it follows that the stress across the plane z is at each point directed radially from the point of application of the load. The magnitude of this stress is

$$\frac{W}{2\pi}\frac{\sqrt{AL}\,r}{\{(\sqrt{AC}-L)^2-(F+L)^2\}^{\frac{1}{2}}}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right),$$

where
$$r^2 = x^2+y^2+z^2,$$
$$r_1^2 = x^2+y^2+z^2/\gamma_1,$$
$$r_2^2 = x^2+y^2+z^2/\gamma_2.$$

To pass to the case of the isotropic solid, put

$$\gamma_1 = 1+\epsilon_1, \quad \gamma_2 = 1+\epsilon_2,$$

where ϵ_1, ϵ_2 ultimately vanish. Then

$$\frac{1}{r_1^3}-\frac{1}{r_2^3} = \frac{3z^2}{2r^5}(\epsilon_1-\epsilon_2)+\&c.$$

and
$$\frac{1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}} = \frac{2}{\epsilon_2-\epsilon_1}+\&c.$$

Hence in the limit the stress becomes

$$\frac{3W}{2\pi}\frac{z^2}{r^4},$$

and this is Boussinesq's result.

Some Quadrature-Formulæ. *By* W. F. Sheppard, M.A., LL.M.

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I. *Equidistant Bounding Ordinates.*

1. The formulæ to be considered are those giving the area of a curve in terms of a series of equidistant ordinates, the two extreme ordinates forming the boundary of the area in question. Thus, if

$$z_0,\ z_1,\ z_2,\ \ldots,\ z_m$$

denote the ordinates, at successive distances h, and if the corresponding abscissæ are

$$x_0,\ x_1 = x_0+h,\ x_2 = x_0+2h,\ \ldots,\ x_m = x_0+mh,$$

the area to be determined is

$$A = \int_{x_0}^{x_m} f(x)\,dx,$$

where

$$\left.\begin{array}{l} z_0 = f(x_0) \\ z_1 = f(x_1) \\ \vdots \quad\ \vdots \\ z_m = f(x_m) \end{array}\right\}.$$

It is assumed that $z \equiv f(x)$ is positive throughout the whole range considered.

The most simple rule is the *trapezoidal rule*, which gives

$$A = h\left(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-1}+\tfrac{1}{2}z_m\right). \tag{1}$$

Simpson's rule, for cases in which m is even, gives

$$A = \tfrac{1}{3}h\left(z_0+4z_1+2z_2+4z_3+2z_4+\ldots+2z_{m-2}+4z_{m-1}+z_m\right), \tag{2}$$

the coefficients from z_1 to z_{m-2} being alternately 4 and 2.

Simpson's second rule, for cases in which m is a multiple of 3, gives

$$A = \tfrac{3}{8}h\left(z_0+3z_1+3z_2+2z_3+3z_4+\ldots+2z_{m-3}+3z_{m-2}+3z_{m-1}+z_m\right), \tag{3}$$

the sequence of coefficients from z_1 to z_{m-3} being 3, 3, 2.

Weddle's rule, for cases in which m is a multiple of 6, gives

$$\begin{aligned} A = \tfrac{3}{8}h\,(z_0+5z_1+z_2+6z_3+z_4+5z_5+2z_6+\ldots \\ \ldots+2z_{m-6}+5z_{m-5}+z_{m-4}+6z_{m-3}+z_{m-2}+5z_{m-1}+z_m), \end{aligned} \tag{4}$$

the sequence of coefficients from z_1 to z_{m-6} being 5, 1, 6, 1, 5, 2.

These are the rules that are best known.

2. The latter three rules are usually regarded as being obtained as follows.

For Simpson's rule we pass a parabola through the tops of the ordinates z_0, z_1, z_2; and we find that the area bounded by this parabola and the ordinates z_0 and z_2 is

$$\tfrac{1}{3}h\,(z_0+4z_1+z_2).$$

This area is substituted for the true area of this portion of the curve. Similarly, for the area between z_2 and z_4, we get

$$\tfrac{1}{3}h\,(z_2+4z_3+z_4),$$

and so on. Adding together all these areas, we get (2).

In the same way, for Simpson's second rule, we pass a parabola of the third degree through the tops of four consecutive ordinates, and obtain, for the area bounded by z_0 and z_3,

$$\tfrac{3}{8}h\,(z_0+3z_1+3z_2+z_3).$$

Proceeding as before, we arrive at (3).

Weddle's rule is obtained in the same way, by passing a parabola of the sixth degree through the tops of seven consecutive ordinates; but the expression for the area of this parabola is slightly altered, so as to give a result which is simpler for purposes of numerical computation.

The formulæ may, however, be regarded in another way. Let A_1 denote the area given by the trapezoidal rule (1), so that

$$A_1 = h\,(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-1}+\tfrac{1}{2}z_m). \tag{5}$$

Now suppose that m is even; and let A_2 denote what A_1 would become if we left out of consideration the alternate ordinates $z_1, z_3, \ldots, z_{m-1}$. Then we have

$$A_2 = 2h\,(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-2}+\tfrac{1}{2}z_m), \tag{6}$$

and Simpson's rule may be written

$$A = \frac{4A_1-A_2}{3}. \tag{7}$$

Again, suppose that m is a multiple of 3; and let A_3 denote what A_1 would become if we left out of consideration the ordinates $z_1, z_2, z_4, z_5, \ldots, z_{m-4}, z_{m-2}, z_{m-1}$, taking only every third ordinate. Then

$$A_3 = 3h\,(\tfrac{1}{2}z_0+z_3+z_6+\ldots+z_{m-3}+\tfrac{1}{2}z_m), \tag{8}$$

and Simpson's second rule may be written

$$A = \frac{9A_1-A_3}{8}. \tag{9}$$

Finally, let m be a multiple of 6. Then Weddle's rule may be written

$$A = \frac{15A_1-6A_2+A_3}{10}. \tag{10}$$

3. Let z', z'', ... denote the differential coefficients of $z \equiv f(x)$ with regard to x. Then the *exact* area under consideration is given by the Euler-Maclaurin formula

$$A = h\left(\tfrac{1}{2}z_0 + z_1 + z_2 + \dots + z_{m-1} + \tfrac{1}{2}z_m\right)$$
$$+ \left[-\frac{B_1}{2!}h^2z' + \frac{B_2}{4!}h^4z''' - \frac{B_3}{6!}h^6z^{\mathrm{v}} + \dots\right]_{x=x_0}^{x=x_m}, \quad (11)$$

where B_1, B_2, B_3, ... are Bernoulli's numbers, and

$$\left[\phi(x)\right]_{x=x_0}^{x=x_m}$$

denotes $\phi(x_m) - \phi(x_0)$.

