obtain

$$x:y:z:w = \Theta_1 : ag_2-ga_2 : ah_2-ha_2 : gh_2-hg_2,$$
$$= bf_2-fb_2 : \Theta_2 : bh_2-hb_2 : hf_2-fh_2,$$
$$= cf_2-fc_2 : cg_2-gc_2 : \Theta_3 : fg_2-gf_2,$$
$$= bc_2-cb_2 : ca_2-ac_2 : ab_2-ba_2 : \Theta;$$

where

$$\Theta_1 = -fa_2-bg_2-ch_2, \quad = af_2+gb_2+hc_2,$$
$$\Theta_2 = -af_2-gb_2-ch_2, \quad = fa_2+bg_2+hc_2,$$
$$\Theta_3 = -af_2-bg_2-hc_2, \quad = fa_2+gb_2+ch_2,$$
$$\Theta = -af_2-bg_2-ch_2, \quad = fa_2+gb_2+hc_2.$$

For the point 3, we have, of course, the same formulæ, with the suffix 3 instead of 2.

Notes on Dualistic Differential Transformations. By E. B. Elliott.

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In these notes I have found it impossible not to introduce much that is well known. I believe, however, that no one else has called attention to the coincidences with established theories, which give completeness to the later results of the first section, and clues towards completing those of the second; and I am not aware that anyone has spent time upon the case of three indepeudent variables, considered in Section III.

I. *Transformation of Ordinary Differential Expressions.*

If, as usual, p denote $\frac{dy}{dx}$, and p' denote $\frac{dy'}{dx'}$, the transformation for x and y, in terms of x' and y',

$$\frac{ax+hy+g}{-p'} = \frac{hx+by+f}{1} = \frac{gx+fy+c}{p'x'-y'},$$

i.e.,
$$-\frac{x}{-Ap'+H+G(p'x'-y')}=\frac{y}{-Hp'+B+F(p'x'-y')}$$
$$=\frac{1}{-Gp'+F+C(p'x'-y')},$$

where $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$ and $\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}$ are non-vanishing reciprocal determinants, is one possessing the quality of duality; in other words, x', y', x, y, p may replace x, y, x', y', p' in the formulæ given. The transformation, in fact, effects polar reciprocation with regard to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

In general the transformed expressions, for second and higher derivatives of y with regard to x, involve the variables x', y' themselves as well as derivatives. This is not the case, however, if $h=0$ and $b=0$, *i.e.*, if the base conic be a parabola with axis parallel to $x=0$. In such a case

$$\frac{d^2y}{dx^2}\frac{d^2y'}{dx'^2}=\frac{a^2}{f^2}.$$

Since we can at any time re-introduce the right power of $-\frac{a}{f}$ as a factor by mere consideration of dimensions, we lose no real generality by putting $a=1$, $f=-1$, and so taking as our base parabola

$$x^2-2y+2gx+c=0.$$

The formulæ of transformation then become

$$x=p'-g, \quad y=gp'+B+p'x'-y',$$

$$x'=p-g, \quad y'=gp+B+px-y,$$

where $B=c-g^2$. It is, of course, generally convenient to take g and B zero.

We notice at once that $y_2, y_3, y_4, \ldots$, the second and higher derivatives of y with regard to x, are merely $\frac{dx'}{dx}, \frac{d^2x'}{dx^2}, \frac{d^3x'}{dx^3}, \ldots$, while

$y_2', y_3', y_4', \ldots$, those of y' with regard to x', are $\frac{dx}{dx'}, \frac{d^2x}{dx'^2}, \frac{d^3x}{dx'^3}, \ldots$. Thus the transformation of second and higher derivatives amounts simply to the interchange of dependent and independent variables in a set of first and higher derivatives. Now, of this interchange a complete theory is at our disposal. It tells us that

$$y_2 = \frac{1}{y_2'},$$

and that
$$\frac{d}{dx} = \frac{1}{y_2'}\frac{d}{dx'};$$

so that
$$y_n = \left(\frac{1}{y_2'}\frac{d}{dx'}\right)^{n-2}\left(\frac{1}{y_2'}\right).$$

