n -ic can be made to pass through the $\frac{1}{2}p(p+1)$ points of the cluster, and its equation will be of the form

$$S_n \equiv \kappa^p u_0 + \kappa^{p-1} u_1 + \dots + \kappa u_{p-1} + u_p + \dots + u_n = 0,$$

where the coefficients of $u_p + u_{p+1} + \dots + u_n$ may be chosen arbitrarily, and those of $\kappa^p u_0 + \kappa^{p-1} u_1 + \dots + \kappa u_{p-1}$ are known in terms of them. Hence the coefficients of $u_p + u_{p+1} + \dots + u_n$ can all be chosen finite, and those of u_0, u_1, \dots, u_{p-1} are then also finite, and are known in terms of the coefficients of u_p ; for when the coordinates $\kappa x, \kappa y$ of a point of the cluster are substituted in $S_n = 0$, the terms arising from $u_{p+1} + \dots + u_n$ will contain higher powers of κ than the preceding terms, and can be neglected, since the cluster does not lie on a $(p-1)$ -ic. The curve S_n is then an ordinary n -ic with a p -ple point

at the origin, having all its finite coefficients, viz. those of $u_p + \dots + u_n$, at disposal. There is thus no necessity that a curve drawn through a cluster should not be a curve of finite dimensions, as we might be naturally disposed to assume.

When, on the other hand, the given cluster is of special form; which is the case, for example, when the cluster lies on a $(p-1)$ -ic; the coefficient of $u_p + \dots + u_n$ will not be entirely arbitrary. In fact, if ρ_q be the q -ic excess of the cluster (Art. 2, *h*), and the coordinates of all the points of the cluster be substituted in $S_n = 0$, no terms being neglected; then by the theorem of the last article we can obtain ρ_{q-1} equations among the coefficients of $u_q + u_{q+1} + \dots + u_n$; in which it is clear that κ will occur with all the coefficients of u_{q+1} , κ^2 with those of u_{q+2} , and so on. Hence it evidently follows that there are $\rho_{q-1} - \rho_q$ equations among the coefficients of u_q alone. If ρ_q is zero, the coefficients of $u_{q+1} + \dots + u_n$ are not affected by the cluster.

It follows from the above that the only alteration required in the equation of a given curve C_n , in order that the new curve may pass through an arbitrary general cluster of degree $\frac{1}{2}p(p+1)$ at each and every p -ple point of C_n , is the addition of a series of infinitely small terms, possibly extending beyond the terms of highest order in C_n . We may then replace C_n by the new curve thus obtained; and those of its properties, which do not become indeterminate in the limit, will hold for the given curve C_n . The shape of this curve at a p -ple point (as may be easily seen by changing x, y to $\kappa x, \kappa y$ and dividing out κ^p in the equation of S_n given above) is, to a first approximation, that of a p -ic on an infinitely small scale; whose asymptotes coincide in direction with the branches of C_n , but which may in all other respects be chosen at will. This then is the shape we assign to an ordinary p -ple point. By this means we resolve the intersection of two given curves at any common multiple point into a cluster of separate points; and, as a consequence, the theorems proved in Arts. 3-6 may be extended so as to include the case of curves with common multiple points.

Thus a given curve C_n with an ordinary p -ple point at A may be supposed ipso facto to contain any given or arbitrarily chosen general cluster at A of any degree not higher than $\frac{1}{2}p(p+1)$.

Also, if two given curves have a common multiple point at A with p and q branches respectively, but without common tangents, their common cluster pq at A may be regarded as forming the complete intersection of two infinitely small curves of orders p, q , whose asymptotes coincide in direction with the branches of the two curves at A .

9. NOETHER'S THEOREM.—By a curve *non-adjoined* to C_m we shall understand any curve which has not exactly $(i-1)$ branches at each and every i -ple point on C_m . Such a curve may not pass at all through some of the multiple points, or may not pass through any of them. If the ordinary points in which any curve C cuts C_m be divided into any two point-groups Q, R , then Q, R are called *adjoined* or *non-adjoined residuals* according as the curve C is *adjoined* or *non-adjoined* to C_m . Similarly, if any number of curves be drawn through Q , and have the same number of branches as C at each multiple point of C_m , cutting C_m again in groups of ordinary points R, R', \dots , then R, R', \dots are called *adjoined* or *non-adjoined coresiduals* respectively. Thus *adjoined* and *non-adjoined residuals* are ordinary or true residuals deprived of the clusters which belong to them; and similarly for *coresiduals*. Noether's theorem consists of two parts, of which the first is as follows:—

(i.) *Adjoined coresiduals on C_m have the same system of adjoined residuals.**

In other words, if two point-groups R, R' on C_m have a common adjoined residual Q , then any adjoined residual of R is also an adjoined residual of R' . Let a general cluster $Q_{\frac{1}{2}i(i-1)}$ of degree $\frac{1}{2}i(i-1)$ be chosen arbitrarily at each i -ple point on C_m , and let ΣQ denote the point-group made up of Q and all the clusters $Q_{\frac{1}{2}i(i-1)}$. Then the adjoined curve through Q, R may be supposed to pass through the whole point-group ΣQ (Art. 8), and cuts C_m again in ΣR , which is made up of R and general clusters $R_{\frac{1}{2}i(i-1)}$. Thus R is part of a point-group ΣR which is a true residual of ΣQ . Similarly R' is part of a point-group $\Sigma R'$ residual to ΣQ , and $\Sigma R, \Sigma R'$ are true coresiduals. Again, by similar reasoning, if Q' is any adjoined residual of R , then Q' is part of a point-group $\Sigma Q'$ which is residual to ΣR , and therefore residual to $\Sigma R'$; hence Q', R' are adjoined residuals, which proves the theorem.

If we consider a point-group by itself, as we shall do later on, without reference to any particular curve C_m on which it lies, we must consider the whole point-group, including the ordinary points and clusters. If a point-group contains a general cluster of degree $\frac{1}{2}i(i-1)$, any curve through the point-group has simply an $(i-1)$ -ple point at the cluster (Art. 8); and the points of the cluster may therefore be supposed to have any arbitrarily chosen general position, without affecting the character of the point-group.

* BRILL-NOETHER, *loc. cit.*, p. 497.

(ii.) *Non-adjoined coresiduals on C_m are either specialized or general adjoined coresiduals.*

Let C, O' be two curves drawn through a point-group Q on C_m , each having k branches ($k \geq 0$) at any i -ple point A , and cutting C_m again in two non-adjoined coresiduals R, R' .

Then, if C, O' have at least $i-1$ branches at each and every i -ple point on C_m , the point-groups R, R' are general adjoined coresiduals.

For the curve O' , having a k -ple point at A ($k \geq i-1$), passes through $ki - \frac{1}{2}i(i-1)$ points of the cluster ki common to C and C_m at A , this being the number of points of the cluster ki which supply independent conditions for $(k-1)$ -ics. Thus C, O', C_m have a common cluster at A of degree $ki - \frac{1}{2}i(i-1)$, which belongs to a whole point-group ΣQ containing Q . Also, C, O' cut C_m again at A in two different general clusters of degree $\frac{1}{2}i(i-1)$, which belong respectively to point-groups $\Sigma R, \Sigma R'$ containing R, R' . Thus R, R' are parts of two coresidual point-groups $\Sigma R, \Sigma R'$ each of which contains a general cluster of degree $\frac{1}{2}i(i-1)$ at each i -ple point of C_m ; i.e., R, R' are general adjoined coresiduals. (Cf. Art. 2, e.)

Again, if the curves C, O' have less than $(i-1)$ branches at a single i -ple point of C_m , then R, R' are specialized adjoined coresiduals.

For, supposing k less than $i-1$, the curves C, O', C_m have a common cluster at A of degree $\frac{1}{2}k(k+1)$ which belongs to ΣQ ; and C, O' cut C_m again at A in two clusters of degree $ki - \frac{1}{2}k(k+1)$, or $k(i-1) - \frac{1}{2}k(k-1)$, which belong to $\Sigma R, \Sigma R'$ respectively. But any adjoined curve through R contains the cluster $k(i-1) - \frac{1}{2}k(k-1)$ of ΣR (Art. 8, ii.); and therefore contains the whole point-group ΣR , and cuts C_m again in a point-group $\Sigma Q'$ which is residual to $\Sigma R'$. Hence R, R' are adjoined coresiduals.

The point-groups R, R' are however specialized, because curves could be drawn through $\Sigma R, \Sigma R'$ which have less than $(i-1)$ branches at the point A ; which would not be the case if R, R' were general adjoined coresiduals.*

Thus, adjoined coresiduals include all other kinds of coresiduals, such as are usually considered. If however the curves C, O' through Q do not have the same number of branches at each multiple point

* The properties of non-adjoined coresiduals are fully worked out by NOETHER, "Ueber die Schnittpunktsysteme einer algebraischen Curve mit nicht-adjungirten Curven" (*Math. Ann.*, xv, p. 507).

of C_m , they will determine point-groups R, R' on C_m , which may be considered as non-adjoined coresiduals of the most general kind. The properties of such point-groups are more complicated than those of the non-adjoined coresiduals considered above; but may be investigated from the results of Arts. 8, 21.

Theorem II (iv) gives the characterization of a point-group which forms the complete intersection of two curves of given order; but we have not yet determined the characterization of any point-groups formed by the partial intersections of curves. Before doing this, we give a few examples on the theorem of residuation; which will help to illustrate the general method of Section III.

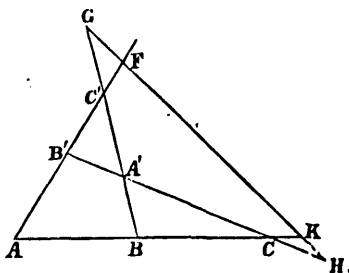
II.

EXAMPLES ON RESIDUATION.

10. *If a complete 5-side can be inscribed to a quartic, then any number of complete 5-sides can be inscribed, all of which are circumscribed to the same conic.*

Adopting German nomenclature, we define a complete n -side as the figure formed by n straight lines, having $\frac{1}{2}n(n-1)$ corners; and a complete n -corner as the figure formed by n points, having $\frac{1}{2}n(n-1)$ sides.

Let the figure represent a complete 5-side inscribed to a quartic, the letters denoting its ten corners. Then, if any four straight lines be drawn through F, G, H, K , they will cut the quartic again in twelve more points, which lie on a cubic (Theorem III); and if the angles made by the four lines with $F'A, GB, HC, KA$ respectively be very small, the twelve points become six pairs close to A, B, C, A', B', C' . Hence a cubic can be drawn touching the quartic at these points. Similarly a conic can be drawn touching the cubic at A', B', C' , and therefore also touching the quartic.