Thus, with our previous notation, we have

$$A = A_1 + \left[-\frac{B_1}{2!}h^2z' + \frac{B_2}{4!}h^4z''' - \frac{B_3}{6!}h^6z^{\mathrm{v}} + \dots\right]_{x=x_0}^{x=x_m} \quad (12)$$

The formula (12) is exact; and we shall also get an exact expression by the same formula in terms of z_0, z_2, ..., z_m, when m is even. Thus, writing $2h$ for h, we have

$$A = A_2 + \left[-\frac{B_1}{2!}2^2h^2z' + \frac{B_2}{4!}2^4h^4z''' - \frac{B_3}{6!}2^6h^6z^{\mathrm{v}} + \dots\right]_{x=x_0}^{x=x_m}. \quad (13)$$

Multiplying (12) and (13) by $\frac{4}{3}$ and $-\frac{1}{3}$ respectively, and adding, we find

$$A = \frac{4A_1 - A_2}{3} + \tfrac{1}{3}\left[-\frac{B_2}{4!}(2^4-2^2)\,h^4z''' + \frac{B_3}{6!}(2^6-2^2)\,h^6z^{\mathrm{v}} - \dots\right]_{x=x_0}^{x=x_m}. \quad (14)$$

Comparing this with (7), we see the amount of error involved in using Simpson's rule (2). Similarly, for Simpson's second rule, we have as the true formula

$$A = \frac{9A_1 - A_3}{8} + \tfrac{1}{8}\left[-\frac{B_2}{4!}(3^4-3^2)\,h^4z''' + \frac{B_3}{6!}(3^6-3^2)\,h^6z^{\mathrm{v}} - \dots\right]_{x=x_0}^{x=x_m}, \quad (15)$$

and for Weddle's rule

$$A = \frac{15A_1 - 6A_2 + A_3}{10} + \tfrac{1}{10}\left[-\frac{B_3}{6!}(3^6 - 6.2^6 + 15.1^6)\,h^6z^{\mathrm{v}} + \dots\right]_{x=x_0}^{x=x_m}. \quad (16)$$

4. To extend these formulæ, let f denote any one of the factors of m (including 1 and m), and let us write

$$A_f = fh\left(\tfrac{1}{2}z_0 + z_f + z_{2f} + \ldots + z_{m-f} + \tfrac{1}{2}z_m\right). \tag{17}$$

Then, if a, b, c, ... are different values of f, and if, for convenience of printing, we write

$$\lambda_r = B_r/(2r)! \tag{18}$$

we have
$$\left.\begin{aligned} A &= A_a + \Big[-\lambda_1 a^2h^2z' + \lambda_2 a^4h^4z''' - \lambda_3 a^6h^6z^{\text{v}} + \ldots\Big]_{x=x_0}^{x=x_m} \\ A &= A_b + \Big[-\lambda_1 b^2h^2z' + \lambda_2 b^4h^4z''' - \lambda_3 b^6h^6z^{\text{v}} + \ldots\Big]_{x=x_0}^{x=x_m} \\ A &= A_c + \Big[-\lambda_1 c^2h^2z' + \lambda_2 c^4h^4z''' - \lambda_3 c^6h^6z^{\text{v}} + \ldots\Big]_{x=x_0}^{x=x_m} \\ &\vdots \qquad\quad \vdots \end{aligned}\right\}. \tag{19}$$

Multiplying these equations by p, q, r, ..., and adding, we get

$$\begin{aligned}(p+q+r+\ldots)A &= pA_a + qA_b + rA_c + \ldots \\ &\ldots + \Big[-(pa^2+qb^2+rc^2+\ldots)\lambda_1 h^2z' \\ &\qquad + (pa^4+qb^4+rc^4+\ldots)\lambda_2 h^4z''' - \ldots\Big]_{x=x_0}^{x=x_m}.\end{aligned} \tag{20}$$

Hence, if there are i factors a, b, c, ..., and if we take p, q, r, ..., to satisfy the equations

$$\left.\begin{aligned} pa^2 + qb^2 + rc^2 + \ldots &= 0 \\ pa^4 + qb^4 + rc^4 + \ldots &= 0 \\ \vdots \qquad\qquad &\;\;\vdots \\ pa^{2i-2} + qb^{2i-2} + rc^{2i-2} + \ldots &= 0 \end{aligned}\right\}, \tag{21}$$

we have

$$\begin{aligned}(p+q+r+\ldots)A &= pA_a + qA_b + rA_c + \ldots \\ &+ \Big[(-)_i (pa^{2i}+qb^{2i}+rc^{2i}+\ldots)\frac{B_i}{(2i)!}h^{2i}\frac{d^{2i-1}z}{dx^{2i-1}} \\ &+ (-)^{i+1}(pa^{2i+2}+qb^{2i+2}+rc^{2i+2}+\ldots)\frac{B_{i+1}}{(2i+2)!}h^{2i+2}\frac{d^{2i+1}z}{dx^{2i+1}} + \ldots\Big]_{x=x_0}^{x=x_m}.\end{aligned} \tag{22}$$

Thus, by using
$$A = \frac{pA_a + qA_b + rA_c + \ldots}{p+q+r+\ldots} \tag{23}$$

as an approximate formula, we introduce an error which only involves differential coefficients of z from the $(2i-1)$th onwards. Or, if we regard the differential coefficients as expressed in terms of differences, the error introduced depends only on the $(2i-1)$th and higher differences.

The equations (21) give

$$\frac{1}{p} : \frac{1}{q} : \frac{1}{r} : \dots :: a^2(a^2-b^2)(a^2-c^2) \dots : b^2(b^2-a^2)(b^2-c^2) \dots$$
$$\dots : c^2(c^2-a^2)(c^2-b^2) \dots : \dots ; \quad (24)$$

and, in particular, for $i = 2$,

$$p : q :: b^2 : -a^2. \quad (25)$$

Substituting from (24) in (23) and (22), we get the approximate formula and the error in the approximation.

5. The following are some special cases, a being taken equal to 1 throughout.

(i.) $m = M(2)$; $a = 1$, $b = 2$. *Simpson's rule.*—

$$p : q :: 4 : -1,$$
$$A = \tfrac{1}{3}(4A_1 - A_2) = A_1 + \tfrac{1}{3}(A_1 - A_2).$$

(ii.) $m = M(3)$; $a = 1$, $b = 3$. *Simpson's second rule.*—

$$p : q :: 9 : -1,$$
$$A = \tfrac{1}{8}(9A_1 - A_3) = A_1 + \tfrac{1}{8}(A_1 - A_3).$$

(iii.) $m = M(4)$; $a = 1$, $b = 2$, $c = 4$.—

$$p : q : r :: 64 : -20 : 1,$$
$$A = \tfrac{1}{45}(64A_1 - 20A_2 + A_4) = A_1 + \tfrac{4}{9}(A_1 - A_2) - \tfrac{1}{45}(A_1 - A_4).$$

This is equivalent to the continued repetition of the formula for the area of a curve in terms of five equidistant ordinates [Boole and Moulton, *Finite Differences*, p. 47, formula (21)].

