thus we have the theorem

$$\frac{1}{r}\log_e\sqrt{\frac{1+\mu}{1-\mu}} = \int_0^\infty e^{-\lambda z}\,Y_0(\lambda\rho)\,d\lambda \quad\text{.................(44)},$$

which corresponds to the known theorem

$$\frac{1}{r} = \int_0^\infty e^{-\lambda z}\,J_0(\lambda\rho)\,d\lambda.$$

From (44), we obtain, by differentiation n times with respect to r, the formula

$$\frac{Q_n(\mu)}{r^{n+1}} = \frac{1}{n!}\int_0^\infty \lambda^n e^{-\lambda z}\,Y_0(\lambda\rho)\,d\lambda \quad\text{.................(45)},$$

where $Q_n(\mu)$ is the zonal harmonic of the second kind. As in the case of the harmonics of the first kind, we find

$$\frac{Q_n^m(\mu)}{r^{n+1}} = \frac{1}{(n-m)!}\int_0^\infty \lambda^n e^{-\lambda z}\,Y_m(\lambda\rho)\,d\lambda \quad\text{...........(46)};$$

thus the tesseral harmonic $\frac{Q_n^m(\mu)}{r^{n+1}}\cos m\phi$ is expressed as a definite integral involving the elements $e^{-\lambda z}\,Y_m(\lambda\rho)\cos m\phi$.

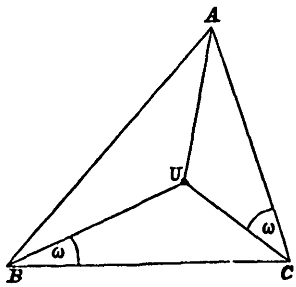
Note on a Variable Seven-points Circle, analogous to the Brocard Circle of a Plane Triangle. By JOHN GRIFFITHS, M.A. Received December 13th, 1893. Read December 14th, 1893.

The object of this note is to show that a seven-points circle can be constructed from a variable point U taken on one of three given circles connected with a triangle ABC.

1. On the side BC of a triangle ABC describe a circular arc BUC touching AC in C, and let U be any point on this arc. This con-

struction gives

$$\angle UBC = \angle UCA = \omega, \quad \text{and} \quad \angle BUC = \pi - C.$$

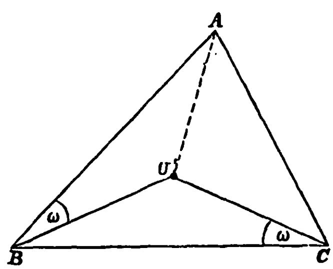


[If $\angle CUA$ be denoted by $\pi - \theta$, then

$$\cot\omega = \cot\theta + \cot B + \cot C.]$$

2. Let U' be the isogonal point of U; *i.e.*, let U' lie on the circular arc described on BC and touching AB in B, so that

$$\angle U'BA = \angle U'CB = \omega, \quad \text{and} \quad \angle BU'C = \pi - B.$$



[$\angle BU'A = \pi - \theta$,

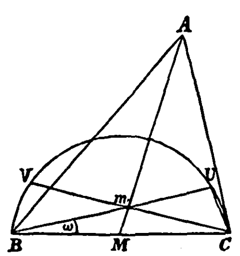
and $\quad \cot\omega = \cot\theta + \cot B + \cot C$,

as before.

If $\theta = A$, then ω is the Brocard angle of ABC.]

This gives U' as a point dependent on U.

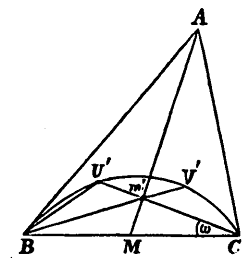
3. Let BU intersect the median AM of $\triangle ABC$ in m; produce the join of C and m to meet the circle $CUVB$ in V; then V is a second point dependent on U.



[The points U, V form two homographic divisions on the circle $CUVB$, in which C and B are corresponding points. Hence the envelope of UV is a conic (a hyperbola), having double contact with the circle, along the homographic axis AM.]

4. Similarly, let CU' intersect the median AM in m'; produce the

join of B and m' to meet the circle $BU'V'C$ in V'; then V' is a third point dependent on U.

