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XL. On the determination of the general term of a new class of infinite series

Charles Babbage Esq. M.A.

No. 1. From Preston quarry in the Cleveland dyke. *Mimosite*, fine grained, imperfectly porpheroidal from the salient crystals of pyroxène. It is a *basalt* of the ancient mineralogists. The specimen contains a great abundance of dark-greenish gray felspar, mixed with a very small quantity of pyroxène and titaniferous iron. Some points of pyrites are to be seen. The *paste* also envelops laminar crystals of felspar, having a considerable lustre, which give the *paste* a scaly appearance which distinguishes it from *basalt*.

No. 2. From Coaly Hill dyke near Newcastle. *Mimosite*, small grained, passing into *xerasite*. Many of the cavities contain *green-earth*. It is imperfectly porpheroidal. The crystals of felspar very brilliant.

No. 3. From Walbottle Dean dyke. This has a more decided character of a *dolerite*, very fine grained, the felspar whiter than in the others.

As these distinctive terms are not generally adopted by English mineralogists; it may be proper to state that *mimosite* and *dolerite* are granular rocks. *Xerasite* and *basalt* are composed of the same elements, but microscopic, and having the appearance of a *paste*.

XL. *On the Determination of the General Term of a New Class of Infinite Series.* By CHARLES BABBAGE, *Esq. M.A. Fellow of the Royal Societies of London and Edinburgh, and of the Cambridge Philosophical Society**.

THE subject of investigation on which I have entered in the following paper, had its origin in a circumstance which is, I believe, as yet singular in the history of mathematical science, although there exists considerable probability, that it will not long remain an isolated example of analytical inquiries, suggested and rendered necessary by the progress of machinery adapted to numerical computation. Some time has elapsed since I was examining a small machine I had constructed, by which a table, having its second difference constant, might be computed by mechanical means. In considering the various changes which might be made in the arrangement of its parts, I observed an alteration, by which the calculated series would always have its second difference equal to the unit's figure of the last computed term of the series; other forms of the machine would make the first or the third, or generally any given difference equal to the unit's figure of the

* From the Cambridge Philosophical Transactions, vol. ii. Part I.

the

the term last computed; and a further alteration would make the same difference equal to double, or generally to (a) times the digit in the unit's place: or if it were preferred, the digit fixed upon might be that occurring in the ten's place, or generally in the nth place. I did not, at that time, possess the means of making these alterations which I had contemplated, but I immediately proceeded to write down one of the series which would have been calculated by the machine thus altered; and commencing with one of the most simple, I formed the series.

Series.	Diff.
2	2
4	4
8	8
16	6
22	2
24	4
28	8
.	.
.	.

If u_z represent any term of this series, then the equation which determines u_z is

$$\Delta u_z = \text{unit's figure of } u_z,$$

an equation of differences of a nature not hitherto considered, nor am I aware that any method has been pointed out for the determining u_z in functions of z from such laws. I shall now lay before the Society, the steps which I took for ascertaining the general terms of such series, and of integrating the equations to which they lead. I shall not, however, commence with the general investigation of the subject, but shall simply point out the paths through which I was led to their solution, conceiving this course to be much more conducive to the progress of analysis, although not so much in unison with the taste which at present prevails in that science.

If we examine the series, and its first differences, it will be perceived, that the terms of the latter recur after intervals of four, and that all the changes in the first differences, are comprised in the numbers 2, 4, 8, 6, which recur continually, and the series may be written thus:

	Series.	Diff.
	2	2
	4	4
	8	8
	16	6
5	$22 = 20 + 2$	2
	$24 = 20 + 4$	4

28 =

	Series.	Diff.
	$28 = 20 + 8$	8
	$36 = 26 + 16$	6
	$42 = 40 + 2$	2
10	$44 = 40 + 4$	4
	$48 = 40 + 8$	8
	$56 = 40 + 16$	6
	$62 = 60 + 2$	2
	$64 = 60 + 4$	4
15	$68 = 60 + 8$	8
	$76 = 60 + 16$	6
	$82 = 80 + 2$	2