(iv.) $m = M(5)$; $a = 1$, $b = 5$.—

$$p : q :: 25 : -1,$$
$$A = \tfrac{1}{24}(25A_1 - A_5) = A_1 + \tfrac{1}{24}(A_1 - A_5).$$

(v.) $m = M(6)$; $a = 1$, $b = 2$, $c = 3$.—

$$p : q : r :: 15 : -6 : 1,$$
$$A = \tfrac{1}{10}(15A_1 - 6A_2 + A_3) = A_1 + \tfrac{1}{2}(A_1 - A_2) - \tfrac{1}{10}(A_2 - A_3).$$

This is Weddle's rule.

(vi.) $m = M(6)$; $a = 1$, $b = 2$, $c = 3$, $d = 6$.—

$$p : q : r : s :: 1296 : -567 : 112 : -1,$$

$$A = \tfrac{1}{840}(1296A_1 - 567A_2 + 112A_3 - A_6)$$
$$= A_1 + \tfrac{19}{35}(A_1 - A_2) - \tfrac{2}{15}(A_2 - A_3) + \tfrac{1}{840}(A_2 - A_6).$$

(vii.) $m = M(8)$; $a = 1$, $b = 2$, $c = 4$, $d = 8$.—

$$p : q : r : s :: 4096 : -1344 : 84 : -1,$$

$$A = \tfrac{1}{2835}(4096A_1 - 1344A_2 + 84A_4 - A_8)$$
$$= A_1 + \tfrac{28}{27}\left\{\tfrac{3}{7}(A_1 - A_2) - \tfrac{1}{35}(A_2 - A_4)\right\} + \tfrac{1}{2835}(A_1 - A_8).$$

(viii.) $m = M(9)$; $a = 1$, $b = 3$, $c = 9$.—

$$p : q : r :: 729 : -90 : 1,$$

$$A = \tfrac{1}{640}(729A_1 - 90A_3 + A_9) = A_1 + (1 + \tfrac{1}{8})\tfrac{1}{8}(A_1 - A_3) - \tfrac{1}{640}(A_1 - A_9).$$

(ix.) $m = M(10)$; $a = 1$, $b = 2$, $c = 5$.—

$$p : q : r :: 175 : -50 : 1,$$

$$A = \tfrac{1}{126}(175A_1 - 50A_2 + A_5) = A_1 + (1 + \tfrac{1}{6})\tfrac{1}{3}(A_1 - A_2) - \tfrac{1}{126}(A_2 - A_5).$$

(x.) $m = M(10)$; $a = 1$, $b = 2$, $c = 5$, $d = 10$.—

$$p : q : r : s :: 70000 : -20625 : 528 : -7,$$

$$A = \tfrac{1}{49896}(70000A_1 - 20625A_2 + 528A_5 - 7A_{10})$$
$$= A_1 + (\tfrac{1}{2} - \tfrac{1}{12} - \tfrac{1}{72})(A_1 - A_2) - \tfrac{2}{189}(A_2 - A_5) + \tfrac{1}{7128}(A_1 - A_{10}).$$

(xi.) $m = M(12)$; $a = 1$, $b = 2$, $c = 3$, $d = 4$.—

$$p : q : r : s :: 56 : -28 : 8 : -1,$$

$$A = \tfrac{1}{35}(56A_1 - 28A_2 + 8A_3 - A_4)$$
$$= A_1 + \tfrac{3}{5}(A_1 - A_2) - \tfrac{1}{5}(A_2 - A_3) + \tfrac{1}{35}(A_3 - A_4).$$

This is very convenient for purposes of calculation.

(xii.) $m = M(12)$; $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 6$.—

$$p : q : r : s : t :: 1728 : -945 : 320 : -54 : 1,$$

$$A = \tfrac{1}{1050}(1728A_1 - 945A_2 + 320A_3 - 54A_4 + A_6).$$

(xiii.) $m = M(15)$; $a = 1$, $b = 3$, $c = 5$.—

$$p : q : r :: 150 : -25 : 3,$$

$$A = \tfrac{1}{128}(150A_1 - 25A_3 + 3A_5).$$

6. To illustrate some of these formulæ, let us take

$$m = 12,$$

and

$$A = \int_0^1 \frac{dx}{1+x}.$$

The true value of this integral, to ten places of decimals, is

$$A = \cdot 69314\ 71806.$$

The thirteen ordinates are

x	z	x	z
0	1·00000 00000	7/12	·63157 89474
1/12	·92307 69231	8/12	·60000 00000
2/12	·85714 28571	9/12	·57142 85714
3/12	·80000 00000	10/12	·54545 45455
4/12	·75000 00000	11/12	·52173 91304
5/12	·70588 23529	12/12	·50000 00000
6/12	·66666 66667		

These give

$$A_1 = \cdot 69358\ 08329,$$

$$A_2 = \cdot 69487\ 73449,$$

$$A_3 = \cdot 69702\ 38095,$$

$$A_4 = \cdot 70000\ 00000,$$

$$A_6 = \cdot 70833\ 33333;$$

so that we have

Formula.	Value of A.	Error $\times 10^{10}$.
(i.) (Simpson's rule)	·69314 86622	+14816
(ii.) (Simpson's second rule)	·69315 04608	+32802
(iii.)	·69314 72535	+ 729
(v.) (Weddle's rule)	·69314 72234	+ 428
(xi.)	·69314 71846	+ 40
(xii.)	·69314 71816	+ 10

7. In using rules such as Simpson's, we get rid of the terms in z', but at the expense of increasing subsequent terms; thus, if the factors are 1 and b, the coefficient of h^4z''' is altered from λ_2 to $-b^2\lambda_2$. Similarly, with Weddle's rule, we get rid of the terms in z' and z''', but, if the factors are 1, b, and c, the coefficient of h^6z^{v} is altered from $-\lambda_3$ to $-b^2c^2\lambda_3$.

If, therefore, we know—as in certain cases we do know—that the bounding values of z', and perhaps also those of z''' and z^{v}, are zero, the best value to take is A_1. The use of Simpson's rule will usually lead to a worse result; and the effect of Weddle's rule may be still worse. It may happen, however, that the bounding values of z' are known to be zero, and that nothing is known about those of z''' and z^{v}. We should then modify our formulæ by taking $p, q, r, \ldots$ to satisfy the equations

$$\left.\begin{array}{l} pa^4+qb^4+rc^4+\ldots=0 \\ pa^6+qb^6+rc^6+\ldots=0 \\ \quad\vdots \qquad\qquad\qquad \vdots \\ pa^{2i}+qb^{2i}+rc^{2i}+\ldots=0 \end{array}\right\},$$

which would give

$$\frac{1}{p}:\frac{1}{q}:\frac{1}{r}:\ldots :: a^4(a^2-b^2)(a^2-c^2)\ldots : b^4(b^2-a^2)(b^2-c^2)\ldots$$
$$\ldots : c^4(c^2-a^2)(c^2-b^2)\ldots : \ldots ;$$

and so on.