Let b', c' be two points on the quartic close to B', C' , such that $b'c'$ is a tangent to the conic inscribed to the 5-side; and let the second tangents to the conic from b', c' meet in a' . Then a' lies on the quartic; for the triangles $A'B'C', a'b'c'$ are circumscribed to a conic, and are therefore inscribed to another, viz. the conic which touches

the quartic at A', B', C' . Hence, corresponding to the line $b'c'$, we obtain another complete 5-side inscribed to the quartic and circumscribed to the conic, whence the theorem follows.

Hence if any triangle be circumscribed to the conic, two corners of which lie on the quartic, the third does also. In other words, the 13 corners of a triangle and 5-side circumscribed to the same conic supply only 12 independent conditions for quartics. Quartics through the 13 points are, however, specialized; for a complete 5-side cannot be inscribed to a general quartic. Similarly the 9 corners of a triangle and 4-side circumscribed to the same conic form the base of a pencil of cubics, which are not specialized. Several properties of cubics may be deduced from this, such as, for instance, the fundamental property that the lines joining any fixed point on the curve to pairs of corresponding points of the same kind* are in involution.

The general property, of which the examples given above are particular cases, is the following. *If an m -side and an n -side circumscribe a conic ($m \leq n$), then the $(n-1)$ -ic excess of the point-group formed by their $\frac{1}{2}m(m-1) + \frac{1}{2}n(n-1)$ corners is $\frac{1}{2}(m-1)(m-2)$. When $m < n-1$, the point-group forms the partial intersection of two $(n-1)$ -ics; when $m = n-1$, it forms the base of a pencil of $(n-1)$ -ics; and when $m = n$, it forms the complete intersection of an n -ic and $(n-1)$ -ic.*

11. *To find the general condition that a group of 13 points Q_{13} should supply only 12 independent conditions for quartics.*

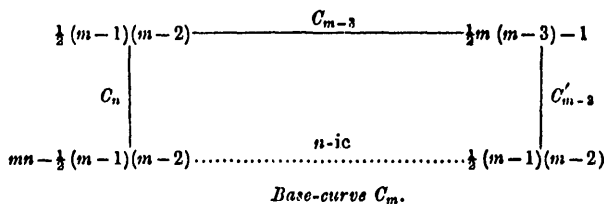
Take any fixed quartic C_4 through the point-group Q_{13} for base-curve. Then, since the 13 given points supply only 12 independent conditions for quartics, it follows that quartics can be drawn through Q_{13} cutting C_4 again in different groups of three points R_3, R'_3, \dots , which form a coresidual system. Now two point-groups R_3, R'_3 on C_4 cannot be coresidual unless they lie on two straight lines which intersect in a point A on the curve. For, if a conic be drawn through R_3 , it will cut C_4 again in a point-group Q_6 , which is residual to R_3 and to R'_3 . Hence more than one conic can be drawn through Q_6 ; and therefore four of the points Q_6 must lie on a straight line, and R_3, R'_3 must lie on two straight lines through the fifth. Hence any two

* Two pairs of corresponding points of the same kind on a cubic are the ends of two diagonals of a complete 4-side inscribed to the curve. (*Higher Plane Curves*, Art. 151.)

quartics through Q_{13} cut again in 3 points on a straight line. Conversely, if any two quartics be drawn through 3 points on a straight line, they will cut again in 13 points which supply only 12 independent conditions for quartics.

12. A group of $mn - \frac{1}{2}(m-1)(m-2)$ given points on a curve C_m ($n \geq m-2$) has an infinite number of residual groups of $\frac{1}{2}(m-1)(m-2)$ points, i.e., does not supply $mn - \frac{1}{2}(m-1)(m-2)$ independent conditions for n -ics, provided one such residual lies on a curve of order $m-3$.

Take C_m as base-curve; then the accompanying figure indicates the proof of the theorem.



Any two point-groups connected by a line in this figure are residual, the order of the curve on which any two residuals lie being marked on the connecting line. Also two point-groups separated by two lines are coresiduals; and two point-groups separated by three lines are residuals (Art. 6).

Starting with the given point-group $mn - \frac{1}{2}(m-1)(m-2)$, a curve C_n is supposed to be drawn through it which cuts the base-curve again in a group of $\frac{1}{2}(m-1)(m-2)$ points, lying on C_{m-3} , by hypothesis; the curve C_{m-3} cuts the base-curve again in a point-group $\frac{1}{2}m(m-3) - 1$, through which a second $(m-3)$ -ic C'_{m-3} can be drawn, determining a second group of $\frac{1}{2}(m-1)(m-2)$ points; this last point-group is residual to the given one, which proves the theorem.

By similar reasoning we can prove the following theorem.

If $n \geq l \geq m < l+m$, $l+m-n = \gamma$; and if $\frac{1}{2}(\gamma-1)(\gamma-2)$ of the points common to C_l , C_m lie on a $(\gamma-3)$ -ic; then n -ics through the remaining $lm - \frac{1}{2}(\gamma-1)(\gamma-2)$ points common to C_l , C_m do not necessarily pass through the $\frac{1}{2}(\gamma-1)(\gamma-2)$ points.*

* BACHARACH, *loc. cit.*, p. 497.

Take C_m as base-curve; then the $(\gamma-3)$ -ic $C_{\gamma-3}$ through the point-group $\frac{1}{2}(\gamma-1)(\gamma-2)$ cuts the base-curve again in the point-group

$$m(\gamma-3) - \frac{1}{2}(\gamma-1)(\gamma-2) = (m-3)(\gamma-3) - \frac{1}{2}(\gamma-4)(\gamma-5),$$

through which a curve C_{m-3} can be drawn, without passing through all points on $C_{\gamma-3}$ (Theorem I), i.e., without passing through the point-group $\frac{1}{2}(\gamma-1)(\gamma-2)$.

$$\begin{array}{ccc} \frac{1}{2}(\gamma-1)(\gamma-2) & \xrightarrow{C_{\gamma-3}} & m(\gamma-3) - \frac{1}{2}(\gamma-1)(\gamma-2) \\ C_l & & C_{m-3} \\ lm - \frac{1}{2}(\gamma-1)(\gamma-2) & \dots\dots\dots (l+m-\gamma)\text{-ic} & m(m-\gamma) + \frac{1}{2}(\gamma-1)(\gamma-2) \\ & \text{Base-curve } C_m. & \end{array}$$

The curve C_{m-3} cuts the base-curve again in a point-group

$$m(m-\gamma) + \frac{1}{2}(\gamma-1)(\gamma-2),$$

residual to $lm - \frac{1}{2}(\gamma-1)(\gamma-2)$. Hence the $lm - \frac{1}{2}(\gamma-1)(\gamma-2)$ points lie on an $(l+m-\gamma)$ -ic, that is an n -ic, which does not pass through the $\frac{1}{2}(\gamma-1)(\gamma-2)$ points; which had to be proved.

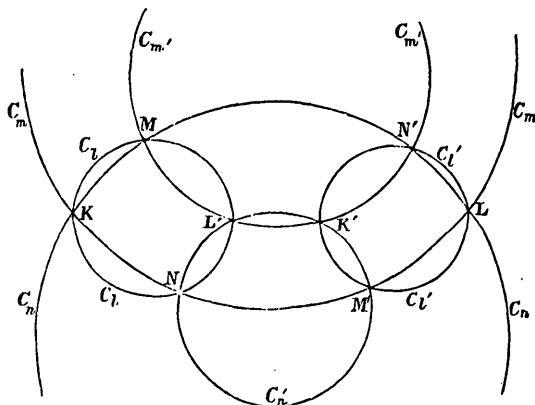
These two theorems are particular cases of more general ones, which are considered in the next section.

13. If $K = 1 + \frac{1}{2}(k-1)(k-2)$ general points be taken in a plane, and three proper curves C_l, C_m, C_n be drawn through them, cutting again in pairs in L, M , and N points; and on C_l any $L_1 = \frac{1}{2}(l-k+1)(l-k+2)$ more fixed general points be taken, and similarly any M_1 and N_1 points on C_m and C_n ; and three other curves C_r, C_m', C_n' , of orders $m+n-k, n+l-k, l+m-k$, be drawn through the $L+M_1+N_1, M+N_1+L_1$, and $N+L_1+M_1$ points respectively; then, just as C_l, C_m, C_n have K points common in the plane, and their remaining points of intersection taken in pairs, viz. L, M, N , on C_r, C_m', C_n' , so also C_r, C_m', C_n' have K' points common in the plane, and their remaining points of intersection taken in pairs, viz. L', M', N' on C_l, C_m, C_n ; where K' is equal to

$$mn + nl + lm - kl - km - kn + \frac{1}{2}(k-1)(k+4).*$$

* This is a generalization of a theorem given by OLIVIER, "Zur Theorie der Erzeugung geometrischen Curven" (*Crelle's Journal*, Bd. LXXI, p. 1). Cf. STUDY, "Ueber Schnittpunktfiguren ebener algebraischer Curven" (*Math. Ann.*, Bd. XXXVI, p. 216). The figure is a copy of one in Professor Study's paper.

The point-groups L_1, M_1, N_1 are not marked separately in the figure, but are included in the larger point-groups L', M', N' ; their numerical values being connected by the equations written below.



$$\begin{aligned} L &= mn - K, & L' &= L_1 + \frac{1}{2}(l-1)(l-2) = m'n' - K', \\ M &= nl - K, & M' &= M_1 + \frac{1}{2}(m-1)(m-2) = n'l' - K', \\ N &= lm - K, & N' &= N_1 + \frac{1}{2}(n-1)(n-2) = l'm' - K'. \end{aligned}$$

We shall first show that the conditions imposed on C_l, C_m, C_n are just sufficient to determine them. The curve C_l is required to pass through the $L + M_1 + N_1$ points; but

$$\begin{aligned} L + M_1 + N_1 &= mn - 1 - \frac{1}{2}(k-1)(k-2) + \frac{1}{2}(m-k+1)(m-k+2) \\ &\quad + \frac{1}{2}(n-k+1)(n-k+2) \\ &= \frac{1}{2}(m+n-k+1)(m+n-k+2) - 1 = \frac{1}{2}l'(l'+3). \end{aligned}$$

Hence a curve C_l can certainly be drawn through the $L + M_1 + N_1$ points; and no other l' -ic can be drawn through the same points, provided they all supply independent conditions for l' -ics. Now the L points, which are common to C_m, C_n , supply L independent conditions; for if they did not, the remaining K points common to C_m, C_n would lie on an $(m+n-l'-3)$ -ic, that is, a $(k-3)$ -ic, by Art. 19 (iv).