To transform any function of the derivatives, it is not, however, necessary to substitute for all derivatives y_n separately the values thus given. We have only to adapt as above a theorem of Mr. Leudesdorf's (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 333). The theorem at which we arrive from it is that, if

$$U \equiv 3y_3^2\frac{d}{dy_4} + 10y_3y_4\frac{d}{dy_5} + (15y_3y_5 + 10y_4^2)\frac{d}{dy_6} + \ldots$$
$$\equiv \sum_{m \not< 3}\sum_{r \not< 3}^{r \not> m} (m)_{r-1}\, y_r\, y_{m-r+3}\frac{d}{dy_{m+1}},$$

where $(m)_{r-1}$ denotes the number of combinations of m things $r-1$ together, and, if f denote a homogeneous isobaric function of second and higher derivatives, of degree i and weight w,

$$f(y_2', y_3', y_4', \ldots) = (-1)^i y_2^{-w} e^{-1/y_2 \cdot U} f(-y_2, y_3, y_4, \ldots).$$

Thus, if any differential expression is the sum of such terms as

$$\phi(x', y', p') f(y_2', y_3', y_4', \ldots),$$

where f is homogeneous and isobaric, its dual consists of such corresponding terms as

$$(-1)^i \phi(p-g,\, gp+B+px-y,\, x+g)\, y_2^{-w} e^{-1/y_2 \cdot U} f(-y_2, y_3, y_4, \ldots).$$

Another remark, of very comprehensive importance, to which the simple fact we have noticed leads, is that the whole theory of reciprocants has its direct application to our dualistic transformation. If, in fact,

$$R(y_1, y_2, y_3, \ldots)$$

be any reciprocant whatever, pure or mixed, and if in it we replace each derivative by the next higher, we thus obtain

$$R(y_2, y_3, y_4, \ldots),$$

a function of the second and higher derivatives which, but for a power of y_2 as factor, is its own dual. In other words, such functions are the criteria of curves whose reciprocals, with regard to a parabola with axis parallel to that of y_1, constitute the same family.

In particular, functions of $y_3, y_4, y_5, \ldots$ without y_2, or with only odd or only even powers of y_2, which U annihilates, are such functions. They correspond to pure reciprocants, or sums of products of pure reciprocants and odd or even powers of the first derivative.

II. *Partial Differential Expressions with Two Independent Variables.*

1. In this, the case of ordinary three-dimensional geometry, let, as usual, p, q denote $\frac{dz}{dx}$, $\frac{dz}{dy}$.

Two classes of transformations, expressing x, y, z homographically in terms of $p', q', p'x'+q'y'-z'$, have the dualistic property, and correspond to two kinds of geometrical reciprocity of surfaces. The homographic formulæ have been given by Chasles (*Aperçu Historique*, Note xxx.) for the first class, and he has also indicated the second class by means of its most important case. The classes are (A) that of transformations in which the coordinates of a point of a surface are expressed by means of a corresponding point on the surface which is the polar reciprocal of the former surface with regard to a fixed quadric, and (B) the class in which the coordinates of a point of a surface are expressed by means of the coordinates of the point of contact with the surface, which it envelopes, of a plane passing through the first point and orthogonal to the direction of its motion on an instantaneous screw of given pitch and axis.

(A) The formulæ given by Chasles for the first class are in effect

$$\frac{ax+hy+gz+l}{-p'} = \frac{hx+by+fz+m}{-q'} = \frac{gx+fy+cz+n}{1} = \frac{lx+my+nz+d}{p'x'+q'y'-z'},$$

i.e.,

$$\frac{x}{-Ap'-Hq'+G+L(p'x'+q'y'-z')}$$

$$=\frac{y}{-Hp'-Bq'+F+M(p'x'+q'y'-z')}$$

$$=\frac{z}{-Gp'-Fq'+C+N(p'x'+q'y'-z')}$$

$$=\frac{1}{-Lp'-Mq'+N+D(p'x'+q'y'-z')},$$

where $\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix}$ and $\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix}$ are non-vanishing reciprocal determinants.