II. *Equidistant Mid-Ordinates.*

8. In some cases the known ordinates are not the bounding ordinates $z_0, z_1, z_2, \ldots, z_m$, but the intermediate ordinates $z_{\frac{1}{2}}, z_{\frac{3}{2}}, \ldots, z_{m-\frac{1}{2}}$. To express the area in terms of the latter, we have, by (11),

$$A = h\left(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-1}+\tfrac{1}{2}z_m\right)+\left[-\frac{B_1}{2!}h^2z'+\frac{B_2}{4!}h^4z'''-\ldots\right]_{x=x_0}^{x=x_m},$$

and, taking intervals $\frac{1}{2}h$,

$$A = \tfrac{1}{2}h\left(\tfrac{1}{2}z_0+z_{\frac{1}{2}}+z_1+z_{\frac{3}{2}}+\ldots+z_{m-\frac{1}{2}}+\tfrac{1}{2}z_m\right)$$
$$+\left[-\frac{B_1}{2^2.2!}h^2z'+\frac{B_2}{2^4.4!}h^4z'''-\ldots\right]_{x=x_0}^{x=x_m}.$$

Multiplying the second of these formulæ by 2, and subtracting the first, we get

$$A = S_1 + \Big[(1-1/2)\frac{B_1}{2!}h^2z' - (1-1/2^3)\frac{B_2}{4!}h^4z'''$$
$$+(1-1/2^5)\frac{B_3}{6!}h^6z^{v} - \dots\Big]_{x=x_0}^{x=x_m}, \quad (26)$$

where $$S_1 = h(z_{\frac{1}{2}} + z_{\frac{3}{2}} + \dots + z_{m-\frac{1}{2}}). \quad (27)$$

The leading term in square brackets in (26) is half the corresponding term in (12), and therefore, if these terms are not eliminated, the use of mid-ordinates will generally give a better result than the use of bounding ordinates. But the mid-ordinates not convenient for more accurate formulæ. The reasoning of § 4 applies, but it is necessary that the factors $a, b, c, \dots$ should be odd numbers. Thus, if m is a multiple of 3, Simpson's second rule applies; so that

$$A = S_1 + \tfrac{1}{8}(S_1 - S_3)$$

gives a better result than $A = S_1$, where

$$S_3 = 3h(z_{\frac{3}{2}} + z_{\frac{9}{2}} + \dots + z_{m-\frac{3}{2}}).$$

III. *Miscellaneous Formulæ.*

9. Various methods have been devised for getting rid of the terms in z' in the Euler-Maclaurin formula, or in the corresponding formula for mid-ordinates, so as to retain the trapezoidal rule as the main basis of calculation. Thus, in Parmentier's rule, the ordinates taken are $z_{\frac{1}{2}}, z_{\frac{3}{2}}, z_{\frac{5}{2}}, \dots, z_{m-\frac{1}{2}}$, and also z_0 and z_m. The values of $\frac{1}{2}hz'$ are then taken as equal to $z_{\frac{1}{2}} - z_0$ and $z_m - z_{m-\frac{1}{2}}$ respectively, so that the formula becomes

$$A = S_1 + \tfrac{1}{12}h(z_0 - z_{\frac{1}{2}} - z_{m-\frac{1}{2}} + z_m). \quad (28)$$

This, however, is not a very good rule; for, although it gets rid of the terms in z', it introduces terms in z''. These latter terms can be kept out by a slight alteration in the formula.

10. Let $$z_{\frac{1}{2}m}$$

denote the mid-ordinate of the whole area, the magnitude of this ordinate not being necessarily known. Then, if we take two ordinates

$$z_{\frac{1}{2}m(1-a)}, \quad z_{\frac{1}{2}m(1+a)}$$

at equal distances from $z_{\frac{1}{2}m}$, and if we write

$$z = f(x)$$

and $$\tfrac{1}{2}mhd/dx = D,$$

we have

$$\begin{aligned}
z_{\frac{1}{2}m(1-a)} + z_{\frac{1}{2}m(1+a)} &= f(x_{\frac{1}{2}m} - \tfrac{1}{2}mah) + f(x_{\frac{1}{2}m} + \tfrac{1}{2}mah) \\
&= 2\cosh aD . f(x_{\frac{1}{2}m}) \\
&= 2f(x_{\frac{1}{2}m}) + \left[\frac{\cosh aD - 1}{\sinh D} f(x)\right]_{x=x_0}^{x=x_m} \\
&= 2z_{\frac{1}{2}m} + \left[\frac{a^2}{2!}\tfrac{1}{2}mhz' - \frac{2a^2-a^4}{4!}(\tfrac{1}{2}mh)^3 z''' + \frac{7a^2-5a^4+a^6}{6!}(\tfrac{1}{2}mh)^5 z^{\mathrm{v}}\right. \\
&\qquad \left. - \frac{124a^2-98a^4+28a^6-3a^8}{3.8!}(\tfrac{1}{2}mh)^7 z^{\mathrm{vii}} + \dots\right]_{x=x_0}^{x=x_m}. \quad (29)
\end{aligned}$$

Taking another pair of ordinates

$$z_{\frac{1}{2}m(1-\beta)}, \quad z_{\frac{1}{2}m(1+\beta)},$$

we get a similar formula. Eliminating $z_{\frac{1}{2}m}$ between the two formulæ, we find

$$\begin{aligned}
&z_{\frac{1}{2}m(1-a)} - z_{\frac{1}{2}m(1-\beta)} - z_{\frac{1}{2}m(1+\beta)} + z_{\frac{1}{2}m(1+a)} \\
&= (a^2-\beta^2)\left[\tfrac{1}{4}mhz' - \frac{2-(a^2+\beta^2)}{192}m^3h^3z''' + \frac{7-5(a^2+\beta^2)+(a^4+a^2\beta^2+\beta^4)}{23040}m^5h^5z^{\mathrm{v}}\right. \\
&\qquad \left. - \frac{124-98(a^2+\beta^2)+28(a^4+a^2\beta^2+\beta^4)-3(a^2+\beta^2)(a^4+\beta^4)}{15482880}m^7h^7z^{\mathrm{vii}} + \dots\right]_{x=x_0}^{x=x_m}.
\end{aligned} \quad (30)$$