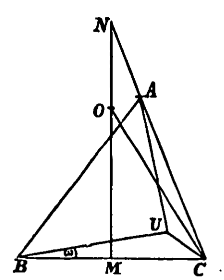


5. Let CV and BV' intersect in K. This gives K as a fourth point dependent on U.

6. Let the perpendicular to BC at M, the mid-point of BC, meet the side CA or CA produced in N; on CN describe a circular arc CON similar to the circular arc CUA, *i.e.*, such that

$$\angle CON = \angle CUA,$$

and let this arc CON meet the perpendicular MN in O; then O is a fifth point dependent on U.



If $$\angle CUA = \pi - \theta \text{ (see 1)},$$

then $$\angle CON = \pi - \theta \quad \text{and} \quad \angle COM = \theta,$$

where $$\cot \omega = \cot \theta + \cot B + \cot C.$$

If $$\theta = A,$$

U will be the positive Brocard point, and O will be the centre of the circumcircle ABC.

7. Lastly, let BU and CU' intersect in A'; then A' is a sixth point dependent on U.

Theorem I.

The seven points U, U', V, V', O, K, A' as defined above all lie on the circumference of a circle whose diameter is OK.

If U be the positive Brocard point, the circle in question is, in fact, the Brocard circle.

By similar constructions to the above with regard to the two other sides CA, AB of ABC, it is clear that we have in all three systems of variable seven-points circles, each of which has properties analogous to those of the Brocard circle of ABC.

Theorem II.

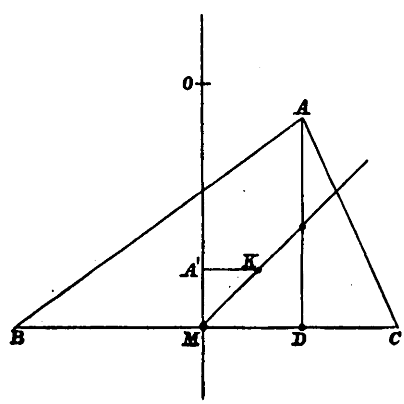
1. The centre of the circle through O, K, U, U', V, V', A' describes a hyperbola as U moves along the arc BUC.

2. The circle OKU is cut orthogonally by a fixed circle having its centre on the side BC.

3. Hence the envelope of the circle OKU is a bicircular quartic.

Theorem III.

As U moves along the fixed circle BUC, the points O and A' describe the right line MO perpendicular to BC at its mid-point M, while K moves on the line through M which bisects the perpendicular AD from A on BC.



If U be the positive Brocard point of ABC, then K is the symmedian point of ABC.

Theorem IV.

The angle subtended at O by UU' is equal to twice the angle UBC; *i.e.*, to 2ω. See 1.

Theorem V.

If from O we draw OA', OB', OC' perpendicular to the sides BC, CA, AB, meeting the circle OKU in A', B', C', then the triangle $A'B'C'$ is inversely similar to ABC.

As U moves on the fixed circular arc BUC, the points B' and C' each describe a hyperbola. [They can also be directly derived from the primary point U, since B' lies on OU and C' on BU'.]

Theorem VI.

Let $U'A$ meet the circle OKU again in B''; then the triangle $A'B''C'$ is similar to the pedal triangle of U with respect to ABC.

The above are some of the principal results which I have arrived at with regard to the variable seven-points circle in question. They have been deduced partly by elementary geometry, and partly by the use of isogonal coordinates, which I have employed in previous notes on the triangle.

The isogonal coordinates x, y, z of a point P are connected with its trilinear coordinates by the relations

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{a\,(a\alpha + b\beta + c\gamma)}{\Sigma a\beta\gamma},$$

which give $$\Sigma ax = \Sigma ayz.$$

It thus follows that the equation of a circle in this system of coordinates is expressed by a linear relation

$$\lambda x + \mu y + \nu z = \delta.$$

If we take U as expressed by its isogonal coordinates

$$x = \frac{\sin C}{\sin B}, \quad y = \frac{\sin \theta}{\sin(\theta + B)}, \quad z = \frac{\sin(\theta + C)}{\sin \theta},$$

I have found that the equation of the above seven-points circle OKU

is $x\dfrac{\sin B\sin C}{\sin A}\sin(2\theta-A)+y\sin\theta\sin(\theta+B)+z\sin\theta\sin(\theta+C)$

$$=2\sin^2\theta\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right)=2\sin^2\theta\frac{\sin B\sin C}{\sin A}\cot\omega,$$

where $\cot\omega=\cot\theta+\cot B+\cot C$. (See 1.)