Of these the Monge–De Morgan formulæ

$$x = p', \quad y = q', \quad z = p'x'+q'y'-z'$$

are a particular case.

(B) The general formulæ for the second class of dualistic transformations are found by identifying

$$p'(\xi-x')+q'(\eta-y')-(\zeta-z') = 0$$

with $\{vl+\omega(m\bar{z}-n\bar{y})\}(\xi-x)+\{vm+\omega(n\bar{x}-l\bar{z})\}(\eta-y)$

$$+\{vn+\omega(l\bar{y}-m\bar{x})\}(\zeta-z) = 0,$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ stand for $x-a$, $y-b$, $z-c$, where

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

is the axis of the screw, and $v\omega^{-1}$ its pitch. They are of the form

$$\frac{-p'}{mz-ny+\lambda}=\frac{-q'}{nx-lz+\mu}=\frac{1}{ly-mx+\nu}=\frac{z'-p'x'-q'y'}{\lambda x+\mu y+\nu z},$$

which may be written

$$\frac{x}{\nu q'+\mu-l(z'-p'x'-q'y')}=\frac{y}{-\nu p'-\lambda-m(z'-p'x'-q'y')}$$

$$=\frac{z}{\mu p'-\lambda q'-n(z'-p'x'-q'y')}=\frac{1}{lp'+mq'-n}.$$

The only restrictions upon the constants are that they be finite and such that $l\lambda+m\mu+n\nu$, *i.e.*, $\nu\omega^{-1}$, do not vanish.

The included transformation discussed by Chasles is

$$x = q', \quad y = -p', \quad z = z'-p'x'-q'y'.$$

2. Regarding Class (A) first, I have considered the transformation of second and higher derivatives only for the included cases where x', y', z' are linear in p, q, $px+qy-z$. With altered notation we have

$$x = ap'+hq'-g,$$
$$y = hp'+bq'-f,$$
$$z = gp'+fq'+c+p'x'+q'y'-z',$$

leading to

$$x' = ap+hq-g,$$
$$y' = hp+bq-f,$$
$$z' = gp+fq+c+px+qy-z.$$

We have at once

$$\frac{d(x, y)}{d(x', y')} = (ab-h^2)(r't'-s'^2) = \frac{1}{(ab-h^2)(rt-s^2)},$$

where r, s, t, r', s', t' denote $\frac{d^2z}{dx^2}, \frac{d^2z}{dxdy}, \frac{d^2z}{dy^2}, \frac{d^2z'}{dx'^2}, \frac{d^2z'}{dx'dy'}, \frac{d^2z'}{dy'^2}$, so that

$$(ab-h^2)^2(rt-s^2)(r't'-s'^2) = 1.$$

We have also, since

$$\frac{du}{dx'} = \frac{d(u, y')}{d(x, y)} \cdot \frac{d(x, y)}{d(x', y')},$$

and

$$\frac{du}{dy'} = \frac{d(x', u)}{d(x, y)} \cdot \frac{d(x, y)}{d(x', y')},$$

the operator equivalents, sufficing to produce derivatives of any order from those of the order preceding,

$$\frac{d}{dx'} = \frac{1}{(ab-h^2)(rt-s^2)}\left\{(hs+bt)\frac{d}{dx}-(hr+bs)\frac{d}{dy}\right\},$$

$$\frac{d}{dy'} = \frac{1}{(ab-h^2)(rt-s^2)}\left\{(ar+hs)\frac{d}{dy}-(as+ht)\frac{d}{dx}\right\}.$$

By use of these, or by the method of Forsyth's "Differential Equations," § 242, we readily obtain

$$r' = \frac{1}{(ab-h^2)^2} \cdot \frac{h^2r+2hbs+b^2t}{rt-s^2},$$

$$s' = -\frac{1}{(ab-h^2)^2} \cdot \frac{ahr+(ab+h^2)\,s+hbt}{rt-s^2},$$

$$t' = \frac{1}{(ab-h^2)^2} \cdot \frac{a^2r+2ahs+h^2t}{rt-s^2},$$

whence the derivation of the transformed values of higher derivatives is direct, though soon laborious.