Also the remaining $M_1 + N_1$ points supply $M_1 + N_1$ more independent conditions; for if they did not, any curve S_r through the L points and all but one of the $M_1 + N_1$ points would necessarily pass through the last; and supposing this last to be one of the N_1 points, then S_r

passes necessarily through a point chosen arbitrarily on C_n , and therefore must be of the form $C_n S_{m-k}$; and S_{m-k} passes through the

$$M_1 = \frac{1}{2} (m-k+1)(m-k+2)$$

points chosen arbitrarily on C_m , which is impossible.* The curves C_l , C_m , C_n are therefore completely determined by the given conditions.

Again, taking C_m as base-curve, C_l passes through the point-group $L + M_1$ on C_m , and cuts C_m again in $\frac{1}{2} (m-1)(m-2)$ points, since

$$L + M_1 = ml' - \frac{1}{2} (m-1)(m-2).$$

Now the $M_1 + \frac{1}{2} (m-1)(m-2)$ points on C_m are residual to the L points, and therefore coresidual to the K points, and therefore residual to the N points. Hence the

$$N + M_1 + \frac{1}{2} (m-1)(m-2) = mn'$$

points lie on an n' -ic; but C_n passes through the $N + M_1$ points (which supply $N + M_1$ independent conditions), and therefore passes through the $\frac{1}{2} (m-1)(m-2)$ points. Thus C_l , C_m intersect in a point-group

$$M' = M_1 + \frac{1}{2} (m-1)(m-2)$$

on C_m . Hence the three curves C_l , C_m , C_n intersect in pairs in three groups of L , M' , N' points on C_l , C_m , C_n respectively, as shown in the figure.

The curves C_m , C_n have the point-group L' common, and cut again in a point-group

$$K' = m'n' - L'.$$

Hence the complete intersection of the two curves $C_m C_n$, $C_n C_l$, is made up of the $K + L + M + N + K' + L' + M' + N'$ points; but of these $K + L' + M + N = l(m+m')$ lie on C_l ; therefore the remaining $K' + L + M' + N' = l'(m+m')$ points lie on a curve of order l' , viz. C_l . This proves the theorem.

The value of K' is given by

$$K' = m'n' - L'$$

$$= (l+m-k)(l+n-k) - \frac{1}{2} (l-k+1)(l-k+2) - \frac{1}{2} (l-1)(l-2)$$

$$= mn + nl + lm - kl - km - kn + \frac{1}{2} (k-1)(k+4).$$

Similar properties hold for curves drawn through a point-group K of special form, in which case K might have any numerical value.

* It is assumed that k is a positive integer; if k were zero, the above reasoning would lead to the result that $C_l \equiv C_m C_n$.

III.

CHARACTERIZATION OF REST-GROUPS.

14. For definitions see Art. 2, ($g-m$). We shall suppose that the point-groups, whose characterization we are about to investigate, are made up of general clusters (including ordinary points), of given degree, finitely separated; that the degree of each cluster is a triangular number $\frac{1}{2}i(i+1)$; and that any curve drawn through a point-group has at a cluster $\frac{1}{2}i(i+1)$ either i or $i+1$ branches. Such a curve has therefore at an ordinary point of the point-group ($i=1$), either one or two branches; and at any point in the plane not belonging to the point-group ($i=0$), either no branch or one branch only. Also two curves C_i, C_m drawn through a point-group N will cut again in a rest-group N' , of the same type as N . Thus, if N has a cluster $\frac{1}{2}i(i+1)$ at A , and C_i, C_m both have i branches at A , then N' has a cluster $\frac{1}{2}i(i-1)$ at A ; if C_i has $i+1$ and C_m i branches, then N' has a cluster $\frac{1}{2}i(i+1)$; and if C_i, C_m both have $i+1$ branches, N' has a cluster $\frac{1}{2}(i+1)(i+2)$; all these clusters being general ones.

We shall define a K point-group on a given curve C_m as one which contains a general cluster of degree $\frac{1}{2}i(i-1)$ at each i -ple point of C_m , but no ordinary points; so that a curve adjoined to C_m , and a curve through a K point-group, are equivalent terms. If p is the deficiency of C_m , the degree of a K point-group is

$$K = \Sigma \frac{1}{2}i(i-1) = \frac{1}{2}(m-1)(m-2) - p.$$

Also the number of ordinary points in which an adjoined $(m-3)$ -ic cuts C_m is

$$m(m-3) - \Sigma i(i-1) = m(m-3) - (m-1)(m-2) + p = 2p-2;$$

and an adjoined $(m-3+r)$ -ic cuts C_m in $2p-2+rm$ ordinary points.

An $(m-3)$ -ic adjoined to C_m cannot be drawn when $p=0$, and does not cut C_m in any ordinary points when $p=1$. Hence, when considering adjoined $(m-3)$ -ics, we shall assume that $p>1$. It should be noticed however, that when $p=0$ or 1 , a K point-group supplies K independent conditions for $(m-3)$ -ics. (Cf. Art. 19, i.)

A square bracket enclosing a triangular number, as in Art. 15 (iii), denotes that the number is to be retained when its factors, in the form in which they are written, are positive; and rejected when negative.

15. (i.) *A simple rest-group (Art. 2, l) can be immediately derived from an incomplete point-group, but not from a redundant one.*

For, if N is incomplete, a simple rest-group N' can be immediately derived from N by drawing two curves through it, of which one at least does not pass through the remaining points of the plane which complete N . If however N is redundant, any immediately derived rest-group N' must be incomplete; since the general points of N , which have no connexion with the rest, will, if added to N' , at the least make N' more complete than before.

(ii.) Since the number of independent conditions supplied by a point-group N for n -ics is $N - r_n$, and the degree of freedom of an n -ic through N is q_n , we have the relation

$$N - r_n + q_n = \frac{1}{2}n(n+3) \dots\dots\dots(1).$$

(iii.) *If n -ics are drawn through any point-group N on a base-curve C_m , the multiplicity of the system of point-groups in which they cut C_m again is*

$$q_n - \left[\frac{1}{2}(n-m+1)(n-m+2) \right] \dots\dots\dots(2).$$

Let S_n be any n -ic through N , cutting C_m again in $mn - N$. Then the n -ic defect of N , viz. q_n , is the number of general points on C_m , together with the number of additional general points in the plane, through which S_n can be drawn; and the former number is the multiplicity of $mn - N$; and the latter is $\frac{1}{2}(n-m+1)(n-m+2)$ or 0, according as $n \geq m$ or $n < m$. Hence the multiplicity of $mn - N$ is

$$q_n - \left[\frac{1}{2}(n-m+1)(n-m+2) \right].$$

16. EXAMPLE.—*The n -ic excess of a point-group $N = lm + a$ on a base-curve C_m , coresidual to a group of $a < \frac{1}{2}(n-l+1)(n-l+2) > 0$ general points, none of which belong to N , is $\frac{1}{2}(l+m-n-1)(l+m-n-2)$; provided $n > l < l+m$, and also $n > m-3$.*

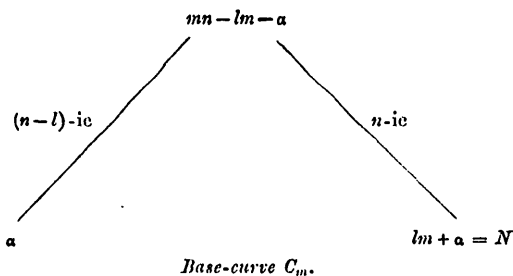
We take $a > 0$, because the theorem has been already proved for the case $a = 0$ (Theorem II, iv); and $a < \frac{1}{2}(n-l+1)(n-l+2)$; because otherwise an $(n-l)$ -ic could not be drawn through a , nor an n -ic through N , except one of the form $C_m S_{n-m}$. Also we take $n > l$, because $N > lm$; $n < l+m$, because the $(l+m-2)$ -ic excess is zero; and $n > m-3$, because no point-group on C_m coresidual to a general point-group can lie on a curve of lower order than $m-2$.*

* It can be proved that no residual of a general points can lie on an $(m-3)$ -ic, or curve of lower order, which does not pass through the point-group a .

If $l < m$, then it can be proved that $a > \frac{1}{2}(m-l-1)(m-l-2)$; if however $l > m-3$, a might have any value from 1 to $\frac{1}{2}m(m-3)$.

Now an $(n-l)$ -ic through the a general points cuts C_m again in a group of $mn-lm-a$ points, whose multiplicity (since $n-l < m$) is

$$\frac{1}{2} (n-l)(n-l+3) - a.$$



The point-groups $mn-lm-a$ and N are residual. Hence an n -ic can be drawn through the point-group N ; and, by Art. 15 (iii), the multiplicity of $mn-lm-a$ (since $n > m-3$) is

$$q_n - \frac{1}{2} (n-m+1)(n-m+2).$$

Equating this to the value found above, we have

$$q_n = \frac{1}{2} (n-m+1)(n-m+2) + \frac{1}{2} (n-l)(n-l+3) - a.$$

But, by Art. 15 (ii),

$$lm+a-r_n+q_n = \frac{1}{2} n(n+3);$$

therefore $r_n = lm+a + \frac{1}{2} (n-m+1)(n-m+2)$

$$+ \frac{1}{2} (n-l)(n-l+3) - a - \frac{1}{2} n(n+3)$$

$$= \frac{1}{2} (l+m-n-1)(l+m-n-2).$$

Thus r_n is zero, when $n = l+m-2$; and therefore also when n has any higher value.

Suppose that a contains a K point-group (Art. 14); then so also does N . Hence, taking $a = K + \beta$ and $N = K' + R$, the result proved may be expressed as follows:—If a group of $K = lm + \beta$ ordinary points on a base-curve C_m , of deficiency p , has a non-specialized adjoined coresidual of $\beta < \frac{1}{2} (n-l+1)(n-l+2) - \frac{1}{2} (m-1)(m-2) + p > 0$ general points on C'_m , then the excess of R for adjoined n -ics is equal to $\frac{1}{2} (l+m-n-1)(l+m-n-2)$; provided a K point-group has no excess for $(n-l)$ -ics, and $n > l < l+m$, $n > m-3$.

17. THEOREM IV.—*The necessary and sufficient condition that a point-group N , on a base-curve C_m , should lie on an $(m-3)$ -ic, is*

$$\rho \geq N - \frac{1}{2}m(m-3);$$

ρ being the multiplicity of N .