This formula can be used for eliminating the terms in z', z''', ... from (11) or (26). Thus, if we take two pairs of ordinates, we should get as our approximate formulæ

$$\begin{aligned}
A = h\,(\tfrac{1}{2}z_0 + z_1 + z_2 + \dots + z_{m-1} + \tfrac{1}{2}z_m) \\
- \frac{1}{3m(a^2-\beta^2)}h\,(z_{\frac{1}{2}m(1-a)} - z_{\frac{1}{2}m(1-\beta)} - z_{\frac{1}{2}m(1+\beta)} + z_{\frac{1}{2}m(1+a)}), \quad (31)
\end{aligned}$$

$$\begin{aligned}
A = h\,(z_{\frac{1}{2}} + z_{\frac{3}{2}} + z_{\frac{5}{2}} + \dots + z_{m-\frac{1}{2}}) \\
+ \frac{1}{6m(a^2-\beta^2)}h\,(z_{\frac{1}{2}m(1-a)} - z_{\frac{1}{2}m(1-\beta)} - z_{\frac{1}{2}m(1+\beta)} + z_{\frac{1}{2}m(1+a)}). \quad (32)
\end{aligned}$$

Or, if we take three pairs of ordinates, we shall get

$$A = h\left(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-1}+\tfrac{1}{2}z_m\right)$$
$$-\frac{2-4/(5m^2)-(\beta^2+\gamma^2)}{3m(a^2-\beta^2)(a^2-\gamma^2)}\,h\left(z_{\frac{1}{2}m(1-a)}+z_{\frac{1}{2}m(1+a)}\right)$$
$$-\frac{2-4/(5m^2)-(\gamma^2+a^2)}{3m(\beta^2-\gamma^2)(\beta^2-a^2)}\,h\left(z_{\frac{1}{2}m(1-\beta)}+z_{\frac{1}{2}m(1+\beta)}\right)$$
$$-\frac{2-4/(5m^2)-(a^2+\beta^2)}{3m(\gamma^2-a^2)(\gamma^2-\beta^2)}\,h\left(z_{\frac{1}{2}m(1-\gamma)}+z_{\frac{1}{2}m(1+\gamma)}\right), \tag{33}$$

$$A = h\left(z_{\frac{1}{2}}+z_{\frac{3}{2}}+z_{\frac{5}{2}}+\ldots+z_{m-\frac{1}{2}}\right)$$
$$+\frac{2-7/(5m^2)-(\beta^2+\gamma^2)}{6m(a^2-\beta^2)(a^2-\gamma^2)}\,h\left(z_{\frac{1}{2}m(1-a)}+z_{\frac{1}{2}m(1+a)}\right)$$
$$+\frac{2-7/(5m^2)-(\gamma^2+a^2)}{6m(\beta^2-\gamma^2)(\beta^2-a^2)}\,h\left(z_{\frac{1}{2}m(1-\beta)}+z_{\frac{1}{2}m(1+\beta)}\right)$$
$$+\frac{2-7/(5m^2)-(a^2+\beta^2)}{6m(\gamma^2-a^2)(\gamma^2-\beta^2)}\,h\left(z_{\frac{1}{2}m(1-\gamma)}+z_{\frac{1}{2}m(1+\gamma)}\right). \tag{34}$$

If we take only the two pairs of ordinates, a and β should be as nearly equal to 1 as possible, in order that the new term in z''' may be as small as possible. Thus, if we only know the ordinates z_0, z_1, …, z_m, we have $a = 1$, $\beta = 1-2/m$, and (31) gives

$$A = A_1-\tfrac{1}{12}\frac{m}{m-1}\,h\left(z_0-z_1-z_{m-1}+z_m\right). \tag{35}$$

Similarly, if we only know the mid-ordinates, we have $a = 1-1/m$, $\beta = 1-3/m$, and (32) gives

$$A = S_1+\tfrac{1}{24}\frac{m}{m-2}\,h\left(z_{\frac{1}{2}}-z_{\frac{3}{2}}-z_{m-\frac{3}{2}}+z_{m-\frac{1}{2}}\right). \tag{36}$$

In the same way, if we know the bounding ordinates z_0 and z_{m-1} and the extreme mid-ordinates $z_{\frac{1}{2}}$ and $z_{m-\frac{1}{2}}$, we find

$$A = A_1-\tfrac{1}{6}\,\frac{2m}{2m-1}\,h\left(z_0-z_{\frac{1}{2}}-z_{m-\frac{1}{2}}+z_m\right), \tag{37}$$

$$A = S_1+\tfrac{1}{12}\frac{2m}{2m-1}\,h\left(z_0-z_{\frac{1}{2}}-z_{m-\frac{1}{2}}+z_m\right). \tag{38}$$

This last is a simple modification of Parmentier's rule, and it will be found to give decidedly better results.

Similarly, from (33) and (34), we can get a number of formulæ, of which the following are examples :—

$$A = A_1 - \tfrac{1}{120}\frac{m(15m-26)}{(m-1)(m-2)}h(z_0 - z_1 - z_{m-1} + z_m)$$

$$+ \tfrac{1}{120}\frac{m(5m-6)}{(m-2)(m-3)}h(z_1 - z_2 - z_{m-2} + z_{m-1}), \qquad (39)$$

$$A = S_1 + \tfrac{1}{960}\frac{m(80m-177)}{(m-2)(m-3)}h(z_{\frac{1}{2}} - z_{\frac{3}{2}} - z_{m-\frac{3}{2}} + z_{m-\frac{1}{2}})$$

$$- \tfrac{1}{960}\frac{m(40m-57)}{(m-3)(m-4)}h(z_{\frac{3}{2}} - z_{\frac{5}{2}} - z_{m-\frac{5}{2}} + z_{m-\frac{3}{2}}), \qquad (40)$$

$$A = A_1 - \tfrac{1}{60}\frac{m(30m-29)}{(m-1)(2m-1)}h(z_0 - z_{\frac{1}{2}} - z_{m-\frac{1}{2}} - z_m)$$

$$+ \tfrac{1}{60}\frac{m(10m-9)}{(m-1)(2m-3)}h(z_{\frac{1}{2}} - z_1 - z_{m-1} + z_{m-\frac{1}{2}}), \qquad (41)$$

$$A = S_1 + \tfrac{1}{90}\frac{m(40m-57)}{(2m-1)(2m-3)}h(z_0 - z_{\frac{1}{2}} - z_{m-\frac{1}{2}} + z_m)$$

$$- \tfrac{1}{180}\frac{m(5m-6)}{(m-2)(2m-3)}h(z_{\frac{1}{2}} - z_{\frac{3}{2}} - z_{m-\frac{3}{2}} + z_{m-\frac{1}{2}}). \qquad (42)$$