3. The Monge–De Morgan case

$$x' = p, \quad y' = q, \quad z' = px+qy-z,$$

or the very slightly more general

$$x' = p-g, \quad y' = q-f, \quad z' = gp+fq+c+px+qy-z,$$

$$x = p'-g, \quad y = q'-f, \quad z = gp'+fq'+c+p'x'+q'y'-z',$$

which is that of reciprocation with regard to a paraboloid of revolution with its axis parallel to that of z, is of special interest; and here a contact with an existing theory affords, as in the corresponding case in Section I., a means of elaborating many interesting results.

We notice, in fact, that the second unaccented derivatives are merely $\frac{dx'}{dx}$, $\frac{dx'}{dy} \equiv \frac{dy'}{dx}$, $\frac{dy'}{dy}$, and that the corresponding accented derivatives are $\frac{dx}{dx'}$, $\frac{dx}{dy'} \equiv \frac{dy}{dx'}$, $\frac{dy}{dy'}$; and generally that the theory of the expression of the second and higher derivatives of z', with regard to x' and y' in terms of those of z with regard to x and y, is included in the theory of the interchange of the dependent and the independent pairs in functions of the first and higher partial derivatives of one pair of variables with regard to another pair of which they are regarded as functions. A paper by the present author on this latter theory will be found in the *Proc. Lond. Math. Soc.*, Vol. XXII., pp. 79, &c. It must be remembered that the theory whose application to the purposes of the present article is immediate is included in, but less general than, that developed in the paper referred to. The fact that, as above noticed, $\frac{dx'}{dy} = \frac{dy'}{dx}$, has nothing corresponding to it in the paper quoted. The $\frac{d^{m+n-1}x'}{dx^{m-1}dy^n}$ and $\frac{d^{m+n-1}y'}{dx^m dy^{n-1}}$ of that paper have here both the same meaning, namely, $\frac{d^{m+n}z}{dx^m dy^n}$. Similarly $\frac{d^{m+n-1}x}{dx'^{m-1}dy'^n}$ and $\frac{d^{m+n-1}y}{dx'^m dy'^{n-1}}$ both mean $\frac{d^{m+n}z'}{dx'^m dy'^n}$.

For instance, what is given by the last result of the paper is that

$$4(z_{30}z_{12}-z_{21}^2)(z_{21}z_{03}-z_{12}^2)-(z_{30}z_{03}-z_{21}z_{12})^2,$$

the discriminant of $\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^3 z$, is, but for a power of $rt-s^2$ as factor, unaltered by our dualistic transformation, *i.e.*, is the criterion of a family of surfaces whose polar reciprocals, with regard to a paraboloid with axis of revolution parallel to the axis of z, are surfaces of the same family. The fact is otherwise at once clear from the equalities, written down by the means already described,

$$z'_{30} = \left\{s^3z_{03}-3s^2tz_{12}+3st^2z_{21}-t^3z_{30}\right\}\div,$$

$$z'_{21} = \left\{-rs^2z_{03}+(2rst+s^3)\,z_{12}-(rt^2+2s^2t)\,z_{21}+st^2z_{30}\right\}\div,$$

$$z'_{12} = \left\{r^2sz_{03}-(2rs^2+r^2t)\,z_{12}+(2rst+s^3)\,z_{21}-s^2tz_{30}\right\}\div,$$

$$z'_{03} = \left\{-r^3z_{03}+3r^2sz_{12}-3rs^2z_{21}+s^3z_{30}\right\}\div,$$

the common denominator being $(rt-s^2)^3$.