We regard the given point-group N as determining a series of other point-groups N , forming a complete coresidual system, of multiplicity ρ (Art. 2, m).

(i.) Suppose that the given point-group N lies on an $(m-3)$ -ic; and that a fixed $(m-3)$ -ic is drawn through it, cutting C_m again in N' ($N+N' = m \cdot \overline{m-3}$). Then every point-group N is residual to N' ; therefore $\rho \geq \frac{1}{2}m(m-3) - N' \geq N - \frac{1}{2}m(m-3)$.

(ii.) Conversely, suppose that $\rho \geq N - \frac{1}{2}m(m-3)$. Then we have to prove that N lies on an $(m-3)$ -ic. All the point-groups N may possibly have a certain number of fixed points in common; but, whether they have or not, it is clear that ρ points, say $A, B, C, \dots H$, can be chosen out of any point-group N , which do not all belong to any other point-group of the system.

Take another point-group of the system containing $B, C, \dots H$, but not A ; and denote the remainders of the two point-groups, when $B, C, \dots H$ are taken away, by $R_{N-\rho+1}$, $R'_{N-\rho+1}$ respectively. Then

$$R_{N-\rho+1} \equiv R'_{N-\rho+1}. \quad (\text{Art. 5})$$

Draw any straight line through A , cutting C_m again in Q_{m-1} ; and any $(m-3)$ -ic through the remaining $N-\rho$ ($\leq \frac{1}{2}m \cdot \overline{m-3}$) points of $R_{N-\rho+1}$, cutting C_m again in the point-group $Q_{m(m-3)-N+\rho}$. Then

$$Q_{m-1} + Q_{m(m-3)-N+\rho} + R_{N-\rho+1} \equiv 0;$$

$$\text{therefore} \quad Q_{m-1} + Q_{m(m-3)-N+\rho} + R'_{N-\rho+1} \equiv 0.$$

These last $m(m-2)$ points therefore lie on an $(m-2)$ -ic, which must break up into the straight line containing the $(m-1)$ points Q_{m-1} and an $(m-3)$ -ic. But the straight line through Q_{m-1} cuts C_m again in A ; hence the point A must be included in

$$Q_{m-1} + Q_{m(m-3)-N+\rho} + R'_{N-\rho+1};$$

i.e., A is necessarily included in $Q_{m(m-3)-N+\rho}$. Hence any $(m-3)$ -ic through the $N-\rho$ points of N passes necessarily through A , and

similarly through all the ρ points. Thus N lies on an $(m-3)$ -ic, and its $(m-3)$ -ic excess is not less than ρ .

To apply the theorem to the case of adjointed curves, we suppose that N consists of a K point-group (Art. 14), and R ordinary points on C_m . Then

$$N = K + R = \frac{1}{2}(m-1)(m-2) - p + R;$$

therefore

$$N - \frac{1}{2}m(m-3) = R - p + 1.$$

Hence the necessary and sufficient condition that N lies on an $(m-3)$ -ic, or that R lies on an adjointed $(m-3)$ -ic, is

$$\rho \geq R - p + 1;$$

where ρ is the multiplicity of N , and therefore also of R .

18. THE RIEMANN-ROCH THEOREM.—*The multiplicity of any point-group on a base-curve C_m is equal to its $(m-3)$ -ic excess.*

If the point-group N does not lie on an $(m-3)$ -ic, then the number of independent conditions it supplies for $(m-3)$ -ics is $\frac{1}{2}(m-1)(m-2)$, by Art. 7; and its $(m-3)$ -ic excess is therefore $N - \frac{1}{2}(m-1)(m-2)$. Also its multiplicity on C_m is less than $N - \frac{1}{2}m(m-3)$, by Theorem IV; and is not less than $N - \frac{1}{2}(m-1)(m-2)$, since this number of points at least can be chosen arbitrarily for determining a point-group on C_m belonging to a given coresidual system of degree N (Theorem I). Hence the multiplicity of N is $N - \frac{1}{2}(m-1)(m-2)$; i.e., is equal to its $(m-3)$ -ic excess.

If the point-group N lies on an $(m-3)$ -ic, let an $(m-3)$ -ic be drawn through it, cutting C_m again in N' ($N + N' = m(m-3)$); and let q_{m-3} , r_{m-3} , ρ and q'_{m-3} , r'_{m-3} , ρ' denote the $(m-3)$ -ic defects, $(m-3)$ -ic excesses, and multiplicities, of N and N' respectively.

Then, by Art. 15 (iii), $\rho = q'_{m-3}$, $\rho' = q_{m-3}$;

and, by Art. 17, $r_{m-3} \geq \rho \geq q'_{m-3}$, $r'_{m-3} \geq \rho' \geq q_{m-3}$;

and, by Art. 15 (ii), $r_{m-3} - q_{m-3} = N - \frac{1}{2}m(m-3)$

$$= N - \frac{1}{2}(N + N') = \frac{1}{2}(N - N') = q'_{m-3} - r'_{m-3}.$$

But neither $r_{m-3} - q'_{m-3}$ nor $r'_{m-3} - q_{m-3}$ is negative, from above;

therefore $r_{m-3} = q'_{m-3} = \rho$, and $r'_{m-3} = q_{m-3} = \rho'$.

Thus, if $N + N' = m(m-3) \equiv 0$, the $(m-3)$ -ic defect and excess of N are equal respectively to the $(m-3)$ -ic excess and defect of N' ; and the difference of the multiplicities of N, N' , viz. $\rho - \rho'$, is equal to $\frac{1}{2}(N - N')$.

19. The theorem of residuation and the Riemann-Roch theorem express the two fundamental properties of point-groups on curves. Some of the immediate consequences of the latter theorem are the following.

(i.) A K point-group on C_m supplies K independent conditions for $(m-3)$ -ics; and an $(m-3)$ -ic through K , i.e., an adjoined $(m-3)$ -ic, does not of necessity pass through any fixed ordinary point on C_m .

Draw any $(m-3)$ -ic through K , cutting C_m again in a point-group $K' + 2p - 2$, of which $2p - 2$ are ordinary points on C_m , and K' is a point-group of the same kind as K . Then, if the $(m-3)$ -ic excess of K is not zero, the $(m-3)$ -ic defect of $K' + 2p - 2$ is not zero (Art. 18), and an $(m-3)$ -ic can be drawn through $K' + 2p - 2$, and one or more other arbitrary points on C_m . We should then have an adjoined $(m-3)$ -ic cutting C_m in more than $2p - 2$ ordinary points, which is impossible. Hence the $(m-3)$ -ic excess of a K point-group is zero.

Again, if an $(m-3)$ -ic through K passes necessarily through a fixed ordinary point A on C_m , the $(m-3)$ -ic excess of the point-group consisting of K and the point A is 1; and the $(m-3)$ -ic defect of a residual point-group $K' + 2p - 3$ is also 1. Hence an $(m-3)$ -ic through $K' + 2p - 3$, which passes necessarily through A , by hypothesis, can be made to pass through another arbitrary point on C_m , and cuts C_m in $2p - 1$ ordinary points altogether, which is impossible.

Since the $(m-3)$ -ic excess of K is zero, the multiplicity of the groups of $2p - 2$ ordinary points cut from C_m by adjoined $(m-3)$ -ics is

$$\frac{1}{2}m(m-3) - K = p - 1.$$

(ii.) If the $(m-3)$ -ic excess of any point-group $K + 2$ on C_m is 1, the two ordinary points of $K + 2$ belong to a system of adjoined co-residual point-pairs on C_m . For the multiplicity of $K + 2$ is 1 (Art. 18); and any co-residual of $K + 2$, of the same degree, is a point-group $K' + 2'$, containing 2 ordinary points.

Also any adjoined $(m-3)$ -ic through one point of a point-pair passes necessarily through the other, by (i); and therefore the $2p - 2$ ordinary points, in which any adjoined $(m-3)$ -ic cuts C_m , consist of $(p-1)$ point-pairs. Curves which contain such systems of point-pairs are called *hyperelliptic* curves, when $p > 1$. A curve whose deficiency is 2 is

necessarily hyperelliptic. If N is any point-group for which $r_{m-3} = 1$; and an m -ic C_m can be described having an i -ple point ($i > 2$) at each cluster of N of degree $\frac{1}{2}i(i-1)$, and a double point at each of the ordinary points of N except two, and passing through the last two; then C_m is hyperelliptic.

(iii.) If an n -ic C_n cuts C_m in two residual point-groups N , N' , and r_n is not zero; then N' lies on an $(m-3)$ -ic, which does not pass through N .

For, if $n \geq m-2$, the multiplicity of N' on C_m is

$$\begin{aligned} \rho' &= q_n - \frac{1}{2}(n-m+1)(n-m+2) & (\text{Art. 15, iii}) \\ &= \frac{1}{2}n(n+3) - (mn - N') + r_n - \frac{1}{2}(n-m+1)(n-m+2) \\ &= N' - \frac{1}{2}m(m-3) + (r_n - 1) \\ &\geq N' - \frac{1}{2}m(m-3). \end{aligned}$$

Hence N' lies on an $(m-3)$ -ic (Theorem IV), which does not pass through N ; since $n \geq m-2$.

Similarly, if $n \leq m-3$,

$$r'_{m-3} = \rho' = q_n = \frac{1}{2}n(n+3) - (mn - N') + r_n;$$

$$\begin{aligned} \text{therefore } N' - r'_{m-3} &= mn - \frac{1}{2}n(n+3) - r_n \\ &= (m-3)n - \frac{1}{2}(n-1)(n-2) - (r_n - 1) \\ &\leq (m-3)n - \frac{1}{2}(n-1)(n-2). \end{aligned}$$

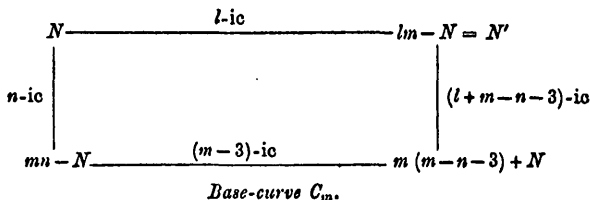
But this is the condition that an $(m-3)$ -ic can be drawn through N' , which is not of the form $C_n S_{m-3-n}$, or which does not pass through N .

If the point-group N is redundant, any $(m-3)$ -ic through N' passes necessarily through the redundant points of N , but not through the remainder.

Conversely, if r_n is zero, the point-group N' does not lie on an $(m-3)$ -ic, when $n \geq m-2$; and can only lie on an $(m-3)$ -ic which passes through N , when $n \leq m-3$.