By writing this equation in the form

$$\xi\cos 2\theta+\eta\sin 2\theta+\zeta=0,$$

where $\xi=0$, $\eta=0$, and $\zeta=0$ denote circles, it is seen at once that the envelope of the seven-points circle is the bicircular quartic expressed by

$$\xi^2+\eta^2=\zeta^2.$$

Since the points U', V, V', O, K, A' are all dependent on U, there is no particular difficulty in expressing their coordinates as functions of those of U.

If U be given by

$$x=\frac{\sin C}{\sin B},\quad y=\frac{\sin\theta}{\sin(\theta+B)},\quad z=\frac{\sin(\theta+C)}{\sin\theta},$$

the coordinates of the six points in question are:—

$$U';\ X=\frac{1}{x},\quad Y=\frac{1}{y},\quad Z=\frac{1}{z}.$$

$$V;\ X=x,\quad Y=xz,\quad Z=\frac{1}{z}+\frac{\sin(B-C)}{\sin B}.$$

$$V';\ X=\frac{1}{x},\quad Y=y+\frac{\sin(C-B)}{\sin C},\quad Z=\frac{1}{xy}.$$

$$O;\ \frac{X}{\cos\theta}=\frac{-Y}{\cos(\theta+C)}=\frac{-Z}{\cos(\theta+B)}=\frac{2\sin\theta}{\sin(2\theta-A)}.$$

$$K;\ \frac{X}{\sin\theta}=\frac{Y}{\sin(\theta+C)}=\frac{Z}{\sin(\theta+B)}=k.$$

where $$k = 2(\sin\theta \sin A + \sin B \sin C \cos\theta)$$

$$\div \{2\sin(\theta+B)\sin(\theta+C)\sin A + \sin B \sin C \sin(2\theta - A)\}.$$

$$A'; \quad \frac{X}{xy} = \frac{Y}{x^2} = \frac{Z}{yz} = \frac{xy\sin A + x^2 \sin B + yz \sin C}{xy(xz\sin A + yz \sin B + x^2 \sin C)},$$

or $$\frac{X}{\sin B \sin C \sin\theta} = \frac{Y}{\sin^2 C \sin(\theta+B)} = \frac{Z}{\sin^2 B \sin(\theta+C)}.$$

If $\theta = A$, we have the known point

$$\frac{X}{abc} = \frac{Y}{c^3} = \frac{Z}{b^3},$$

which lies on the Brocard circle.

As I have stated above, the coordinates X, Y, Z must, in all cases, satisfy the relation $\Sigma(X - YZ)\sin A = 0$.

[*Appendix.*

At the suggestion of Mr. Tucker I append some additional details with regard to the isogonal coordinates of the seven points U, U', V, V', O, K, A', in order to verify the fact that they are concyclic.

1. If we take the coordinates of the primary point U to be

$$x = \frac{\sin C}{\sin B}, \quad y = \frac{\sin\theta}{\sin(\theta+B)}, \quad z = \frac{\sin(\theta+C)}{\sin\theta},$$

the condition that this point shall lie on the circle expressed by the equation

$$X\frac{\sin B \sin C}{\sin A}\frac{\sin(2\theta - A)}{\sin^2\theta} + Y\frac{\sin(\theta+B)}{\sin\theta} + Z\frac{\sin(\theta+C)}{\sin\theta}$$

$$= 2\left(1 + \frac{\sin B \sin C}{\sin A}\cot\theta\right)$$

is $\sin^2 C \sin(2\theta - A) + \sin A \sin^2\theta + \sin A \sin^2(\theta+C)$

$$= 2\sin A \sin^2\theta + \sin B \sin C \sin 2\theta.$$

There is no difficulty in proving that this is a trigonometrical identity.

2. Since

$$z^2-\frac{1}{y^2}=\frac{\sin^2(\theta+C)-\sin^2(\theta+B)}{\sin^2\theta}=\frac{\sin(B-C)\sin(2\theta-A)}{\sin^2\theta},$$

it appears that the equation of the circle in question can also be written in the form

$$\left(z^2-\frac{1}{y^2}\right)(X-x)\frac{\sin B\sin C}{\sin A\sin(B-C)}+\frac{1}{y}(Y-y)+z(Z-z)=0.$$