The transformed expressions for higher derivatives are deduced by aid of the operator equivalences

$$\frac{d}{dx'} = \frac{1}{rt-s^2}\left\{t\frac{d}{dx}-s\frac{d}{dy}\right\},$$

$$\frac{d}{dy'} = \frac{1}{rt-s^2}\left\{r\frac{d}{dy}-s\frac{d}{dx}\right\}.$$

Elementary absolute self-dual functions are

$$(rt-s^2)^{-\frac{1}{2}}s,\quad (rt-s^2)^{-\frac{1}{2}}(r+t),\quad (rt-s^2)^{-\frac{1}{2}}(r-t),$$

of characters $-$, $+$, $-$ respectively; and, u, v being any two absolute self-dual functions, another is

$$(rt-s^2)^{-\frac{1}{2}}\frac{d(u, v)}{d(x, y)}.$$

We may, for instance, conveniently take for u and v the two $\frac{r \pm t}{s}$ in the first place, and successively afterwards others derived from those two.

4. For the linear case of Class (B),

$$x' = hq+g,$$

$$y' = -hp+f,$$

$$z' = gp+fq+z-px-qy,$$

which is only in appearance more general than that of Chasles, we have

$$h^4(rt-s^2)(r't'-s'^2) = 1,$$

and

$$\frac{r'}{r} = \frac{s'}{s} = \frac{t'}{t} = -\frac{1}{h^2}\cdot\frac{1}{rt-s^2}.$$

We have also

$$\frac{d}{dx'} = \frac{1}{h(rt-s^2)}\left\{r\frac{d}{dy} - s\frac{d}{dx}\right\},$$

and

$$\frac{d}{dy'} = \frac{1}{h(rt-s^2)}\left\{s\frac{d}{dy} - t\frac{d}{dx}\right\}.$$

Hence, quite readily,

$$z'_{30} = \left\{-s^3z_{30}+3s^2rz_{21}-3sr^2z_{12}+r^3z_{03}\right\} \div,$$

$$z'_{21} = \left\{-s^2tz_{30}+(s^3+2rst)\,z_{21}-(2rs^2+r^2t)\,z_{12}+r^2sz_{03}\right\} \div,$$

$$z'_{12} = \left\{-st^2z_{30}+(2s^2t+rt^2)\,z_{21}-(2rst+s^3)\,z_{12}+rs^2z_{03}\right\} \div,$$

$$z'_{03} = \left\{-t^3z_{30}+3s^2tz_{21}-3st^2z_{12}+s^3z_{03}\right\} \div,$$

the common denominator of all four being $h^3(rt-s^2)^3$.

If h be unity,

$$(rt-s^2)^{-\frac{1}{2}}r, \quad (rt-s^2)^{-\frac{1}{2}}s, \quad (rt-s^2)^{-\frac{1}{2}}t$$

are elementary self-dual functions, absolute, and all of negative character. From any two such, u and v say, we can at once derive another of higher order, viz.,

$$(rt-s^2)^{-\frac{1}{2}}\frac{d\,(u,\,v)}{d\,(x,\,y)}.$$

For instance, we may, in the first place, take $\frac{r}{s}$ and $\frac{t}{s}$ for u and v.

III. *Partial Differential Expressions with Three Independent Variables.*

1. If u is a function of three variables x, y, z, and if p, q, r denote now $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, dualistic formulæ of transformation analogous to those of Class (A), Section II., are with ease developed. In what follows attention is confined to those in which new independent variables x', y', z' are expressed linearly in terms of p, q, r and $px+qy+rz-u$. They are

$$\begin{aligned} x' &= ap+hq+gr-l, \\ y' &= hp+bq+fz-m, \\ z' &= gp+fq+cz-n, \\ u' &= lp+mq+nr+d+px+qy+rz-u. \end{aligned}$$