(iv.) If two curves C_l, C_m through a point-group N cut again in a rest-group N' , and if $q_n \geq 0$, $r_n \geq 1$, $r_{n+1} = 0$; then the curve of lowest order through N' , which does not pass through N , is an $(l+m-n-3)$ -ic.

For an n -ic through N cuts C_m again in a point-group $mn-N$; which lies on an $(m-3)$ -ic, by (iii); this cuts C_m again in a point-



group $N+m(m-n-3)$, which does not contain all the N points. Hence N' lies on an $(l+m-n-3)$ -ic, which does not pass through N ; and by similar reasoning, since r_{n+1} is zero, N' cannot lie on an $(l+m-n-4)$ -ic which does not pass through N .

20. THEOREM V.—If any two curves C_l, C_m be drawn through a point-group N , cutting again in a rest-group N' ($N+N' = lm$); then, provided $q_n \geq 0$, $r_n \geq 1$,

$$r'_{l+m-n-3} = q_n + 1 - \left[\frac{1}{2}(n-l+1)(n-l+2) \right] - \left[\frac{1}{2}(n-m+1)(n-m+2) \right] \dots\dots\dots(3),$$

$$q'_{l+m-n-3} = r_n + 1 + \left[\frac{1}{2}(l-n-1)(l-n-2) \right] + \left[\frac{1}{2}(m-n-2)(m-n-2) \right] \dots\dots\dots(4).$$

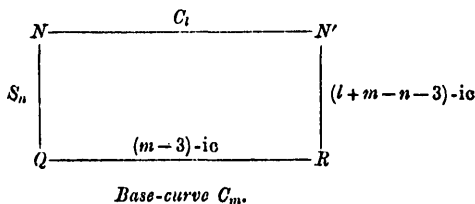
This theorem practically determines the complete characterization of N' , if that of N is known; for $q_n \geq 0$, $r_n \geq 1$ are the necessary and sufficient conditions that an $(l+m-n-3)$ -ic can be drawn through N' , without passing through N (Art. 19, iv). In the case of these curves drawn through N' which, as a consequence, pass through N , the excess of N' is that of $N+N'$ diminished by N * (See Note, p. 504.)

Also, assuming $l \leq m$, it should be noticed that the general n -ic S_n through N , although it may necessarily pass through N' , has no factor in common with C_m , and therefore cuts C_m again in a finite rest-group Q .

Equation (4) is only another form of (3), and can be deduced from it by means of Art. 15 (ii). Hence it is only necessary to prove (3).

* By taking $q_n = -1$ when N does not lie on an n -ic (Note, p. 507), the equations (3), (4) can be proved to hold for all values of n .

(i.) Suppose that S_n does not pass through N' , so that Q does not contain all the N' points. Then, since r_n is not zero, Q lies on an



$(m-3)$ -ic (Art. 19, iii). Let any $(m-3)$ -ic through Q cut C_m again in R . Then N', R are residual and lie on an $(l+m-n-3)$ -ic.

By Art. 15 (iii), the multiplicity of R is

$$q'_{l+m-n-3} - \left[\frac{1}{2} (l-n-1)(l-n-2) \right];$$

and the multiplicity of Q is

$$q_n - \left[\frac{1}{2} (n-m+1)(n-m+2) \right].$$

But, by Art. 18, the difference of these is $\frac{1}{2} (R-Q)$; hence

$$\begin{aligned} q'_{l+m-n-3} - \left[\frac{1}{2} (l-n-1)(l-n-2) \right] - q_n + \left[\frac{1}{2} (n-m+1)(n-m+2) \right] \\ = \frac{1}{2} (R-Q) = R - \frac{1}{2} (Q+R) \\ = m(l+m-n-3) - N' - \frac{1}{2} m(m-3); \end{aligned}$$

therefore

$$\begin{aligned} r'_{l+m-n-3} &= N' + q'_{l+m-n-3} - \frac{1}{2} (l+m-n)(l+m-n-3) \quad (\text{Art. 15, ii.}) \\ &= q_n + \left[\frac{1}{2} (l-n-1)(l-n-2) \right] - \left[\frac{1}{2} (n-m+1)(n-m+2) \right] \\ &\quad + m(l+m-n-3) - \frac{1}{2} m(m-3) - \frac{1}{2} (l+m-n)(l+m-n-3) \\ &= q_n + \left[\frac{1}{2} (l-n-1)(l-n-2) \right] - \left[\frac{1}{2} (n-m+1)(n-m+2) \right] \\ &\quad - \frac{1}{2} (l-n)(l-n-3) \\ &= q_n + 1 - \left[\frac{1}{2} (n-l+1)(n-l+2) \right] - \left[\frac{1}{2} (n-m+1)(n-m+2) \right]. \end{aligned}$$

(ii.) Suppose that S_n necessarily passes through N' . Then Q contains N' , and any $(m-3)$ -ic through R passes through a point-group Q , and therefore passes through all the N' points. Hence the $(l+m-n-3)$ -ic excess of N' is zero (Art. 19, iii). Again, since S_n necessarily passes through N' , n cannot be less than l ; and, if n is less than m , S_n must contain C_l as a factor; therefore

$$q_n = \frac{1}{2} (n-l)(n-l+3) + \left[\frac{1}{2} (n-m+1)(n-m+2) \right].$$

Hence the relation (3) gives $r'_{l+m-n-3} = 0$, which is the correct value. Conversely, if $r'_{l+m-n-3} = 0$, an n -ic drawn through N must also pass through N' .

Hence an n -ic ($n \geq l \geq m < l+m-2$) through N points common to two given curves C_l, C_m does or does not necessarily pass through the remaining N' points according as the $(l+m-n-3)$ -ic excess of N' is or is not zero.

21. EXAMPLE.—Suppose that three proper curves C_l, C_m, C_n ($l \leq m, m \leq n$) have N points common in all, which are moreover N general points of intersection of an l -ic and m -ic. To find the excesses of the rest-group N' common to C_m, C_n .

From the conditions of the question we must have

$$N < lm - \left[\frac{1}{2} (l+m-n-1)(l+m-n-2) \right];$$

but this, and the condition $N \geq \frac{1}{2}l(l+3)$, are included in those found below. From the relation (3) of the last article, we have

$$\begin{aligned} r'_{m+n-p-3} &= q_p + 1 - \left[\frac{1}{2} (p-m+1)(p-m+2) \right] \\ &\quad - \frac{1}{2} [(p-n+1)(p-n+2)], \end{aligned}$$

provided $q_p \geq 0$, and $r_p \geq 1$.

Now the condition $q_p \geq 0$ is equivalent to $p \geq l$. Also the condition $r_p \geq 1$ is equivalent to $N + q_p \geq \frac{1}{2}(p+1)(p+2)$, by Art. 15 (ii); and requires p to be less than n , since $r_n = 0$. Hence the last term in the value of $r'_{m+n-p-3}$ above, must be omitted.

Again, since the N points are general points of intersection of an l -ic and m -ic, and r_p is not zero; a p -ic through N must have C_l for a factor if $p < m$, and must pass through all points common to C_l, C_m , if $p \geq m$; hence we have

$$q_p = \frac{1}{2} (p-l)(p-l+3) + \left[\frac{1}{2} (p-m+1)(p-m+2) \right].$$

Substituting this in the value of $r'_{m+n-p-3}$, we have

$$r'_{m+n-p-3} = \frac{1}{2} (p-l+1)(p-l+2),$$

provided $p \geq l < n$, and $N + q_p \geq \frac{1}{2} (p+1)(p+2)$.

Changing p to $m+n-p-3$, we may express the result in the form

$$r'_p = \frac{1}{2}(m+n-l-p-1)(m+n-l-p-2),$$

provided $N > l(m+n-p) - \frac{1}{2}l(l+3) - [\frac{1}{2}(n-p-1)(n-p-2)]$, and $p < m+n-l > m-3$.

This result is a similar one to that in Art. 16, but more general.

22. THEOREM VI.—If a point-group of given degree N has, as a consequence of satisfying certain given conditions, a definite number k of absolute connexions, and a definite absolute n -ic multiplicity x_n (Art. 2, m); then

$$N - x_n + r_n = k \dots\dots\dots(5).$$

The number k is the number of independent geometrical connexions that exist between any N points which satisfy the given conditions; and $2N-k$ is the least number of parameters in terms of which the $2N$ coordinates of any such N points can be expressed.

The given conditions might be of such a kind that, although n -ics could be drawn through any point-group N , such n -ics would be necessarily specialized. In that case the equation (5) will still hold if x_n and r_n denote the absolute multiplicity and excess of N for the specialized n -ics. A simple example of this is the point-group formed by the vertices and orthocentre of any triangle ($N=4$, $k=2$); all conics through any such point-group being equilateral hyperbolas.

Since x_n has a definite value, it follows that a point-group N can be placed on a general n -ic; and this implies that the given conditions do not necessitate the point-groups having any clusters. This, however, does not prevent some of the x_n arbitrary points on a given n -ic being chosen at the multiple points (if there are any) so as to form clusters; although it will generally happen that there is a limit, less than x_n , to the number that can be chosen in this way. As bearing on this, it should be noticed that x_n is the number of *general* points that can be chosen arbitrarily on a given n -ic (Art. 2, m); so that, if the x_n points, or part of them, are chosen in a special position, the problem of finding the remainder may become a porismatic one, either having no solutions, or else having a greater multiplicity of solutions than it would have in the general case.

The relation (5) may be regarded as determining k if r_n and x_n are known for one particular value of n ; and, when k is known, as determining x_n for all values of n for which r_n is known.

The theorem can be applied to a point-group with a given characterization, by finding the simplest construction for such a point-group, and deducing therefrom the value of k (Arts. 26-31).

(i.) Suppose that $r_n = 0$; then we have to prove that

$$N - x_n = k, \quad \text{or} \quad x_n = N - k.$$

It is clear that x_n is not less than $N - k$, since the k absolute connexions cannot make more than k points of any point-group N on a given curve to be dependent on the rest.

Again, x_n is not greater than $N - k$. For the coordinates of all the points of any point-group N are expressible in terms of $2N - k$ parameters, and if these coordinates so expressed be substituted in the general equation $S_n = 0$ of a curve of the n^{th} order, we obtain N equations involving the coefficients of S_n and the $2N - k$ parameters; and these equations are independent, since $r_n = 0$. Now, regarding S_n as a given curve, each point on it has one parameter; and each of the parameters of the N points on S_n are expressible in terms of the coefficients of S_n and the $2N - k$ parameters of the point-group N . If therefore more than $N - k$ general points of a group N on S_n could be chosen arbitrarily, we should, by equating their parameters to general arbitrary values α, β, \dots , have more than $N - k$ additional independent equations; making with the N equations more than $2N - k$ altogether. The $2N - k$ parameters of the point-group N could therefore be eliminated, giving rise to one or more equations between the coefficients of S_n and the arbitrary quantities α, β, \dots , which is impossible. Hence x_n must be equal to $N - k$.