If then z be of the form $4v+i$, the value of u_z will be $20v+$ one of four numbers 2, 4, 8, 16, according to the value of i, and if i always represents one of the numbers 1, 2, 3, 4, the value of u_z will be thus expressed,

$$u_z = 20v + 2^i.$$

As a second example, let us consider the series whose first term is 2, its first difference 0, and its second difference always equal the unit's figure of the next term; its equation will be

$$\Delta^2 u_z = \text{unit's figure of } u_z,$$

and the few first terms are

2	28
2	48
4	76
10	110
16	144
	182

This series may be put under the form

	Series.	1 Diff.
0	2	0
	2	2
	4	6
	10	6
	16	12
5	28	20
	48	28
	76	34
	110	34
	144	38
10	182	$40 = 40 + 0$
	222	$42 = 40 + 2$
	264	$46 = 40 + 6$
	310	$46 = 40 + 6$
	356	$52 = 40 + 12$

Table of $(\bar{a})$.

if $a = 0$	$(\bar{a}) = 2$
1	2
2	4
3	10
4	16
5	28
6	48
7	76
8	110
9	144

	Series.	1 Diff.	
15	408	60 =	40 + 20
	468	68 =	40 + 28
	536	74 =	40 + 34
	610	74 =	40 + 34
	684	78 =	40 + 38
20	762	80 =	80 + 0
	842	82 =	80 + 2
	924	86 =	80 + 6
	1010	86 =	80 + 6
	1096	92 =	80 + 12
25	1188	100 =	80 + 20
	1288	108 =	80 + 28
	1396	114 =	80 + 34
	1510	114 =	80 + 34
	1624	118 =	80 + 38
30	1742	120 =	120 + 0
	1862	122 =	120 + 2

In this series it may be observed, that u_z when z is less than 10, is equal to the sum of the first differences of all the preceding terms; and if z be greater than 10, it will be composed of four terms, viz. first the sum of the ten first terms of the first difference, multiplied by the number of tens contained in z; secondly, of the sum of the series $40 + 80 + 120 +$ to as many terms as there are tens in z, this must be multiplied by 10, as each term is ten times added; and thirdly, of the number 40 multiplied by the same number of the tens, and also multiplied by the digit in the unit's place of z; and fourthly, of the sum of so many terms of the series as is equal to the unit's figure of z; this being expressed by $(\bar{a})$ signifying the number opposite a in the previous table. These four parts, if $z = 10b + a$, are thus expressed,

1st $180b$,

2nd $40\frac{b \cdot \overline{b-1}}{2}10$,

3rd $40ba$,

4th $(\bar{a})$.

These added together produce

$$u_z = 20b(10b + 2a - 1) + (\bar{a}).$$

This value of u_z, if diminished by 2, is equal to the sum of $z-1$ term of the series which constitute the first difference.

This inductive process for discovering the nth terms of such series, might be applied to others of the same kind; but it does not admit of an application sufficiently general or direct, to render it desirable that it should be pursued further.

If

If we consider any series in which the first difference is equal to the digit occurring in the unit's place of the corresponding term, as for example, the series

6	6
12	2
14	4
18	8
26	6
32	2

a slight examination will satisfy us, that the value of the digit occurring in the unit's figure of u_z, depends entirely on the value of u_z, at the commencement of the series, and also that whenever the same digit again occurs, there will, at that point, commence a repetition of the same figures which have preceded; consequently, the first difference at those two points will be equal.

In the first example which I have adduced of a series of this kind, it will be found, that this reappearance of the terminal figure, happens at the 5th, at the 9th, at the 13th terms, &c. or that

$$\Delta u_1 = \Delta u_5 = \Delta u_9 = \Delta u_{13} = \ldots.$$

This gives for the equation of the series,

$$\Delta u_z = \Delta u_{z+4},$$

or by integrating

$$u_z = u_{z+4} + b,$$

but when $z = 1$, $u_1 = u$, therefore $b = 0$, and

$$u_{z+1} - u_z = 0,$$

whose integral is

$$u_z = a(-\sqrt{-1})^z + b(-\sqrt{-1})^{z+1} + c(-\sqrt{-1})^{z+2} + d(-\sqrt{-1})^{z+3} + 5z.$$

The four constants being determined, by comparing this value of u_z with the first four terms of the series, we shall find

$$a = 0,\ b = -5,\ c = \tfrac{1}{2} - \sqrt{-1},\ \ d = \tfrac{1}{2} + \sqrt{-1},$$

and the value of u_z becomes

$$u_z = 5(z-1) + (\tfrac{1}{2} - \sqrt{-1})(\sqrt{-1})^z + (\tfrac{1}{2} + \sqrt{-1})(-\sqrt{-1})^z,$$

which expresses any term of the series

2, 4, 8, 16, 22, 24, 28, 36, 42, 44, 48.