Without real loss of generality, we may put x, x', y, y', z, z' for $x+l$, $x'+l$, $y+m$, $y'+m$, $z+n$, $z'+n$, and write

$$\begin{aligned} x' &= ap+hq+gz, \\ y' &= hp+bq+fz, \\ z' &= gp+fq+cz, \\ u' &= px+qy+rz-u+d, \end{aligned}$$

of which the first three, otherwise written, are

$$\begin{aligned} \Delta p &= Ax'+Hy'+Gz', \\ \Delta q &= Hx'+By'+Fz', \\ \Delta r &= Gx'+Fy'+Cz', \end{aligned}$$

where A, B, C, F, G, H are the minors corresponding to a, b, c, f, g, h

in the non-vanishing determinant

$$\Delta \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

By differentiation of u', we have

$$p'dx'+q'dy'+r'dz' = xdp+ydq+zdr$$
$$= \frac{1}{\Delta}\left\{(Ax+Hy+Gz)\,dx'+(Hx+By+Fz)\,dy'+(Gx+Fy+Cz)\,dz'\right\};$$

from which it follows that

$$\Delta p' = Ax+Hy+Gz,$$
$$\Delta q' = Hx+By+Fz,$$
$$\Delta r' = Gx+Fy+Cz,$$

which, with

$$u = px+qy+rz-u'+d$$
$$= \frac{1}{\Delta}\Sigma.(Ax'+Hy'+Gz')(ap'+hq'+gr')-u'+d$$
$$= p'x'+q'y'+r'z'-u'+d,$$

show that the duality is complete.

2. Let $u_{11}, u_{12}, \ldots, u'_{11}, u'_{12}, \ldots$ denote the second derivatives $\frac{d^2u}{dx^2}, \frac{d^2u}{dx\,dy}, \ldots, \frac{d^2u'}{dx'^2}, \frac{d^2u'}{dx'\,dy'}, \ldots$; and let

$$J = \begin{vmatrix} u_{11}, & u_{12}, & u_{13} \\ u_{21}, & u_{22}, & u_{23} \\ u_{31}, & u_{32}, & u_{33} \end{vmatrix},$$

a determinant whose minors will be called $J_{11}, J_{12}, \ldots$. Moreover $J', J'_{11}, J'_{12}, \ldots$ are the like functions of accented derivatives.

By differentiation of x', y', z', we obtain

$$dx' = (au_{11}+hu_{12}+gu_{13})\,dx+(au_{12}+hu_{22}+gu_{23})\,dy+(au_{13}+hu_{23}+gu_{33})\,dz,$$
$$dy' = (hu_{11}+bu_{12}+fu_{13})\,dx+(hu_{12}+bu_{22}+fu_{23})\,dy+(hu_{13}+bu_{23}+fu_{33})\,dz,$$
$$dz' = (gu_{11}+fu_{12}+cu_{13})\,dx+(gu_{12}+fu_{22}+cu_{23})\,dy+(gu_{13}+fu_{23}+cu_{33})\,dz.$$

We have also, by differentiation of $\Delta p'$, $\Delta q'$, $\Delta r'$,

$$\Delta\,(u'_{11}dx'+u'_{12}dy'+u'_{13}dz') = A\,dx+H\,dy+G\,dz,$$
$$\Delta\,(u'_{12}dx'+u'_{22}dy'+u'_{23}dz') = H\,dx+B\,dy+F\,dz,$$
$$\Delta\,(u'_{13}dx'+u'_{23}dy'+u'_{33}dz') = G\,dx+F\,dy+C\,dz,$$

which give other linear expressions for dx', dy', dz', in terms of dx dy, dz. The identification of these with the former gives at once

$$\Delta J'\,(au_{11}+hu_{12}+gu_{13}) = AJ'_{11}+HJ'_{12}+GJ'_{13},$$
$$\Delta J'\,(hu_{11}+bu_{12}+fu_{13}) = AJ'_{12}+HJ'_{22}+GJ'_{23},$$
$$\Delta J'\,(gu_{11}+fu_{12}+cu_{13}) = AJ'_{13}+HJ'_{23}+GJ'_{33},$$