This circle will pass through $U'\left(\frac{1}{x},\ \frac{1}{y},\ \frac{1}{z}\right)$ if

$$\left(z^2-\frac{1}{y^2}\right)\left(\frac{1}{x}-x\right)\frac{\sin B\sin C}{\sin A\sin(B-C)}+\frac{1}{y}\left(\frac{1}{y}-y\right)+z\left(\frac{1}{z}-z\right)=0,$$

i.e., if $$\left(\frac{1}{x}-x\right)\sin B\sin C=\sin A\sin(B-C),$$

or $$\sin^2 B-\sin^2 C=\sin A\sin(B-C).$$

3. The coordinates of V are

$$X=x,\quad Y=xz,\quad Z=\frac{1}{z}+\frac{\sin(B-C)}{\sin B};$$

V will therefore lie on the circle, if

$$\frac{1}{y}(xz-y)+z\left(\frac{1}{z}-z+\frac{\sin(B-C)}{\sin B}\right)=0,$$

or $$\frac{x}{y}-z+\frac{\sin(B-C)}{\sin B}=0.$$

Now, since

$$x=\frac{\sin C}{\sin B},\quad =\frac{\sin\theta}{\sin(\theta+B)},\quad z=\frac{\sin(\theta+C)}{\sin\theta},$$

we have

$$\frac{1}{y\sin B}-\frac{z}{\sin C}=\cot B+\cot\theta-(\cot C+\cot\theta)=\frac{\sin(C-B)}{\sin B\sin C},$$

i.e., $$\frac{x}{y}-z+\frac{\sin(B-C)}{\sin B}=0.$$

4. The coordinates of V' are

$$X=\frac{1}{x},\quad Y=y+\frac{\sin(C-B)}{\sin C},\quad Z=\frac{1}{xy}.$$

Therefore V' will lie on the circle, if

$$\left(z^2-\frac{1}{y^2}\right)\left(\frac{1}{x}-x\right)\frac{\sin B\sin C}{\sin A\sin(B-C)}+\frac{1}{y}\,\frac{\sin(C-B)}{\sin C}+z\left(\frac{1}{xy}-z\right)=0,$$

or
$$\left(z^2-\frac{1}{y^2}\right)+\frac{1}{y}\,\frac{\sin(C-B)}{\sin C}+z\left(\frac{1}{xy}-z\right)=0,$$

i.e., if
$$\frac{x}{y}-z+\frac{\sin(B-C)}{\sin B}=0.$$

This is an equation satisfied by the coordinates of U, as I have just proved.

5. The coordinates of O are

$$\frac{X}{\cos\theta}=\frac{Y}{-\cos(\theta+C)}=\frac{Z}{-\cos(\theta+B)}=\frac{2\sin\theta}{\sin(2\theta-A)}.$$

This point will lie on the circle expressed by

$$X\frac{\sin B\sin C}{\sin A}\,\frac{\sin(2\theta-A)}{\sin^2\theta}+Y\frac{\sin(\theta+B)}{\sin\theta}+Z\frac{\sin(\theta+C)}{\sin\theta}$$
$$=2\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right),$$

if
$$2\frac{\sin B\sin C}{\sin A}\cot\theta-2\,\frac{\sin(\theta+B)\cos(\theta+C)+\sin(\theta+C)\cos(\theta+B)}{\sin(2\theta-A)}$$
$$=2\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right),$$

i.e., if $\sin(\theta+B)\cos(\theta+C)+\sin(\theta+C)\cos(\theta+B)+\sin(2\theta-A)=0$,

or
$$\sin(2\theta+B+C)+\sin(2\theta-A)=0,$$
$$\sin(2\theta-A+\pi)+\sin(2\theta-A)=0.$$

6. The coordinates of K are

$$X=k\sin\theta,\quad Y=k\sin(\theta+C),\quad Z=k\sin(\theta+B),$$

where
$$\Sigma X\sin A=\Sigma YZ\sin A,$$

or
$$k=\frac{\sin\theta\sin A+\sin(\theta+C)\sin B+\sin(\theta+B)\sin C}{\sin(\theta+B)\sin(\theta+C)\sin A+\sin(\theta+B)\sin\theta\sin B+\sin(\theta+C)\sin\theta\sin C}$$
$$=\frac{2(\sin A\sin\theta+\sin B\sin C\cos\theta)}{\sin B\sin C\sin(2\theta-A)+2\sin A\sin(\theta+B)\sin(\theta+C)}.$$