It is necessary, for the success of this method, that we should have continued the given series until we arrive at some term whose unit's figure is the same as that of some term which has preceded it: now if we consider that this figure depends solely

solely on that of the one which occupied the same place in the preceding term, it will appear that the same digit must reappear in the course of ten terms at the utmost, since there are only ten digits, and that it may re-occur sooner. The same reasoning is applicable to the case of series whose first difference is equal to any multiple of the digits found in the unit's place of the corresponding term, or to those contained in the equation

$$\Delta u_z = a \times (\text{unit's figure of } u_z),$$

as also to those in which this is increased by a given quantity, as

$$\Delta u_z = a\,(\text{unit's figure of } u_z) + b.$$

If the second difference is equal to some multiple of the figure occurring in the unit's place of the next term, as in the series

2, 2, 4, 10, 16,

already given, since the unit's figure must always depend on the same figure in the first term of the series, and its first difference

2	0
2	2
4	6
10	6
.	.
.	.

whenever those two figures are the same, a similar period must reappear: now as there are only two figures concerned, they can only admit of 100 permutations, consequently, this is the greatest limit of the periods in such species of series.—In the one in question the period is comprised in ten terms. This reasoning may be extended to other forms of series in which higher differences are given in terms of the digits occurring in the unit's, ten's, or other places of u_z or u_{z+1} or elsewhere, but I am aware that it does not in its present form present that degree of generality which ought to be expected on such a subject: probably the attempt to solve directly that class of equations to which these and similar inquiries lead, may be attended with more valuable results.

As the term "*unit's figure of*" occurs frequently, it will be convenient to designate it by an abbreviation; that which I shall propose is the combination of the two initials, and I shall write the above equation of differences thus

$$\Delta u_z = a\,\mathrm{UF}u_z \ldots\ldots\ldots\ldots (a).$$

This may be reduced to a more usual form by the following method. If S_x represent the sum of the xth powers of unity, divided by ten; then

0 S_x

$$0\,S_x + 1\,S_{x+1} + 2\,S_{x+2} + 3\,S_{x+3} + 4\,S_{x+4} + 5\,S_{x+5} + 6\,S_{x+6} + 7\,S_{x+7} + 8\,S_{x+8} + 9\,S_{x+9},$$

will represent the figure which occurs in the unit's place of any number x: substituting u_z instead of x, we have

$$\frac{1}{a}\Delta u_z = 0\,S_{u_z} + 1\,S_{u_z+1} + 2\,S_{u_z+2} + \ldots 9\,S_{u_z+9} \ldots\ldots (b).$$

an equation in which u_z enters as an exponent.

From the previous knowledge of the form of the general terms of the series we are considering, it would appear that the general solution of the equations (a) and (b) is

$$u = 9z + c\,S_z + c_1\,S_{z+z} + c_2\,S_{z+2} + \ldots\ldots c_9\,S_{z+9},$$

where the constants must be determined from the conditions. In the further pursuit of any inquiries in this direction, much assistance may be derived by consulting a paper of Mr. Herschel's in the Philosophical Transactions for 1818, "On circulating functions."

Amongst the conditions for determining the general terms of series by some relation amongst particular figures, there occurs a curious class, in which, if we consider only whole numbers, the series becomes impossible after a certain number of terms.

Let the equation determining u_z be

$$\Delta u_z = \tfrac{1}{2}(\mathrm{UF}u_{z-1} + \mathrm{UF}u_{z+1}).$$

Then the following series conform to this law,

Series.	Diff.	Series.	Diff.	Series.	Diff.
1	3	4	6	1	9
4	5	10	4	10	1
9		14	4	11	1
		18		12	3
				15	

If the law is restricted to whole numbers, none of these series admit of any prolongation; nor have I, with that restriction, been able to discover any series of the kind possessing more than five terms.

Devonshire Street, Portland Place, March 29, 1824. C. BABBAGE.