$$\Delta J'\,(au_{12}+hu_{22}+gu_{23}) = HJ'_{11}+BJ'_{12}+FJ'_{13},$$
$$\Delta J'\,(hu_{12}+bu_{22}+fu_{23}) = HJ'_{12}+BJ'_{22}+FJ'_{23},$$
$$\Delta J'\,(gu_{12}+fu_{22}+cu_{23}) = HJ'_{13}+BJ'_{23}+FJ'_{33},$$

$$\Delta J'\,(au_{13}+hu_{23}+gu_{33}) = GJ'_{11}+FJ'_{12}+CJ'_{13},$$
$$\Delta J'\,(hu_{13}+bu_{23}+fu_{33}) = GJ'_{12}+FJ'_{22}+CJ'_{23},$$
$$\Delta J'\,(gu_{13}+fu_{23}+cu_{33}) = GJ'_{13}+FJ'_{23}+CJ'_{33}.$$

These express for us u_{11}, u_{12}, ... in terms of J' and its minors. They give us, in fact,

$$\Delta^2 J'u_{11} = A^2J'_{11}+2AHJ'_{12}+2AGJ'_{13}+H^2J'_{22}+2HGJ'_{23}+G^2J'_{33},$$
$$\Delta^2 J'u_{22} = H^2J'_{11}+2HBJ'_{12}+2HFJ'_{13}+B^2J'_{22}+2BFJ'_{23}+F^2J'_{33},$$
$$\Delta^2 J'u_{33} = G^2J'_{11}+2GFJ'_{12}+2GCJ'_{13}+F^2J'_{22}+2FCJ'_{23}+C^2J'_{33},$$

$$\Delta^2 J'u_{12} = AHJ'_{11}+(AB+H^2)\,J'_{12}+(AF+GH)\,J'_{13}+BHJ'_{22}$$
$$+(HF+BG)\,J'_{23}+FGJ'_{33},$$
$$\Delta^2 J'u_{13} = AGJ'_{11}+(AF+GH)\,J'_{12}+(AC+G^2)\,J'_{13}+HFJ'_{22}$$
$$+(CH+FG)\,J'_{23}+CGJ'_{33},$$
$$\Delta^2 J'u_{23} = GHJ'_{11}+(BG+HF)\,J'_{12}+(CH+FG)\,J'_{13}+BFJ'_{22}$$
$$+(BC+F^2)\,J'_{23}+CFJ'_{33}.$$

We may briefly express the facts arrived at by saying that the linear transformation

$$\alpha = a\alpha' + h\beta' + g\gamma',$$
$$\beta = h\alpha' + b\beta' + f\gamma',$$
$$\gamma = g\alpha' + f\beta' + c\gamma'$$

produces $\dfrac{1}{J'}\left\{J'_{11}\alpha'^2 + J'_{22}\beta'^2 + J'_{33}\gamma'^2 + 2J'_{23}\beta'\gamma' + 2J'_{13}\gamma'\alpha' + 2J'_{12}\alpha'\beta'\right\},$

as the equivalent of

$$u_{11}\alpha^2 + u_{22}\beta^2 + u_{33}\gamma^2 + 2u_{23}\beta\gamma + 2u_{13}\gamma\alpha + 2u_{12}\alpha\beta.$$

One conclusion hence is that

$$\Delta^2 JJ' = 1;$$

and this might at once have been written down by noticing that

$$\frac{d(x', y', z')}{d(x, y, z)} = \Delta J,$$

and

$$\frac{d(x, y, z)}{d(x', y', z')} = \Delta J'.$$

3. A few words as to the particular dualistic transformation

$$x' = p,$$
$$y' = q,$$
$$z' = r,$$
$$u' = px + qy + rz - u,$$

which, as in Section II., 3, may be given an apparently slightly more general form, will conclude the present paper. The formulæ of transformation of second derivatives become now

$$\frac{u_{11}}{J'_{11}} = \frac{u_{12}}{J'_{12}} = \frac{u_{13}}{J'_{13}} = \frac{u_{22}}{J'_{22}} = \frac{u_{23}}{J'_{23}} = \frac{u_{33}}{J'_{33}} = \frac{1}{J'} = J$$

$$= \frac{J_{11}}{u'_{11}} = \frac{J_{12}}{u'_{12}} = \frac{J_{13}}{u'_{13}} = \frac{J_{22}}{u'_{22}} = \frac{J_{23}}{u'_{23}} = \frac{J_{33}}{u'_{33}}.$$