(ii.) If $r_n > 0$; choose $N - r_n$ points of any point-group N which supply $N - r_n$ independent conditions for n -ics. Then if $q_n > 0$, i.e., if $N - r_n < \frac{1}{2}n(n+3)$, the remaining r_n points of N are dependent on the $N - r_n$ points, since they lie on all n -ics through the $N - r_n$ points; and the $N - r_n$ points have therefore only $k - 2r_n$ absolute connexions. Hence, applying the result of (i) to the $N - r_n$ points, we have

$$x_n = (N - r_n) - (k - 2r_n), \quad \text{or} \quad N - x_n + r_n = k.$$

If $r_n > 0$, and $N - r_n = \frac{1}{2}n(n+3)$; only one n -ic C_n can be drawn through a given point-group N ; and all the $\frac{1}{2}n(n+3)$ coefficients of C_n , and the x_n parameters of x_n general points on C_n , are expressible in terms of the $2N - k$ parameters of N ; and vice versa. Hence

$$\frac{1}{2}n(n+3) + x_n = 2N - k, \quad \text{or} \quad N - x_n + r_n = k.*$$

* A difficult problem in enumerative geometry, is to find the number of point-groups N on a given n -ic, corresponding to x_n general points chosen arbitrarily on

23. The two following examples illustrate Theorem VI.

(i.) To find the number of absolute connexions of the point-group formed by the complete intersection of any l -ic and m -ic.*

If $l < m$, we have

$$N = lm, \quad x_m = \frac{1}{2}l(l+3), \quad r_m = \frac{1}{2}(l-1)(l-2);$$

$$\text{hence} \quad k = lm - \frac{1}{2}l(l+3) + \frac{1}{2}(l-1)(l-2) = lm - 3l + 1.$$

If $l = m$, we have

$$N = m^2, \quad x_m = \frac{1}{2}m(m+3) - 1, \quad r_m = \frac{1}{2}(m-1)(m-2);$$

$$\text{hence} \quad k = m^2 - \frac{1}{2}m(m+3) + 1 + \frac{1}{2}(m-1)(m-2) = (m-1)(m-2).$$

(ii.) To find the number x_n of general points that can be chosen on a given curve C_n which belong to a point-group on C_n formed by the complete intersection of an l -ic and m -ic.

We assume that an n -ic ($n \geq l \geq m$) through all the points common to a general l -ic and m -ic is not a specialized n -ic; from which it follows that a point-group forming the complete intersection of an l -ic and m -ic can be placed on a given n -ic.

If $l < m$; then

$$N = lm, \quad r_n = \left[\frac{1}{2}(l+m-n-1)(l+m-n-2) \right], \quad k = lm - 3l + 1;$$

therefore

$$lm - x_n + \left[\frac{1}{2}(l+m-n-1)(l+m-n-2) \right] = lm - 3l + 1,$$

$$\text{or} \quad x_n = 3l - 1 + \left[\frac{1}{2}(l+m-n-1)(l+m-n-2) \right].$$

$$\text{If } l = m; \text{ then} \quad k = m^2 - 3m + 2;$$

$$\text{and} \quad x_n = 3m - 2 + \left[\frac{1}{2}(2m-n-1)(2m-n-2) \right]. \dagger$$

the n -ic. The solution will, of course, depend on the nature of the given conditions. CASTELNUOVO (*Rendiconti della Reale Accademia dei Lincei*, v, p. 130) has given a solution for the number of ordinary point-groups, of lowest degree, on a given curve C_m , having a given multiplicity r ; when p , the deficiency of C_m , is divisible by $r+1$. BRILL (*Math. Ann.*, xxxvi, p. 321), and ZEUTHEN (*Math. Ann.*, xl, p. 119), have also given solutions for the number of ordinary point-groups, of given degree, on C_m , cut out by adjoined n -ics; when the excess of each point-group for adjoined n -ics is 1.

* Cf. JACOBI (*Crelle's Journal*, xv, p. 285).

† Cf. CREMONA (*Teoria Geometrica delle Curve Piane*, Bologna, p. 46), and CLEBSCH-LINDEMANN (*Leçons sur la Géométrie*, T. III, p. 129), for the case $l = m$.

24. THEOREM VII.—If k, k' are the numbers of absolute connexions of two point-groups N, N' ($N + N' = lm$), which satisfy such given conditions that any l -ic and m -ic through any point-group N or N' cut again in a point-group N' or N respectively; then

$$k - r_l - r_m = k' - r'_l - r'_m \dots\dots\dots (6).$$

The enunciation imposes a strict limitation to the given conditions, in addition to the limitations explicitly stated in Theorem VI; but the theorem certainly applies when either N or N' is a rest-group, derived by any number of steps, from a general point-group with no absolute connexions.

Suppose that we take any fixed base-curve C_m , and find the least number of parameters in terms of which the positions of all the points of two residuals N, N' on C_m , satisfying the given conditions, can be expressed. The number of general points that can be chosen arbitrarily which belong to a point-group N on C_m , is x_m ; but when N is determined, we can still choose ρ' points of N' arbitrarily, ρ' being the multiplicity of N' on C_m . Hence $x_m + \rho'$ is the least number of parameters required. Similarly $x'_m + \rho$ is also the least number of parameters; hence

$$x_m + \rho' = x'_m + \rho,$$

or

$$x_m - x'_m = \rho - \rho'.$$

But

$$\rho = q'_l - \left[\frac{1}{2} (l - m + 1)(l - m + 2) \right],$$

and

$$\rho' = q_l - \left[\frac{1}{2} (l - m + 1)(l - m + 2) \right]; \quad (\text{Art. 15, iii})$$

therefore

$$\begin{aligned} x_m - x'_m &= q'_l - q_l \\ &= (N - r_l) - (N' - r'_l), \end{aligned} \quad (\text{Art. 15, ii})$$

or

$$N - x_m - r_l = N' - x'_m - r'_l.$$

But

$$N - x_m = k - r_m,$$

and

$$N' - x'_m = k' - r'_m; \quad (\text{Theorem VI})$$

therefore

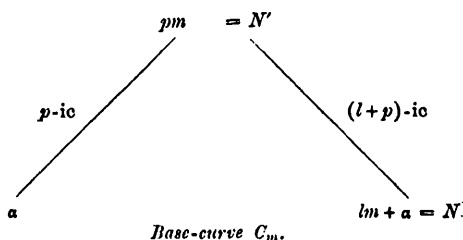
$$k - r_l - r_m = k' - r'_l - r'_m.$$

Some, or all, of the quantities r_l, r_m, r'_l, r'_m , may be zero.

25. EXAMPLE.—To find the number of absolute connexions of a group $N = lm + a$ on C_m , which has a coresidual of $a < \frac{1}{2}(m-1)(m-2)$ general points.

This point-group is considered in Art. 16, where the value of r_n is found. Suppose $p (\leq m-3)$, the order of the lowest curve through a , cutting C_m again in $pm - a = N'$. Then the N' points are general points on a p -ic; and, by Theorem VII,

$$k - r'_m - r_{l+p} = k' - r'_m - r'_{l+p};$$



but $r'_{l+p} = 0$, $k' - r'_m = N - x'_m$

$$= (pm - a) - \frac{1}{2}p(p+3),$$

$$r_{l+p} = \frac{1}{2}(m-p-1)(m-p-2); \quad (\text{Art. 16})$$

hence $k = r_m + \frac{1}{2}(m-p-1)(m-p-2) + pm - a - \frac{1}{2}p(p+3)$

$$= r_m - a + \frac{1}{2}(m-1)(m-2).$$

And, according as only one m -ic C_m , or more than one m -ic can be drawn through N ; the conditions for the latter case being that $m > l$, and $a < \frac{1}{2}(m-l+1)(m-l+2)$; we have

$$r_m = lm + a - \frac{1}{2}m(m+3), \quad \text{or} \quad \frac{1}{2}(l-1)(l-2); \quad (\text{Art. 16})$$

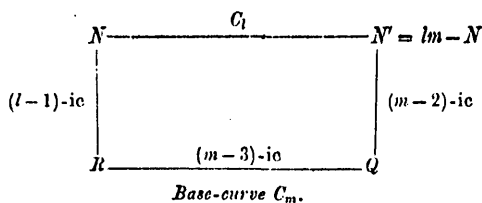
and $k = lm - 3m + 1$, or $\frac{1}{2}(l+m-1)(l+m-2) - N + 1.$ *

Hence, since k is known, and also the value of r_n for all values of n , the value of x_n is also known (Theorem VI).

* In this case, $k > lm - 3m + 1 < lm - 3l + 1$; for $a < \frac{1}{2}(m-l+1)(m-l+2)$, from above; and $a > \frac{1}{2}(m-l-1)(m-l-2)$, since $l < m$ (Note, Art. 16).

26. CONSTRUCTION OF POINT-GROUPS WHOSE CHARACTERIZATION IS GIVEN.—By means of Theorem V, we can prove that from a non-composite point-group N , whose characterization is given, we can derive a series of rest-groups in succession, each one of which has a less complex characterization than the preceding, until we arrive ultimately at a general point-group.

Suppose that m is the order of the lowest curve C_m that passes through N . By means of the relation $N - r_n + q_n = \frac{1}{2}n(n+3)$, we can write down the values of q_m, q_{m+1}, \dots , those of r_m, r_{m+1}, \dots being given. If q_m is not zero, a second m -ic can be drawn through N , cutting C_m again in a rest-group $m^2 - N$; but, if q_m is zero, then no other m -ic besides C can be drawn through N . In this case we take l to be the order of the lowest curve which passes through N , and does not contain C_m as a factor; and we shall consider this case, as being the more general, since it reduces to the former case when $l = m$. The value of l is found from the first term of the series $q_m, q_{m+1}, q_{m+2}, \dots$ which is greater than the corresponding term of the series $0, 2, 5, 9, \dots$. Let n be the order of the highest curve for which the excess of N does not vanish; so that $r_{n+1} = 0, r_{n+2} = 0$, &c. Then the order of the lowest curve which passes



through N' is $l + m - n - 3$ (Art. 19, iv). Again, the point-group N' cannot have an excess for any curve of order higher than $m-3$; for, if it had an $(m-2)$ -ic excess, it will be seen by referring to the figure that Q lies on an $(m-3)$ -ic (Art. 19, iii); and that R, N lie on an $(l-1)$ -ic; which is contrary to the hypothesis that an l -ic is the lowest curve through N which does not contain C_m as a factor. It may, however, happen that the $(m-3)$ -ic, $(m-4)$ -ic, &c., excesses of N' vanish; such, in fact, will be the case when l -ics, $(l+1)$ -ics, &c., which pass through N , necessarily pass through N' .