The method can, of course, be extended to any number of pairs of ordinates. Thus we should get, for three pairs,

$$A = A_1 - \frac{mh}{5040}\left\{\frac{763m^2 - 3444m + 3636}{(m-1)(m-2)(m-3)}(z_0 - z_1 - z_{m-1} + z_m)\right.$$

$$- \frac{4(119m^2 - 504m + 432)}{(m-2)(m-3)(m-4)}(z_1 - z_2 - z_{m-2} + z_{m-1})$$

$$\left. + \frac{133m^2 - 462m + 360}{(m-3)(m-4)(m-5)}(z_2 - z_3 - z_{m-3} + z_{m-2})\right\}, \qquad (43)$$

$$A = S_1 + \frac{mh}{80640}\left\{\frac{9842m^2 - 53970m + 70407}{(m-2)(m-3)(m-4)}(z_{\frac{1}{2}} - z_{\frac{3}{2}} - z_{m-\frac{3}{2}} + z_{m-\frac{1}{2}})\right.$$

$$- \frac{9604m^2 - 46032m + 45810}{(m-3)(m-4)(m-5)}(z_{\frac{3}{2}} - z_{\frac{5}{2}} - z_{m-\frac{5}{2}} + z_{m-\frac{3}{2}})$$

$$\left. + \frac{3122m^2 - 12222m + 10935}{(m-4)(m-5)(m-6)}(z_{\frac{5}{2}} - z_{\frac{7}{2}} - z_{m-\frac{7}{2}} + z_{m-\frac{5}{2}})\right\}$$

$$(44)$$

These formulæ look clumsy; but the coefficients can be calculated, and their values tabulated, for various values of m.

11. The formulæ obtained in the last section are quite different in character from those obtained in § 4; but both sets of formulæ, and all other quadrature-formulæ depending on symmetrically-placed ordinates (whether equidistant or not), can be comprised in one general relation.

Taking, as in § 10, two ordinates

$$z_{\frac{1}{2}m(1-a)}, \quad z_{\frac{1}{2}m(1+a)},$$

we have, with the notation of that section,

$$A = \int_{-\frac{1}{2}mh}^{\frac{1}{2}mh} f(x_{\frac{1}{2}m}+\theta)\,d\theta$$

$$= mh\frac{\sinh D}{D} f(x_{\frac{1}{2}m})$$

$$= mh\cosh aD.f(x_{\frac{1}{2}m}) + \tfrac{1}{2}mh\left[\left(\frac{1}{D}-\frac{\cosh aD}{\sinh D}\right)f(x)\right]_{x=x_0}^{x=x_m}$$

$$= \tfrac{1}{2}mh\left(z_{\frac{1}{2}m(1-a)}+z_{\frac{1}{2}m(1+a)}\right)$$

$$+\left[\frac{1-3a^2}{6}(\tfrac{1}{2}mh)^2 z' - \frac{7-30a^2+15a^4}{360}(\tfrac{1}{2}mh)^4 z'''\right.$$

$$\left. + \frac{31-147a^2+105a^4-21a^6}{15120}(\tfrac{1}{2}mh)^6 z^{\text{v}} - \dots\right]_{x=x_0}^{x=x_m}. \quad (45)$$

Replacing a by $\beta, \gamma, \dots$, we get a series of different expressions for A. Multiplying these by $p, q, r, \dots$, adding, and dividing by $p+q+r+\dots$, we get a general expression for A in terms of a series of pairs of ordinates, symmetrically placed about $z_{\frac{1}{2}m}$, with terms involving the bounding values of $z', z''', z^{\text{v}}, \dots$; and the values of $p, q, r, \dots$ can be chosen so that the coefficients of these latter terms shall satisfy given conditions.

Thus, to get rid of the differential coefficients of z up to the $(2i-1)$th inclusive, we must take $p, q, r, \dots$ to satisfy the first i of the equations

$$\left.\begin{array}{l} p(1-3a^2) \qquad\qquad + q(1-3\beta^2) \qquad\qquad + r(1-3\gamma^2) \qquad\qquad + \dots = 0 \\ p(7-30a^2+15a^4)+q(7-30\beta^2+15\beta^4)+r(7-30\gamma^2+15\gamma^4)+\dots = 0 \\ \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \end{array}\right\}. \quad (46)$$

We have, therefore, as an approximate expression for the area,

$$A = \tfrac{1}{2}mh\left\{p\left(z_{\frac{1}{2}m(1-a)}+z_{\frac{1}{2}m(1+a)}\right)+q\left(z_{\frac{1}{2}m(1-\beta)}+z_{\frac{1}{2}m(1+\beta)}\right)\right.$$

$$\left. + r\left(z_{\frac{1}{2}m(1-\gamma)}+z_{\frac{1}{2}m(1+\gamma)}\right)+\dots\right\}, \quad (47)$$

where $p, q, r, \ldots$ satisfy the first $i+1$ of the equations

$$\left.\begin{aligned} p\ +q\ +r\ +\ldots &= 1 \\ p\alpha^2+q\beta^2+r\gamma^2+\ldots &= \tfrac{1}{3} \\ p\alpha^4+q\beta^4+r\gamma^4+\ldots &= \tfrac{1}{5} \\ p\alpha^6+q\beta^6+r\gamma^6+\ldots &= \tfrac{1}{7} \\ \vdots\qquad &\quad \vdots \end{aligned}\right\}. \tag{48}$$

The formula (47), with the condition given by (48), comprises all the formulæ given in the preceding sections.

12. We have assumed, in the last section, that the values of $\alpha, \beta, \gamma, \ldots$ are known. Suppose, however, that these are arbitrary, and that there are n of them. Then there will also be n of the coefficients $p, q, r, \ldots$. These $2n$ quantities can be determined so as to satisfy the first $2n$ of the equations (48); and, using them in (47), we shall get an expression which agrees with A up to the $(4n-3)$th differential coefficient of z. The first differential coefficient which will appear will be the $(4n-1)$th; and, if z is of the $(4n-1)$th degree in x, this differential coefficient will be a constant, and therefore will vanish when taken between limits. Hence, if z is of the $(4n-1)$th degree in x, or of a lower (integral) degree, we can obtain an accurate value for the area by choosing $2n$ suitable ordinates. Similarly, if z is of the $(4n+1)$th degree, or of a lower degree, we only require $2n+1$ ordinates. This is Gauss's well-known theorem.