As an example of conclusions arrived at it may be noticed that every solution of an equation linear in u_{11}, u_{12}, &c., corresponds to a solution of a corresponding equation linear in J'_{11}, J'_{12}, &c.

In particular, to every spherical harmonic, or solution of

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0,$$

corresponds a single solution of

$$\Sigma\left\{\frac{d^2u}{dx^2}\,\frac{d^2u}{dy^2}-\left(\frac{d^2u}{dx\,dy}\right)^2\right\}=0.$$

For instance, $u=(x^2+y^2+z^2)^{-\frac{1}{2}}=\rho^{-1}$ is a spherical harmonic. The corresponding solution of the dual equation last written is obtained from

$$x'=p=-x\rho^{-3},$$

$$y'=q=-y\rho^{-3},$$

$$z'=r=-z\rho^{-3},$$

$$u'=px+qy+rz-u=-2\rho^{-1}$$

$$=-2\,(x'^2+y'^2+z'^2)^{\frac{1}{4}}.$$

Thus $u'=\rho'^{\frac{1}{2}}$ is a solution as required.

More generally, let u_n be a spherical harmonic of degree n. It is homogeneous in x, y, z. A corresponding solution u' of the dual equation is given by

$$x'=\frac{du_n}{dx},$$

$$y'=\frac{du_n}{dy},$$

$$z'=\frac{du_n}{dz},$$

$$u'=(n-1)\,u_n$$

$$=\text{a homogeneous function of degree }\frac{n}{n-1}\text{ in }x',\,y',\,z'.$$

Thus the degrees n, n' of the spherical harmonic and its dual are connected by the relation $\frac{1}{n}+\frac{1}{n'}=1$. Moreover, to the relation

$n_1 + n_2 = -1$, connecting the positive and negative degrees of two associated spherical harmonics, corresponds the relation

$$3n_1' n_2' - 2(n_1' + n_2') + 1 = 0,$$

connecting the degrees of two associated solutions of the dual equation. Such two associated solutions differ by a factor which is a power of $\left(\frac{du'}{dx'}\right)^2 + \left(\frac{du'}{dy'}\right)^2 + \left(\frac{du'}{dz'}\right)^2$, where u' is either of the two.

On certain Quartic Curves of the Fourth Class and the Porism of the Inscribed and Circumscribed Polygon. By R. A. Roberts.

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In a former communication to the *Proceedings* of the Society, I showed (see Vol. XVI., p. 53) that for certain plane curves of the fourth degree there existed an infinite number of polygons simultaneously inscribed in and circumscribed about the curve; and I gave the means of determining the parameter in the equation of the curve in the same way as Professor Cayley has done for the similar problem of two conics, namely, by equating to zero determinants formed by the coefficients in the expansion of a radical expression. It may be worth while, however, making the investigation directly, with the use of the elliptic integrals by means of which the results are arrived at.

In the place referred to, I showed that the curves are unicursal and possessed of two cusps and a node. Slightly altering the notation, I write the curve now

$$(xy + zx - yz)^2 - m^2 z^2 xy = 0,$$

where xz, yz are the cusps, and xy is the node. This equation, it is easy to verify, is satisfied by taking

$$x = 1 - \theta^2 - m\theta, \quad y = \theta^2(1 - \theta^2 - m\theta), \quad z = \theta^2,$$

where θ is a parameter. Hence, if a line

$$\lambda x + \mu y + \nu z = 0$$