Hence we have only to consider the excesses and defects of N' for curves of all orders from $l + m - n - 3$ up to $m-3$. Now in (4), Theorem V, the factors of the triangular numbers are negative for all these cases; and we therefore have

$$q'_{l+m-n-3} = r_n - 1, \quad q'_{l+m-n-2} = r_{n-1} - 1, \quad \dots \quad q'_{m-3} = r_1 - 1;$$

also $r'_{m-3} = q_l - \frac{1}{2}(l-m+1)(l-m+2),$

by (3), Theorem V. From these results we can arrange the following table of reduction:—

N	m	$m+1$	l	$l+1$	n
Defect	$q_m = 0$	$q_{m+1} = 2$	q_l	q_{l+1}	q_n
Excess	r_m	r_{m+1}	r_l	r_{l+1}	r_n
$N' = lm - N$			$l+m-n-3$	$l+m-n-2$	$m-3$
Defect			$r_n - 1$	$r_{n-1} - 1$	$r_l - 1$
Excess					$q_l - \frac{1}{2}(l-m+1)(l-m+2)$

In this table, there are three rows corresponding to each point-group; the first containing the degree N , and the orders $m, m+1, \dots$ of all the curves for which N has any excess; and the second and third, the excesses and defects of N corresponding to the several curves. The positions of the columns which are not inserted are indicated by the thick vertical lines. The order $l+m-n-3$ of the lowest curve through N' is placed in the same column as the order l of the lowest curve through N which determines a rest-group of N , viz. N' ; and the order $m-3$ of the highest possible curve for which N' can have an excess thus falls in the last column. If $q_m > 0$, so that $l = m$, the defects and excesses of N' will occupy all the columns.

The defects of N' will then be the excesses of N written in the reverse order and diminished by unity. Only the last of the excesses of N' has been inserted, the remainder being more easily obtained from (1), Art. 15, than from (3), Theorem V, as explained in the examples given below.

To give general examples of this method of reduction, in which the n -ic excess of N is expressed as an algebraical function of n , would be too complicated; because, even in simple cases, the functions change in form, when n passes through certain values. We shall therefore only consider particular examples; choosing the excesses or defects of the particular point-group N arbitrarily, within certain limits: Some results, for a general case, are stated in Art. 31.

27. Suppose it is required to find a construction for a point-group N , characterized by the numbers $N = 67$, $r_{12} = 1$, $r_{11} = 3$, $r_{10} = 7$, $r_9 = 13$; from which we obtain $q_9 = 0$, $q_{10} = 5$, $q_{11} = 13$, $q_{12} = 24$. The reduction of this point-group is given in the following table:—

N	m	l		n	
(1) 67	9	10	11	12	
Defect	0	(4) 5	(7) 13	(10) 24	
Excess	13	(6) 7	(4) 3	(2) 1	
(2) $N' = lm - N$ $= 23$		4	5	6	
Defect		0	(1) 2	(3) 6	
Excess		9	(4) 5	(3) 2	
(2') 24		4	5	6	7
Defect		0	(1) 2	(3) 6	(5) 12
Excess		10	(4) 6	(3) 3	(2) 1

The letters N , m , l , n are placed immediately above the numbers in the first row to which they correspond. Between successive defects in the second row the differences *diminished by unity* are interpolated; and when any one of these is subtracted from the order of the curve in the column to the right, it gives the difference of the corresponding excesses. This follows from the relations

$$N - r_n + q_n = \frac{1}{2}n(n+3), \quad N - r_{n-1} + q_{n-1} = \frac{1}{2}(n-1)(n+2);$$

which give by subtraction,

$$r_{n-1} - r_n = n - (q_n - q_{n-1} - 1).$$

The numbers in the three rows of stage (2) of the table are then written down as follows:—The orders of the curves are obtained by writing $m-3 = 6$ in the last column, followed by 5, 4, ... until the

column l is reached; the defects and their interpolated numbers are found by diminishing the numbers in the row of the excesses in stage (1) by unity, and reversing; and the excesses are written down beginning with the last, the differences being found as already explained. In the same way, the numbers in any stage may be written down from those in the preceding stage.

In the given example, the process of reduction may be regarded as finished at stage (2); for N' forms a recognisable point-group, viz. that formed by 23 points of intersection of a 4-ic and 6-ic. Hence, since N is the rest-group of N' determined by an l -ic and m -ic, it follows that the point-group $N = 67$ may be constructed by drawing a 9-ic and 10-ic through 23 points of intersection of a 4-ic and 6-ic. This point-group has the given characterization.

It is important to notice that N' is incomplete, and that the process of reduction cannot be safely used for incomplete or redundant point-groups.* As the derived point-groups are often possibly incomplete, it follows that, by making corrections in the table for them, a variety of ways of reducing a characterized point-group may be obtained; and a corresponding variety of ways of constructing the characterized point-group. This is considered more fully in Art. 29.

Stage (2') in the table represents the complete point-group $N' + 1 = 24$, of which N' forms a part; and only differs from stage (2) in respect to the excesses, and the addition of a 7-ic to the series of curves. For one step further in the reduction, either (2) or (2') might be used.

28. Another example of reduction is that of a point-group 369, whose characterization is given in the first three rows of the table on the next page.

In stage (4), the third derived rest-group 18 is recognisable as 18 points of intersection of a 4-ic and 5-ic, which is an incomplete group, as in the preceding example. Hence we have the following construction:—Draw two 10-ics through 18 points common to a 4-ic and 5-ic; through the rest-group 82 draw two 17-ics; and through the rest-group 207, two 24-ics; these cut again in a rest-group 369, which has the given characterization.

When, as in stage (3), one or more excesses become zero, the curves to which they correspond are simply left out of consideration in forming the next stage (Art. 19, iv).

* The reduction, if no correction were made, would lead eventually to a composite rest-group; to which would correspond composite curves, with a common factor.

(1) 369	24	25	26	27	28
Defect	3	(7) 11	(12) 24	(18) 43	(23) 67
Excess	48	(18) 30	(14) 16	(9) 7	(5) 2
(2) 207	17	18	19	20	21
Defect	1	(4) 6	(8) 15	(13) 29	(17) 47
Excess	38	(14) 24	(11) 13	(7) 6	(4) 2
(3) 82	10	11	12	13	14
Defect	1	(3) 6	(6) 12	(10) 23	(13) 37
Excess	18	(8) 10	(6) 4	(3) 1	(1) 0
(4) 18	4	5	6	7	
Defect	0	(2) 3	(5) 9	(7) 17	
Excess	4	(3) 1	(1) 0	(0) 0	

In this example, no defect is stated for a 23-ic; and a point-group 369, constructed as above, does not lie on a 23-ic. If, however, another group of 369 points has the same characterization, except that the 23-ic defect is zero, it will reduce by the same process; the fourth derived rest-group consisting of 3 points on a straight line.

29. The following example illustrates how an incomplete point-group may be recognised. As explained below, stage (2) gives an incomplete rest-group; which is completed in (2'), by adding 3 to the degree, and all the excesses, without changing the defects. From (2') we obtain (3) and (4); the last representing 19 points of intersection of a 4-ic and 5-ic. We therefore have the following construction for the point-group 569. Draw two 13-ics through 19 points common to a 4-ic and 5-ic; through the rest-group 150 draw two 22-ics; and through 331 points of the rest-group 334 draw two 30-ics; then the rest-group 569 has the given characterization.

(1) 569	30	31	32	33	34	35	
Defect	4	(9) 14	(13) 28	(17) 46	(23) 70	(28) 99	
Excess	78	(22) 56	(19) 37	(16) 21	(11) 10	(7) 3	
(2) 331	22	23	24	25	26	27	
Defect	2	(6) 9	(10) 20	(15) 36	(18) 55	(21) 77	
Excess	58	(17) 41	(14) 27	(10) 17	(8) 9	(6) 3	
(2') 334	22	23	24	25	26	27	28
Defect	2	(6) 9	(10) 20	(15) 36	(18) 55	(21) 77	(24) 102
Excess	61	(17) 44	(14) 30	(10) 20	(8) 12	(6) 6	(4) 2
(3) 150	13	14	15	16	17	18	19
Defect	1	(3) 5	(5) 11	(7) 19	(9) 29	(13) 43	(16) 60
Excess	47	(11) 36	(10) 26	(9) 17	(8) 9	(5) 4	(3) 1
(4) 19	4	5	6	7	8	9	10
Defect	0	(2) 3	(4) 8	(7) 16	(8) 25	(9) 35	(10) 46
Excess	5	(3) 2	(2) 0	0	0	0	0

If, commencing at the last column, the first two or three excesses $r_n, r_{n-1}, r_{n-2}, \dots$ of a point-group N are the triangular numbers 1, 3, 6, ..., or multiples of them $a, 3a, 6a, \dots$; then the a^{th} rest-group derived from N will be found to simplify very much; the excesses of the a^{th} derived rest-group vanishing, in the last columns, when a is even; and the defects of the a^{th} derived rest-group forming the series 0, 2, 5, ..., in the first columns, when a is odd. If then the excesses $r_n, r_{n-1}, r_{n-2}, \dots$ of N are such that when the same number is added to them, they become consecutive (but not necessarily the first) terms

of the series $a, 3a, 6a, \dots$, we should assume N to be incomplete; subject to a condition as to the maximum value that r_n can have.