IV. *Extension to Calculation of Volumes.*

13. The method of § 4 may be extended to the calculation of volumes. Suppose that a volume stands on a rectangular base, the sides being mh and nk, and that it is divided by $m-1$ planes at distances h in one direction and $n-1$ planes at distances k in the other. Then the ordinates which we have to take may be either the edges of the mn constituent prisms, or the mid-ordinates of the faces in one direction, or the central ordinates of the prisms. Generally they will be the edges of the prisms, and they may be denoted by

$$\begin{matrix} z_{0,0} & z_{1,0} & z_{2,0} & \ldots & z_{m,0}, \\ z_{0,1} & z_{1,1} & z_{2,1} & \ldots & z_{m,1}, \\ \vdots & \vdots & \vdots & & \vdots \\ z_{0,n} & z_{1,n} & z_{2,n} & \ldots & z_{m,n}. \end{matrix}$$

Let $V_{1,1}$ denote hk times the sum of these ordinates, taken with coefficients

$$\begin{matrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} \end{matrix}$$

Then the true volume V will differ from $V_{1,1}$ by a rather complicated expression, involving the differences of bounding values of certain differential coefficients of the ordinate z. It is not necessary to write down this expression; we need only note that it consists of

(i.) terms in $h^2k\dfrac{dz}{dx}, \quad h^4k\dfrac{d^3z}{dx^3}, \quad h^6k\dfrac{d^5z}{dx^5}, \quad \ldots;$

(ii.) terms in $hk^2\dfrac{dz}{dy}, \quad hk^4\dfrac{d^3z}{dy^3}, \quad hk^6\dfrac{d^5z}{dy^5}, \quad \ldots;$

(iii.) terms in $h^2k^2\dfrac{d^2z}{dx\,dy}, \quad h^4k^2\dfrac{d^4z}{dx^3\,dy}, \quad h^6k^2\dfrac{d^6z}{dx^5\,dy}, \quad \ldots,$

$$h^2k^4\frac{d^4z}{dx\,dy}, \quad h^4k^4\frac{d^6z}{dx^3\,dy^3}, \quad \ldots,$$

$$h^2k^6\frac{d^6z}{dx\,dy^5}, \quad \ldots,$$

&c.

Now let b and β denote any factors of m and of n respectively, and let $V_{b,1}$, $V_{1,\beta}$, $V_{b,\beta}$ denote the different values of $V_{1,1}$ obtained by altering the intervals from h to bh or from k to βk. Then we may take

$$V = \frac{pV_{1,1}+qV_{b,1}+rV_{1,\beta}+sV_{b,\beta}}{p+q+r+s} \tag{49}$$

as a more accurate value of the volume, and get rid of the terms in

$$\frac{dz}{dx}, \quad \frac{dz}{dy}, \quad \frac{d^2z}{dx\,dy},$$

provided p, q, r, s satisfy the equations

$$\left.\begin{aligned} p+qb^2+r\beta+sb^2\beta &= 0 \\ p+qb+r\beta^2+sb\beta^2 &= 0 \\ p+qb^2+r\beta^2+sb^2\beta^2 &= 0 \end{aligned}\right\}. \tag{50}$$

These equations give

$$p : q : r : s :: b^2\beta^2 : -\beta^2 : -b^2 : 1;$$

so that our approximate formula is

$$V = \frac{b^2\beta^2 V_{1,1} - \beta^2 V_{b,1} - b^2 V_{1,\beta} + V_{b,\beta}}{(b^2-1)(\beta^2-1)}. \tag{51}$$

If, for instance, $b = \beta = 2$, we have

$$V = \tfrac{1}{9}\left(16V_{1,1} - 4V_{2,1} - 4V_{1,2} + V_{2,2}\right), \tag{52}$$

which is Simpson's rule, applied to volumes.

14. To get more accurate formulæ, by using a larger number of factors of m and n, we might proceed in the same way, and write down the equations corresponding to (50). But it is not necessary to do this. The coefficients in (51) are the coefficients of the products of A and A' in the expression

$$\frac{b^2 A_1 - A_b}{b^2 - 1}\,\frac{\beta^2 A'_1 - A'_\beta}{\beta^2 - 1},$$

the first factor of which is the formula giving the area of a section in one direction in terms of the ordinates at intervals h and the ordinates at intervals bh, while the second factor is the corresponding formula for a section in the other direction, the intervals being k and βk. This rule can be extended, as follows:—

The true value of V is

$$\int_{x_0}^{x_m}\int_{y_0}^{y_n} z\,dx\,dy = \frac{\sinh \frac{1}{2}mh\,d/dx}{d/dx}\,\frac{\sinh \frac{1}{2}nk\,d/dy}{d/dy}\,z_{\frac{1}{2}m,\frac{1}{2}n}. \tag{53}$$

Now let $a, b, c, \ldots$ denote any factors of m. Then, if we express each side of the formula (22) as the result of an operation on the mid-ordinate $z_{\frac{1}{2}m}$, we see that

$$\begin{aligned}
&\frac{\sinh \frac{1}{2}mh\,d/dx}{d/dx} \\
&= \left\{ \frac{p.ah\mu_a\sigma_a + q.bh\mu_b\sigma_b + r.ch\mu_c\sigma_c + \ldots}{p+q+r+\ldots} \right. \\
&\quad + (-)^i \frac{pa^{2i} + qb^{2i} + rc^{2i} + \ldots}{p+q+r+\ldots}\,\frac{B_i}{(2i)!}\,h^{2i}\,\frac{d^{2i-1}}{dx^{2i-1}} \\
&\quad \left. + (-)^{i+1} \frac{pa^{2i+2} + qb^{2i+2} + rc^{2i+2} + \ldots}{p+q+r+\ldots}\,\frac{B_{i+1}}{(2i+2)!}\,h^{2i+2}\,\frac{d^{2i+1}}{dx^{2i+1}} + \ldots \right\} \delta_m;
\end{aligned} \tag{54}$$

where μ, δ, σ denote the central-difference operators explained in a previous paper.* These operators are defined by the relations

$$\left.\begin{aligned} \mu_f\phi(x) &= \tfrac{1}{2}\{\phi(x+\tfrac{1}{2}fh)+\phi(x-\tfrac{1}{2}fh)\} \\ \delta_f\phi(x) &= \phi(x+\tfrac{1}{2}fh)-\phi(x-\tfrac{1}{2}fh) \\ \delta_f\sigma_f\phi(x) &= \phi(x) \end{aligned}\right\},$$

whence, if f is a factor of m,

$$\mu_f\sigma_f(z_m-z_0) = \tfrac{1}{2}z_0+z_f+z_{2f}+\ldots+z_{m-f}+\tfrac{1}{2}z_m.$$