K will therefore lie on the circle expressed by the equation

$$X\frac{\sin B\sin C}{\sin A}\,\frac{\sin(2\theta-A)}{\sin^2\theta}+Y\frac{\sin(\theta+B)}{\sin\theta}+Z\,\frac{\sin(\theta+C)}{\sin\theta}$$

$$=2\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right),$$

if $$k\left\{\frac{\sin B\sin C}{\sin A}\,\frac{\sin(2\theta-A)}{\sin\theta}+2\,\frac{\sin(\theta+B)\sin(\theta+C)}{\sin\theta}\right\}$$

$$=2\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right),$$

or $$k=\frac{2(\sin A\sin\theta+\sin B\sin C\cos\theta)}{\sin B\sin C\sin(2\theta-A)+2\sin A\sin(\theta+B)\sin(\theta+C)}.$$

7. The coördinates of A' are

$$X=k\sin B\sin C\sin\theta,\quad Y=k\sin^2 C\sin(\theta+B),\quad Z=k\sin^2 B\sin(\theta+C).$$

Now, since $$\Sigma X\sin A=\Sigma YZ\sin A,$$

we have $$k=\frac{N}{D},$$

where $$N=2(\sin A\sin\theta+\sin B\sin C\cos\theta),$$

$$D=\sin A\sin B\sin C\sin(\theta+B)\sin(\theta+C)$$
$$+\left\{\sin^3 B\sin(\theta+C)+\sin^3 C\sin(\theta+B)\right\}\sin\theta;$$

A' will therefore lie on the circle expressed by

$$X\frac{\sin B\sin C}{\sin A}\,\frac{\sin(2\theta-A)}{\sin^2\theta}+Y\frac{\sin(\theta+B)}{\sin\theta}+Z\frac{\sin(\theta+C)}{\sin\theta}$$

$$=2\left(1+\frac{\sin B\sin\theta\cot\theta}{\sin A}\right),$$

if $$k\left\{\sin^2 B\sin^2 C\sin(2\theta-A)+\sin^2 C\sin^2(\theta+B)\sin A\right.$$
$$\left.+\sin^2 B\sin^2(\theta+C)\sin A\right\}$$

$$=2(\sin A\sin\theta+\sin B\sin C\cos\theta)=N.$$

It thus appears that the circle will pass through A', if

$$\sin^2 B\sin^2 C\sin(2\theta-A)+\left\{\sin^2 C\sin^2(\theta+B)+\sin^2 B\sin^2(\theta+C)\right\}\sin A$$
$$=D=\sin A\sin B\sin C\sin(\theta+B)\sin(\theta+C)$$
$$+\left\{\sin^3 B\sin(\theta+C)+\sin^3 C\sin(\theta+B)\right\}\sin\theta.$$

There is no especial difficulty in proving that this is a trigonometrical identity.

8. It may be noticed here that, since the above note was communicated to the Society, I have found that the circle $UU'V\ldots$ passes through an eighth point dependent on U; viz., the intersection of the circles AOU, ABU'.

The coordinates of this point, W, say, are

$$x = \frac{\sin 2\theta}{\sin(2\theta - A)}, \quad y = \frac{\sin\theta}{\sin(\theta + B)}, \quad z = \frac{\sin\theta}{\sin(\theta + C)}.$$

As θ varies, or the circle $UU'V\ldots$ moves, the locus of W is a circular cubic having A for a double point.

The equation of this curve is

$$\frac{1}{y\sin B} - \frac{1}{z\sin C} = \cot B - \cot C,$$

which proves that it passes through B, C, and the foot of the perpendicular from A upon BC, and has an asymptote parallel to the median AM.]

On a Theorem of Liouville's. By Mr. G. B. Mathews.

Read December 14th, 1893.

In the first of the series of papers "Sur quelques Formules Générales qui peuvent être Utiles dans la Théorie des Nombres" [*Journ. de Math.*, (2) iii. (1858), p. 143], Liouville has stated without proof the following remarkable proposition:—

Let $2m$, the double of any odd integer, be expressed in all possible ways as the sum of two odd numbers, a and b, where the decompositions $2m = a+b$ and $2m = b+a$ are considered distinct, unless $a = b = m$; let α denote any divisor of a, and β any divisor of b, and let $f(x)$ be any even function of x, that is, such that

$$f(-x) = f(x).$$

Then, if μ denotes any divisor of m,

$$\Sigma\{f(\alpha-\beta) - f(\alpha+\beta)\} = \Sigma\mu\{f(0) - f(2\mu)\},$$

where the summation on the left applies to all pairs of divisors α and