Hence N would be *possibly* incomplete if a positive number ρ could be found such that $r_n + \rho, r_{n-1} + \rho$ are respectively the same multiple of two consecutive triangular numbers, and r_n does not exceed a certain limit; and such a number ρ could be found as often as not. We should not, however, generally consider that this alone would be a sufficient indication that N is incomplete; although the effect of adding on ρ to N, r_n, r_{n-1}, \dots might be tried, if the reduction of N appears to lead to an impossible result. We may take as a test for an incomplete point-group that a positive number ρ can be found, such that the three following equations shall hold at least. ($p \geq 0$):—

$$r_n + \rho = \frac{a}{2} (p+1)(p+2), \quad r_{n-1} + \rho = \frac{a}{2} (p+2)(p+3),$$

$$r_{n-2} + \rho = \frac{a}{2} (p+3)(p+4).$$

This only requires

$$r_{n-1} - r_n = a(p+2), \quad r_{n-2} - r_{n-1} = a(p+3).$$

Hence, if the differences $r_{n-1} - r_n, r_{n-2} - r_{n-1}$, which are given in the table, are such that, when their H.C.F. a is divided out, they become consecutive integers $p+2, p+3$ ($p \geq 0$), we should assume N to be incomplete. We should, in that case, substitute a complete point-group for N , by increasing the degree and all the excesses by

$$\rho = \frac{a}{2} (p+1)(p+2) - r_n;$$

at the same time adding p curves of orders $n+1, n+2, \dots, n+p$ with excesses, $\frac{a}{2} p(p+1), \dots, 3a, a$, to the table. The condition that must be satisfied, if the point-group N is fully characterized, is that the $(n+1)$ -ic excess of the completed point-group, viz. $\frac{a}{2} p(p+1)$,

must be equal to or less than ρ . This requires $r_n \leq a(p+1)$; or that r_n should not exceed the first term of the A.P. whose second and third terms are $r_{n-1} - r_n$ and $r_{n-2} - r_{n-1}$.

The rest-group 331, in the example above, has been replaced by the complete point-group 334; with the result that we obtain, two stages later, a much simplified rest-group.

The original point-group N should be tested not only for incompleteness, but also for redundancy; for, if N were redundant, the process of reduction would eventually lead to composite rest-groups. If N is a simple point-group, then among the derived rest-groups it is only necessary to test for incompleteness; since an incomplete rest-group will always precede a redundant one (Art. 15, i).

To test whether or not the original point-group N is redundant, we examine the differences of the defects from the beginning. If $q_{m+1} - q_m$, $q_{m+2} - q_{m+1}$ have the values $a(p+2)$, $a(p+3)$, where $p \geq 0$, then N should be assumed redundant; and the number

$$\rho = \frac{a}{2} (p+1)(p+2) - q_m - 1$$

should be added to all the defects, and subtracted from the degree N ; while p curves of orders $m-p$, $m-p+1$, ... $m-1$, with defects $a-1$, $3a-1$, ... $\frac{a}{2} p(p+1) - 1$, should be added at the beginning of the table. If N is fully characterized, it is necessary that the condition $q_m + 1 \leq a(p+1)$ should be satisfied.

Unless the first given defect and last given excess of a point-group N are comparatively small, it should be assumed that the characterization of N is only partially given; and curves should be added to the table, with such assumed values for the defects and excesses as will make N reduce as rapidly as possible. (Cf. Art. 31.)

30. If the point-group N contains a general cluster $\frac{1}{2}i(i+1)$ at a point A , or any number of such clusters; we may choose the two lowest curves C_i , C_m through N to have i -ple points at A , and to have no common multiple points except at the clusters of N . Then C_i , C_m will intersect again at A in a general cluster $\frac{1}{2}i(i-1)$, which will belong to N' . Thus, if the reduction can be continued far enough, we shall arrive eventually at a rest-group which has only a single point at A . The process of reduction may however finish before the cluster disappears, the result being that the general point-group at which we finally arrive contains one or more general clusters. In this case the reduction may possibly lead to a point-group which has no excess for curves of any order, but which has necessarily a specialized form, which is not determined by the reduction. (See Note, p. 496.) In reversing the process, in order to construct N , the two curves drawn through a general cluster $\frac{1}{2}i(i-1)$ at

A will each have an i -ple point at A , and intersect again in a general cluster $\frac{1}{2}i(i+1)$ at A .

Thus it follows, as a consequence of Theorem V, that any non-composite point-group of special form is a rest-group, whose construction can in general be found, if its characterization is given. At the same time, Theorem V determines the complete characterization of a point-group, if its construction is given.

31. THE NUMBER OF ABSOLUTE CONNEXIONS OF A POINT-GROUP.—When the construction of a point-group is known, we can employ Theorems VI and VII to determine the number of its absolute connexions, and its absolute multiplicity for a curve of any order.

Suppose it required to find the smallest possible number of absolute connexions of a point-group N , of which nothing more is known than that its n -ic defect and n -ic excess are q and r respectively.

Suppose that the n -ic excess r of N is put into the form

$$r = \frac{a_1}{2} (p_1+1)(p_1+2) + \frac{a_2}{2} (p_2+1)(p_2+2) + \dots + \frac{a_r}{2} (p_r+1)(p_r+2),$$

where p_1, p_2, \dots, p_r are all different, and in descending order; and that we assume for the complete characterization of N that given by the equations (a having both positive and negative integral values)

$$r_{n+a} = \frac{a_1}{2} (p_1-a+1)(p_1-a+2) + \dots + \frac{a_r}{2} (p_r-a+1)(p_r-a+2),$$

.....(a)

the triangular numbers disappearing from the end as soon as they become zero. Thus the highest curve for which N has an excess is of order $n+p_1$, and $r_{n+p_1} = a_1$.

The number k of absolute connexions of a point-group, with this characterization, may be determined by finding its construction, and by repeated applications of Theorem VII. The work is too long to be given here, and we merely state the result, viz.,

$$k = (q+1) \sum a - \frac{1}{2} (\sum a)^2 + \frac{1}{2} \sum a^2 - \frac{1}{2} (\sum ap)^2 + \frac{1}{2} \sum ap^2 + n \sum ap$$

$$- \frac{1}{2} \{ a_1 \cdot \overline{a_1-1} \cdot \overline{p_1-p_2+a_1+a_2} \cdot \overline{a_1+a_2-1} \cdot \overline{p_2-p_3} + \dots$$

$$+ \overline{a_1+\dots+a_r} \cdot \overline{a_1+\dots+a_r-1} \cdot \overline{p_r} \}, \dots (A)$$

which may also be written in the form

$$k = a_1 (q_{n+p_1}+1) + a_2 (q_{n+p_2}+1) + \dots + a_r (q_{n+p_r}+1); \dots (A')$$

with the condition that the minimum value of

$$\frac{q}{c} + \frac{1}{2} (c+3)(1+\Sigma a) + \Sigma ap,$$

for positive integral values of c , must not exceed $n+2$.

Again, if $q+1$ be put into the form

$$q+1 = \frac{a_1}{2} (p_1+1)(p_1+2) + \dots + \frac{a_\mu}{2} (p_\mu+1)(p_\mu+2),$$

with the same conditions regarding the p 's as before; and we assume the characterization of N to be that given by the equations

$$q_{n-c}+1 = \frac{a_1}{2} (p_1-a+1)(p_1-a+2) + \dots + \frac{a_\mu}{2} (p_\mu-a+1)(p_\mu-a+2),$$

.....(b)

the triangular numbers being omitted if their factors are negative; then the number of connexions is

$$k = r\Sigma a - \frac{1}{2} (\Sigma a)^2 + \frac{1}{2} \Sigma a^2 - \frac{1}{2} (\Sigma ap)^2 - \frac{1}{2} \Sigma ap^2 + n\Sigma ap$$

$$- \frac{3}{2} \{ a_1 \cdot \overline{a_1-1} \cdot \overline{p_1-p_2} + \overline{a_1+a_2} \cdot \overline{a_1+a_2-1} \cdot \overline{p_2-p_3} + \dots$$

$$\dots + \overline{a_1+a_2+\dots+a_\mu} \cdot \overline{a_1+\dots+a_\mu-1} \cdot \overline{p_\mu} \}, \dots (B)$$

which may also be written in the form

$$k = a_1 r_{n-p_1} + a_2 r_{n-p_2} + \dots + a_\mu r_{n-p_\mu}; \quad \dots (B')$$

with the condition that the minimum value of

$$\frac{r}{c} + \frac{1}{2} (c+3)(\Sigma a-1) + \Sigma ap,$$

for positive integral values of c , must not exceed n .

The smallest possible number of connexions is not greater than either of the minimum values of k given by (A), (B), subject to the conditions stated.

The condition given for (A) and (A') is obtained by assuming that $n-c+1$ is the order of the lowest curve through the point-group N , and that q_{n-c} would turn out negative according to the characterization given in (a).

Thus $N - r_{n-c} > \frac{1}{2}(n-c)(n-c+3)$, $N - r_n + q_n = \frac{1}{2}n(n+3)$;

and, by subtracting,

$$c(n + \frac{3}{2}) - \frac{1}{2}c^2 > q_n + r_{n-c} - r_n,$$

$$\text{or } c(n + \frac{3}{2}) > q + \frac{1}{2}\Sigma a(p+c+1)(p+c+2) - \frac{1}{2}\Sigma a(p+1)(p+2) + \frac{1}{2}c^2,$$

$$\text{or } n+3 > \frac{q}{c} + \Sigma a(p + \frac{3}{2}) + \frac{1}{2}c\Sigma a + \frac{1}{2}(c+3),$$

$$\text{or } n+2 \geq \frac{q}{c} + \frac{1}{2}(c+3)(\Sigma a+1) + \Sigma ap.$$

This is a sufficient condition that a point-group N , with the characterization given in (a), can be placed on an n -ic.

Similarly, the condition given for (B) and (B') is obtained by assuming that $n+c-1$ is the order of the highest curve for which the point-group N has an excess, and that r_{n+c} would turn out zero or negative according to the characterization given in (b).

Point-groups with the characterization given in (a) or (b) are simple point-groups, all of whose successive derived rest-groups, of lowest degree, are also simple. If the point-group has an excess for n -ics only, and not for curves of any other order; then $p_1 = 0$, $a_2 = a_3 = \dots = 0$, and $c = 1$. In this case both (A) and (B) give $k = r(q+1)$, with the conditions $n \geq q+2r$, $n \geq r+2q$ respectively. Hence it would appear that the known formula $r(q+1)$ does not always give the correct value of k ; and that it certainly does not, if n is less than the smaller of the two numbers $q+2r$, $r+2q$. This can be easily proved independently.

The following presents to the Library were received during the Recess :—

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APPENDIX.

The Society's death-roll for the Session 1894-5 is a heavy one. We have lost one of our most distinguished and oldest members, Professor Cayley. For the following sketch, which has been drawn up at the request of the Council, we are indebted to Mr. Samuel Roberts, who sat for so many years at the Council-table with Professor Cayley:—

The death of Professor Cayley has naturally been the subject of world-wide regret. At home and abroad numerous biographical notices and appreciative comments on his life and work constitute a