Similarly, if α, β, γ, ... are j factors of n, and if μ', δ', σ' denote the operations μ, δ, σ, performed with regard to y, we shall have a formula

$$\frac{\sinh\frac{1}{2}nk\,d/dy}{d/dy} = \left\{\frac{p'.\alpha k\mu'_\alpha\sigma'_\alpha+q'.\beta k\mu'_\beta\sigma'_\beta+r'.\gamma k\mu'_\gamma\sigma'_\gamma+\ldots}{p'+q'+r'+\ldots}\right.$$
$$\left.+\text{terms in } \frac{d^{2j-1}}{dy^{2j-1}},\ \frac{d^{2j+1}}{dy^{2j+1}},\ \ldots\right\}\delta'_n, \quad (55)$$

where p', q', r', ... satisfy equations similar to (21). If, now, we substitute from (54) and (55) in (53), the two sets of operators μ, σ, δ and μ', σ', δ' will combine with one another, and with powers of d/dx and d/dy, according to the ordinary laws of algebra, and we shall have, as an approximate formula,

$$V = \frac{p.ah\mu_a\sigma_a+q.bh\mu_b\sigma_b+r.ch\mu_c\sigma_c+\ldots}{p+q+r+\ldots}$$
$$\times\frac{p'.\alpha k\mu'_\alpha\sigma'_\alpha+q'.\beta k\mu'_\beta\sigma'_\beta+r'.\gamma k\mu'_\gamma\sigma'_\gamma+\ldots}{p'+q'+r'+\ldots}\,\delta_m\delta'_n z_{\frac{1}{2}m,\frac{1}{2}n}$$

$$\begin{aligned} = \{&pp'V_{a,\alpha}+pq'V_{a,\beta}+pr'V_{a,\gamma}+\ldots \\ &+qp'V_{b,\alpha}+qq'V_{b,\beta}+qr'V_{b,\gamma}+\ldots \\ &+rp'V_{c,\alpha}+rq'V_{c,\beta}+rr'V_{c,\gamma}+\ldots \\ &+\ldots\}/(p+q+r+\ldots)(p'+q'+r'+\ldots), \end{aligned}\quad (56)$$

the ratios $p:q:r:\ldots$ and $p':q':r':\ldots$ being given by the equations

$$\left.\begin{aligned} pa^2 \quad +qb^2 \quad +rc^2 \quad +\ldots &= 0 \\ pa^4 \quad +qb^4 \quad +rc^4 \quad +\ldots &= 0 \\ \vdots \qquad\qquad\qquad &\ \vdots \\ pa^{2i-2}+qb^{2i-2}+rc^{2i-2}+\ldots &= 0 \end{aligned}\right\} \quad (57)$$

* *Proceedings*, Vol. xxxi., p. 449.

and

$$\left.\begin{array}{l} p'\alpha^2 \;\; +q'\beta^2 \;\; +r'\gamma^2 \;\; +\ldots = 0 \\ p'\alpha^4 \;\; +q'\beta^4 \;\; +r'\gamma^4 \;\; +\ldots = 0 \\ \vdots \hspace{4cm} \vdots \\ p'\alpha^{2\nu-2}+q'\beta^{2\nu-2}+r'\gamma^{2\nu-2}+\ldots = 0 \end{array}\right\} . \tag{58}$$

Suppose, for instance, that m is a multiple of 4, and n a multiple of 6, and that we take (iii.) of § 5 in the one case, and (v.) (Weddle's rule) in the other. Then, developing the product

$$\frac{64A_1-20A_2+A_4}{45} \cdot \frac{15A_1'-6A_2'+A_3'}{10},$$

we find, as our approximate formula,

$$\begin{aligned} V = \tfrac{1}{450}\,(&960V_{1,1}-384V_{1,2}+64V_{1,3} \\ &-300V_{2,1}+120V_{2,2}-20V_{2,3} \\ &+\;15V_{4,1}-\;\;6V_{4,2}+\;\;V_{4,3}). \end{aligned} \tag{59}$$

15. In the last section we have only considered the case in which the given ordinates are the edges of the constituent prisms; but the method applies equally to the other cases, subject to the limitations pointed out in § 8. Thus, if the ordinates are the central ordinates

$$\begin{array}{ccccc} z_{\frac12,\frac12} & z_{\frac32,\frac12} & z_{\frac52,\frac12} & \ldots & z_{m-\frac12,\frac12} \\ z_{\frac12,\frac32} & z_{\frac32,\frac32} & z_{\frac52,\frac32} & \ldots & z_{m-\frac12,\frac32} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{\frac12,n-\frac12} & z_{\frac32,n-\frac12} & z_{\frac52,n-\frac12} & \ldots & z_{m-\frac12,n-\frac12} \end{array}$$

and if $W_{1,1}$ denotes hk times the sum of these ordinates, we shall have a formula

$$\begin{aligned} V = \{\,&pp'W_{a,\alpha}+pq'W_{a,\beta}+pr'W_{a,\gamma}+\ldots \\ &+qp'W_{b,\alpha}+qq'W_{b,\beta}+qr'W_{b,\gamma}+\ldots \\ &+rp'W_{c,\alpha}+rq'W_{c,\beta}+rr'W_{c,\gamma}+\ldots \\ &+\ldots\}\,/\,(p+q+r+\ldots)(p'+q'+r'+\ldots), \end{aligned} \tag{60}$$

the ratios $p:q:r:\ldots$ and $p':q':r':\ldots$ being given, as before, by (57) and (58); but the factors $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ must all be odd. Similarly, if the ordinates are

$$\begin{array}{ccccc} z_{0,\frac{1}{2}} & z_{1,\frac{1}{2}} & z_{2,\frac{1}{2}} & \dots & z_{m,\frac{1}{2}} \\ z_{0,\frac{3}{2}} & z_{1,\frac{3}{2}} & z_{2,\frac{3}{2}} & \dots & z_{m,\frac{3}{2}} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{0,n-\frac{1}{2}} & z_{1,n-\frac{1}{2}} & z_{2,n-\frac{1}{2}} & \dots & z_{m,n-\frac{1}{2}} \end{array}$$

the factors of m may be odd or even, but the factors of n must only be odd. The coefficients by which the respective ordinates are to be multiplied in this latter case are

$$\begin{array}{cccccc} \frac{1}{2} & 1 & 1 & \dots & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & \dots & 1 & \frac{1}{2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{2} & 1 & 1 & \dots & 1 & \frac{1}{2} \end{array}$$

Second Complément à l'Analysis Situs. Par H. Poincaré. Read at request of the President, June 14th, 1900, and received June 30th, 1900.

Introduction.

J'ai publié dans le *Journal de l'Ecole Polytechnique* (Tome c, N° 1) un travail intitulé "Analysis Situs"; je me suis occupé une seconde fois du même problème dans un mémoire portant pour titre "Complément à l'Analysis Situs," et qui a été imprimé dans les *Rendiconti del Circolo Matematico di Palermo* (Tome XIII, 1899).

Cependant la question est loin d'être épuisée, et je serai sans doute forcé d'y revenir à plusieurs reprises. Pour cette fois, je me bornerai à certaines considérations qui sont de nature à simplifier, à éclaircir et à compléter les résultats précédemment acquis.

Les renvois portant simplement une indication de paragraphe ou de page se rapporteront au premier mémoire, celui du *Journal de l'Ecole Polytechnique;* les renvois où ces indications seront précédées de la lettre c se rapporteront au mémoire des *Rendiconti.*